Probabilistic analysis of an approximation algorithm for the m-peripatetic salesman problem on random instances unbounded from above.

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But most of this problems are NP-hard, like the well-known Travelling Salesman Problem.

m-Peripatetic Salesman Problem (m-PSP)[Krarup 1974]

The problem is to find

m edge-disjoint Hamiltonian cycles $H_1, \ldots, H_m$
in a given complete graph $G = (V, E)$
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$$W_1(H_1) + \ldots + W_m(H_m) = \sum_{i=1}^{m} \sum_{e \in H_i} w_i(e) \to \min(\max).$$

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Known results for the m-PSP

Some previous results for m-PSP

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- We have obtained the performance guarantees of this algorithm for certain classes of random inputs of the problem.
- We have justified the conditions for the algorithm to be asymptotically exact on the considered classes of inputs.

Algorithm $\widetilde{A}$ for minimum-weight m-PSP

Input:

A complete n-vertex graph $G = (V, E)$ with weight functions $w_i : E \to \mathbf{R}_+, \; i = 1, \ldots, m$, where $m < n/4$

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Main idea:

modification of the greedy algorithm; finding each Hamiltonian cycle by turns.

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In Stage i we consider given graph G with weight function w_i and construct Hamiltonian cycle H_i in 3 steps.

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For the formation of all further Hamiltonian cycles $i + 1, \ldots, m$ forbid all edges in H_i and the corresponding reverse edges.

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1. given undirected $\hat{n}$-vertex graph $H(V_H, E_H)$ such that $\forall v \in V_H \quad deg(v) > \hat{n}/2$,
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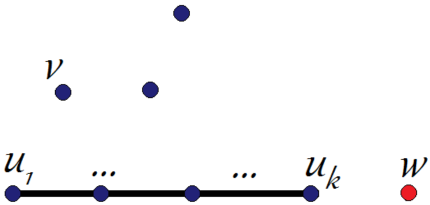
While $1 \leq k \leq \hat{n}$

- Let $P = \{u_1, \ldots, u_k\}$ be the constructed path.



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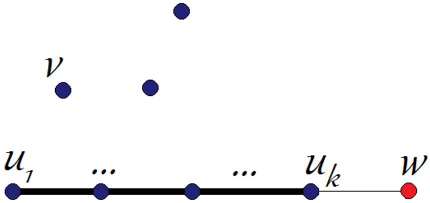

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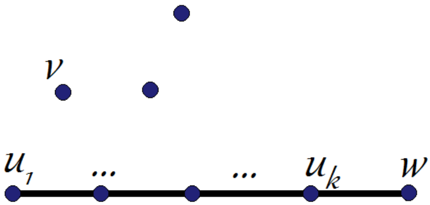

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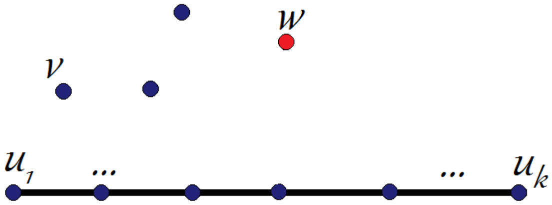

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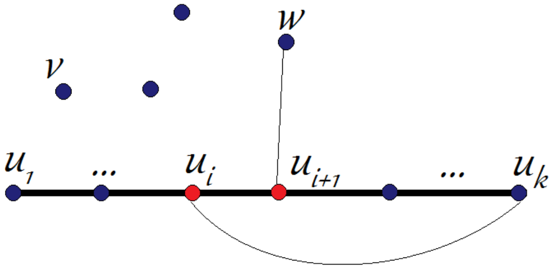

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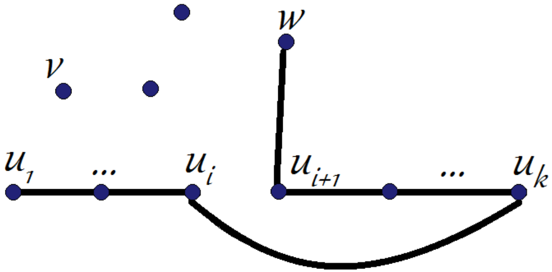

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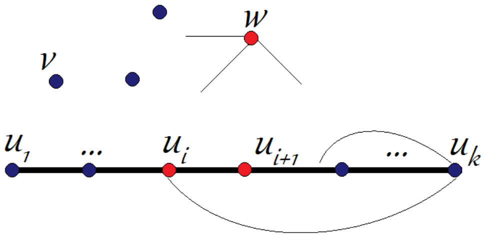
Correctness of the procedure $\mathbb{P}$



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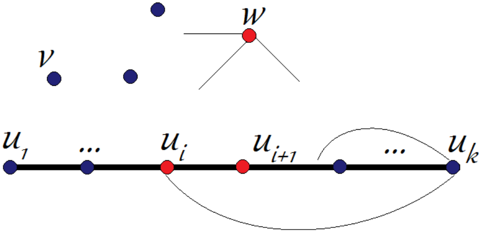

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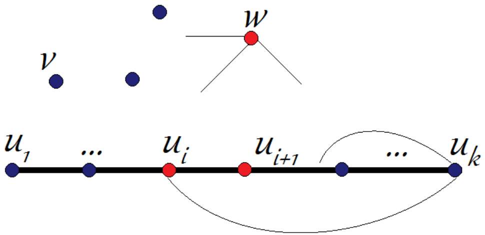

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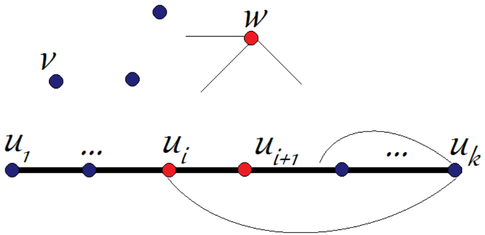

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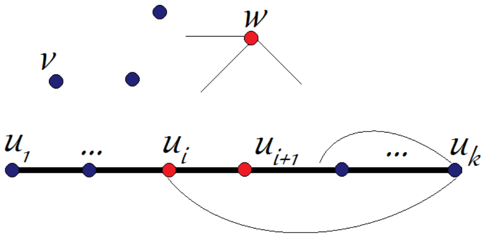

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Contradiction.

Random inputs for the m-PSP

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$m \times n \times n$ cost matrix $C = \left(c_{ijk}\right)$, where c_{ijk} is equal to the i-th weight function $w_i(e)$ of edge $e = (j, k)$, $i = \overline{1, m}$,$j, k = \overline{1, n}$.

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- of $\mathcal{F}_\beta$-majorizing type,
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Denote $\xi_{is} = w_i(e_s^{(i)})$. Then

$$Pr\Big\{ \sum_{i=1}^{m} \sum_{s=1}^{n} \xi_{is} > (1+\varepsilon_{\widetilde{A}})OPT \Big\} \le \delta_{\widetilde{A}}.$$

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- Use Petrov's Theorem

Petrov V.V. 'Limit theorems for sums of independent random variables', 1987

Petrov's Theorem

Consider independent random variables $\eta_1, \ldots, \eta_n$ and $S = \Sigma_{k=1}^{n} \eta_k$. Let there be positive constants $g_1, \ldots, g_n$ and T, such that

$$\mathbf{E} e^{t\eta_k} \leq e^{\frac{g_k t^2}{2}},\ 0 \leq t \leq T,\ k = 1, \ldots, n$$

Denote $\mathcal{G} = \Sigma_{k=1}^{n} g_k$. Then

$$\mathbf{Pr}\{S \geq x\} \leq \begin{cases} e^{\frac{-x^2}{2G}}, & 0 \leq x \leq \mathcal{G}T, \\ e^{\frac{-Tx}{2}}, & x \geq \mathcal{G}T \end{cases}$$

Where $\mathbf{E}X$ is the expected value of random variable X.

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Probabilistic analysis of Algorithm $\widetilde{A}$

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The performance bounds of the algorithm obtained for random inputs of m-PSP with some distribution function $F(x)$ will also be true for random inputs with any distribution function of $F(x)$-majorizing type.

Statement 1

Let $\xi_1, ..., \xi_k$ be the independent random variables with distribution function $F(x)$,
Let $\hat{F}(x)$ be the distribution function of $\xi = \min(\xi_1, ..., \xi_k)$,
Let $\eta_1, ..., \eta_k$ be the independent random variables with distribution function $G(x)$,
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Then for any x

$$F(x) \le G(x) \Rightarrow \hat{F}(x) \le \hat{G}(x).$$

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The statement follows directly from the equations

$$\hat{F}(x) = 1 - (1 - F(x))^k \quad \text{and} \quad \hat{G}(x) = 1 - (1 - G(x))^k.$$

Distribution functions of majorizing type

Statement 2

Let $P_\xi, P_\eta, P_\zeta, P_\chi$ be the distribution functions of random variables ξ, η, ζ, χ, respectively. And let ξ and ζ be independent, η and χ be independent. Then

$$(\forall x\ P_\xi(x) \leq P_\eta(x)) \wedge (\forall y\ P_\zeta(y) \leq P_\chi(y)) \Rightarrow (\forall z\ P_{\xi+\zeta}(z) \leq P_{\eta+\chi}(z)).$$

Proof

$$P_{\xi+\zeta}(x) = \int\limits_{-\infty}^{\infty} P_\xi(x-y)dP_\zeta(y) \leq \int\limits_{-\infty}^{\infty} P_\eta(x-y)dP_\zeta(y)$$

$$= P_{\eta+\zeta}(x) = \int\limits_{-\infty}^{\infty} P_\zeta(x-y)dP_\eta(y) \leq \int\limits_{-\infty}^{\infty} P_\chi(x-y)dP_\eta(y) = P_{\eta+\chi}(x).$$

Distribution functions of majorizing type

Theorem

Let the distribution function $F(x)$ of random inputs of m-PSP be s.t.

$$F(x) \geq P(x).$$

Then Algorithm $\widetilde{A}$ has the same performance guarantees $(\varepsilon_{\widetilde{A}}, \delta_{\widetilde{A}})$ on these random inputs, as it would have on random inputs with distribution function $P(x)$.

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Corollary (for example)

The performance guarantees of Algorithm $\widetilde{A}$ obtained in the case of random inputs with **exponential** distribution with a parameter β will also hold in case of random inputs with **truncated normal** distribution function with a certain parameter σ_n.

The conditions of the asymptotic optimality of Algorithm $\widetilde{A}$

For the random inputs of m-PSP with the distribution function of **UNI**$[a_n, b_n]$-**majorizing type**, $0 < a_n < b_n$, Algorithm $\widetilde{A}$ is asymptotically exact with the following performance guarantees

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for $2 \le m \le \ln n$

$$\varepsilon_{\widetilde{A}} = O\left(\frac{b_n/a_n}{n/\ln n}\right), \quad \delta_{\widetilde{A}} = n^{-9},$$

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Last publication of authors on the theme of the report

The paper

Э.Х. Гимади, А.М. Истомин, И.А. Рыков, О.Ю. Цидулко. Вероятностный анализ приближённого алгоритма для решения задачи нескольких коммивояжеров на случайных входных данных, неограниченных сверху // Труды ИММ УрО РАН. 2014. Т. 20, № 2, С. 88-98.
Probabilistic analysis of an approximation algorithm for the m-peripatetic salesman problem on random instances unbounded from above.

Thank you!

Thank you for your attention!

Algorithm $\widetilde{A}$ solving the m-PSP

- **Input:** A complete n-vertex graph $G = (V, E)$ with weight functions $w_i : E \to \mathbf{R}_+,\ i = 1, \ldots, m$, where $m < n/4$
- **Output:** m edge disjoint Hamiltonian cycles $H_1, \ldots, H_m$
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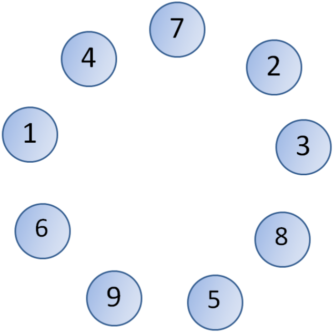
- **Output:** m edge disjoint Hamiltonian cycles $H_1, \ldots, H_m$
- **Time complexity:** $O(mn^2)$
- **Main idea:** modification of the greedy algorithm; finding each Hamiltonian cycle by turns.

Step 0

i – number of current Hamiltonian cycle.
F – set of forbidden edges (at first $F = \emptyset$).

1. Consider the traveling salesman problem for graph $G \setminus F$ with weight function w_i.



2. Randomly choose the first vertex to start with. Let it be vertex 1.
3. Among all neighbors of 1 randomly choose a vertex v.

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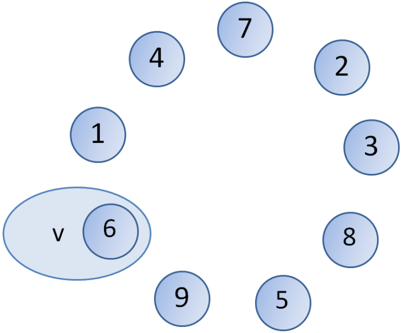

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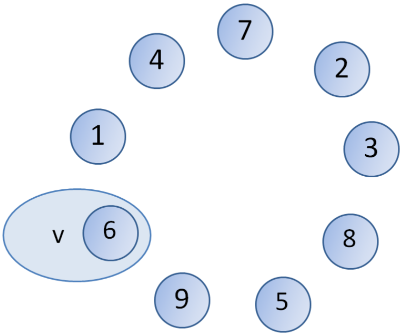
Step 1

i – number of current Hamiltonian cycle.
s – number of processed vertices.

1. While $s < n - 4i$



$i = 1, \ \ s = 1.$

2. "go to the nearest unvisited vertex, except vertex v.
3. $s := s + 1$.

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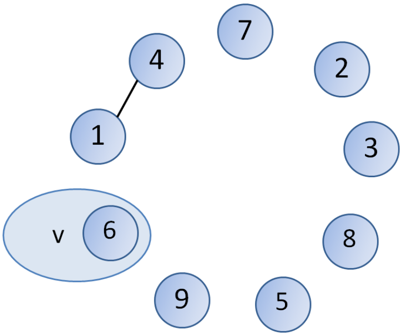

$$i = 1, \; s = 2.$$

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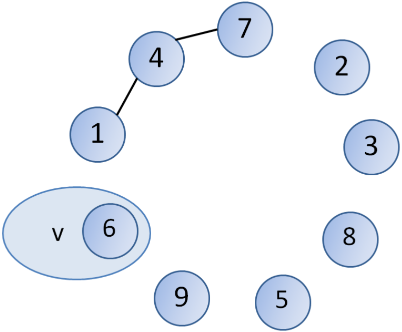

$i = 1, \;\; s = 3.$

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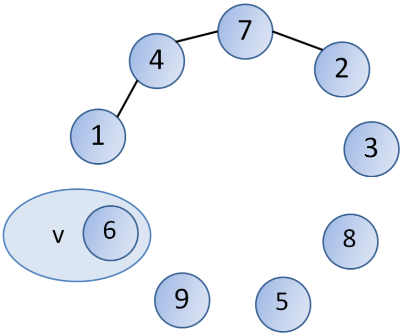

$i = 1, \; s = 4.$

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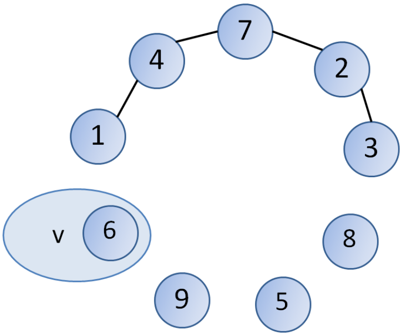

$i = 1, \ s = 5.$

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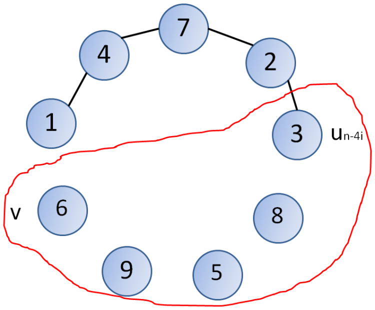
Step 2

- Consider a subgraph H induced by all unprocessed vertices, and the last processed vertex:



- Using procedure $\mathbb{P}$ build a path with endpoints u_{n-4i}, v,
- Complete the Hamiltonian cycle H_i.
- For further stages forbid all edges $\in H_i$ and the corresponding reverse edges.

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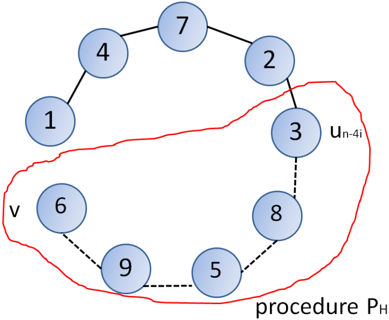

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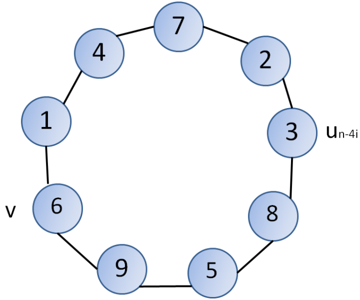

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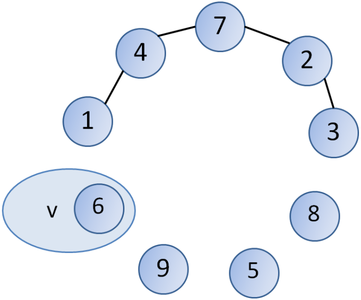

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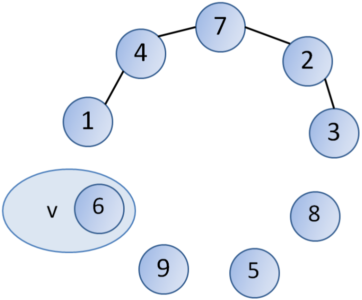

- the degree of each vertex at the beginning of Step 1: $deg(v) = n - 2 - 2(i - 1) = n - 2i$

Relevance of Algorithm $\widetilde{A}$

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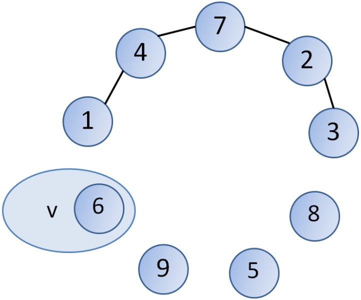
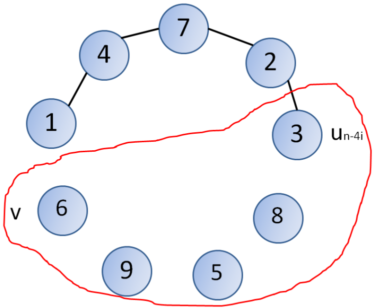


- the degree of each vertex at the beginning of Step 1: $deg(v) = n - 2 - 2(i - 1) = n - 2i$
- the greedy algorithm makes $n - 4i$ steps, so it is always possible to make the next step.

Consider subgraph H constructed in Step 2.



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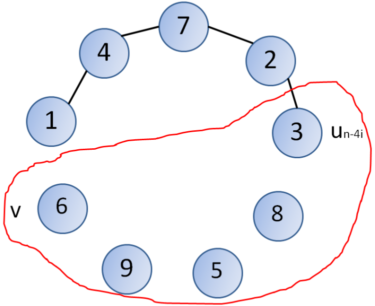

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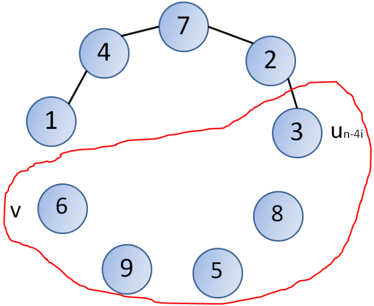

- Since $s = n - 4i$, $|V_H| = n - s + 1 = 4i + 1$.
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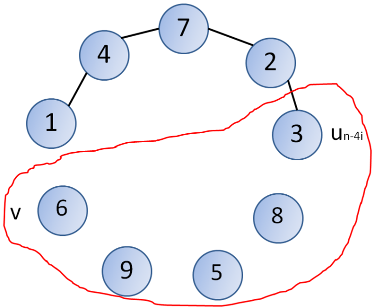

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- $\forall v \in V_H \; deg(v) \geq (4i + 1 - 1) - 2(i - 1) = 2i + 2$
- Thus we can use procedure $\mathbb{P}$ for this graph.

For each Hamiltonian cycle $H_1, \ldots, H_m$ we have:

Time complexity of Algorithm $\widetilde{A}$

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Total time complexity: $O(mn^2)$.