APPROXIMATION OF MINIMUM WEIGHT k-SIZE CYCLE COVER PROBLEM

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Abstract

- For a given natural k, a problem of k collaborating salesmen sharing the same set of cities (nodes of graph) to serve is studied.
- We call it Minimum Weight k-Size Cycle Cover Problem (Min-k-SCCP).
- Related problems
  - Min-1-SCCP is Traveling Salesman Problem (TSP)
  - **Vertex-Disjoint Cycle Cover Problem**
  - k-Peripatetic Salesmen Problem
  - Min-L-CCP
- Min-k-SCCP can be considered as a special case of Vehicle Routing Problem (VRP)

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- Nuclear Power Plant dismantling problem

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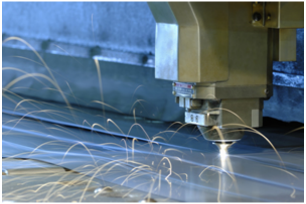
- high-precision metal shape cutting problem



Abstract - ctd.

Results

1. Min-k-SCCP is strongly NP-hard and hardly approximable in the general case
2. Metric and Euclidean cases are intractable as well
3. 2-approximation algorithm for Metric Min-k-SCCP is proposed
4. Polynomial-time approximation scheme (PTAS) for Min-2-SCCP on the plane is constructed

Contents

Definitions and Notation

Standard notation is used

- $\mathbb{R}$ — field of real numbers
- $\mathbb{N}$ — field of rational numbers
- $\mathbb{N}_m$ — integer segment $\{1, ..., m\}$,
- $\mathbb{N}_m^0$ — segment $\{0, ..., m\}$.
- $G = (V, E, w)$ is a simple complete weighted (di)graph with loops, edge-weight function $w : E \to \mathbb{R}$

Minimum Weight k-Size Cycle Cover Problem (Min-k-SCCP)

Input: graph $G = (V, E, w)$.

Find: a minimum-cost collection $\mathcal{C} = C_1, ..., C_k$ of vertex-disjoint cycles such that $\bigcup_{i \in \mathbb{N}_k} V(C_i) = V$.

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$$
\begin{aligned}
\min \quad & \sum_{i=1}^{k} W(C_i) \equiv \sum_{i=1}^{k} \sum_{e \in E(C_i)} w(e) \\
s.t. \quad & \\
& C_1, \ldots, C_k \text{ are cycles in } G \\
& C_i \cap C_j = \varnothing \\
& V(C_1) \cup \ldots \cup V(C_k) = V
\end{aligned}
$$

Metric and Euclidean Min-k-SCCP

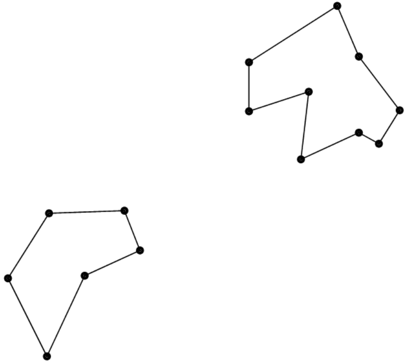
Metric Min-k-SCCP

- $w_{ij} \geqslant 0$
- $w_{ii} = 0$
- $w_{ij} = w_{ji}$
- $w_{ij} + w_{jk} \geqslant w_{ik} \quad (\{i, j, k\})$

Euclidean Min-k-SCCP

- For some $d > 1$, $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$
- $w_{ij} = \|v_i - v_j\|_2$

Instance of Euclidean Min-2-SCCP



Complexity

Known facts

- (Karp, 1972) TSP is strongly NP-hard
- (Sahni and Gonzales, 1976) TSP can not be approximated within $O(2^n)$ (unless $P = NP$)
- (Papadimitriou, 1977) Euclidean TSP is NP-hard

Complexity

Theorem 1

For any $k \geqslant 1$, Min-k-SCCP is strongly NP-hard.

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Proof idea

- Reduce TSP to Min-k-SCCP by cloning the instance
- Spread them apart
- Show that any optimal solution of Min-k-SCCP consists of cheapest Hamiltonian cycles for the initial TSP

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Corollary

- Min-k-SCCP also can not be approximated within $O(2^n)$ (unless $P = NP$)
- Metric Min-k-SCCP and Euclidean Min-k-SCCP are NP-hard as well

Minimum spanning forest

- k-forest is an acyclic graph with k connected components
- For any k-forest F, weight (cost)

$$W(F) = \sum_{e \in E(F)} w(e)$$

- k-Minimum Spanning Forest (k-MSF) Problem

Kruskal's algorithm for k-MSF

1. Start from the empty n-forest F_0.
2. For each $i \in \mathbb{N}_{n-k}$ add the edge
$$e_i = \arg\min\{w(e) \colon F_{i-1} \cup \{e\} \text{ remains acyclic}\}$$
to the forest F_{i-1}.
3. Output k-forest F^*.

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Theorem 2

F^* is k-Minimum Spanning Forest.

2-approximation algorithm for Metric Min-k-SCCP

Following to the scheme of well-known 2-approx. algorithm for Metric TSP.
Wlog. assume $k < n$.
Algorithm:

1. Build a k-MSF F
2. Take edges of F twice
3. For any non-trivial connected component, find a Eulerian cycle
4. Transform them into Hamiltonian cycles
5. Output collection of these cycles adorned by some number of isolated vertices

Correctness proof

Assertion

Approximation ratio:

$$2(1-2/n) \leqslant \sup_I \frac{APP(I)}{OPT(I)} \leqslant 2(1-1/n)$$

Running-time:

$$O(n^2 \log n).$$

Proof sketch

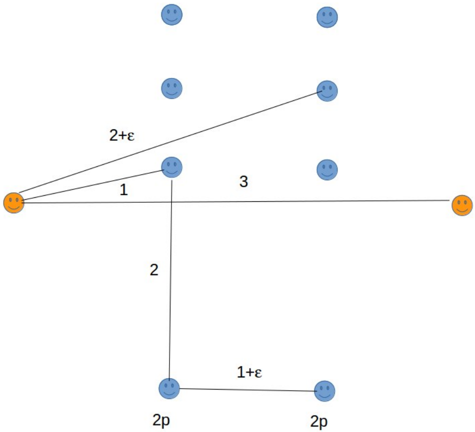
Consider optimal cycle cover $\mathcal{C}$ (with weight OPT).
Removing the most heavy edge from any non-empty cycle transform it into some spanning forest $F(\mathcal{C})$ with cost SF.
Then

$$MSF \leqslant SF \leqslant OPT(1-1/n),$$

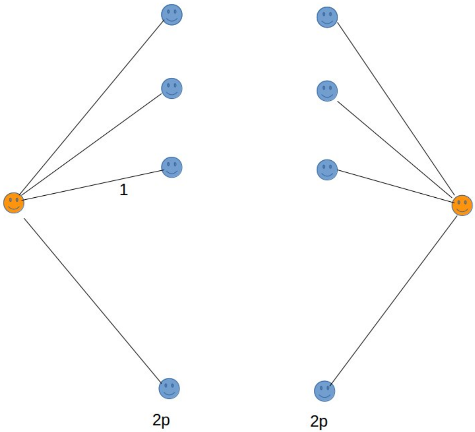
where

$$APP \leqslant 2 \cdot MSF \leqslant 2(1-1/n)OPT.$$

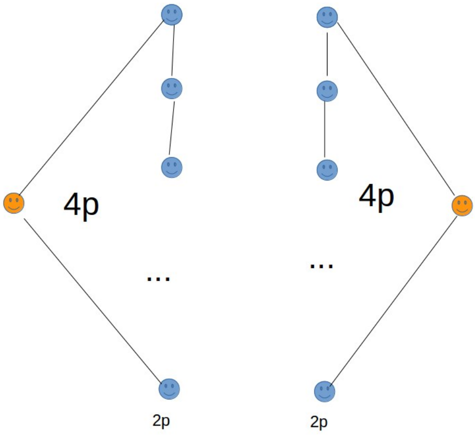
Lower bound - instance



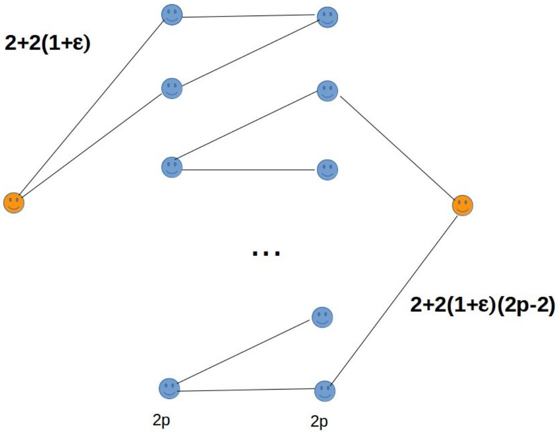
Lower bound - 2-forest



Lower bound - approximation



Lower bound - better approximation



Lower bound - discussion

- number of nodes $n = 4p + 2$
- $APP = 8p$
- $OPT \leq 4p + 2 + 2\varepsilon(2p-1)$
- for approximation ratio r we have

$$r \geq \sup_{\varepsilon \in (0,1)} \frac{8p}{4p + 2 + 2\varepsilon(2p-1)} = \frac{4p}{2p+1} = 2(1 - 2/n)$$

PTAS for Euclidean Min-2-SCCP on the plane

Definition

For a combinatorial optimization problem, Polynomial-Time Approximation Scheme (PTAS) is a collection of algorithms such that for any fixed $c > 1$ there is an algorithm finding a $(1 + 1/c)$-approximate solution in a polynomial time depending on c.

Instance preprocessing

For an arbitrary instance of Min-2-SCCP, there exists one of the following alternatives (each of them can be verified in polynomial time)

1. The instance in question can be decomposed into 2 independent TSP instances;
2. Inter-node distance can be overestimated using some function that depends on OPT linearly.

Young's inequality

Consider a set S of diameter D in d-dimensional Euclidean space, let R be a radius of the smallest containing sphere.
Then

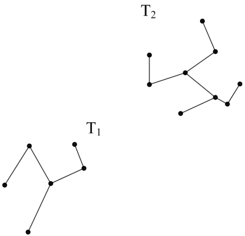
$$\frac{1}{2}D \leqslant R \leqslant \left(\frac{d}{2d+2}\right)^{\frac{1}{2}} D.$$

In particular, in the plane:

$$\frac{1}{2}D \leqslant R \leqslant \frac{\sqrt{3}}{3}D. \tag{1}$$

Instance preprocessing - ctd.

- Construct 2-MSF consisting of trees T_1 and T_2.



- let D_1, D_2 be diameters of T_1 and T_2, and R_1, R_2 be radii of the smallest circles $B(T_1)$ and $B(T_2)$ containing the trees T_1 and T_2. Denote $D = \max\{D_1, D_2\}$ and $R = \max\{R_1, R_2\}$.

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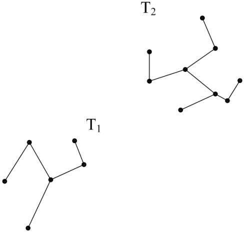

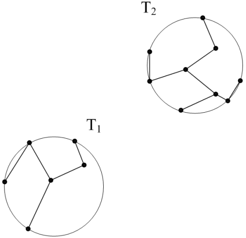

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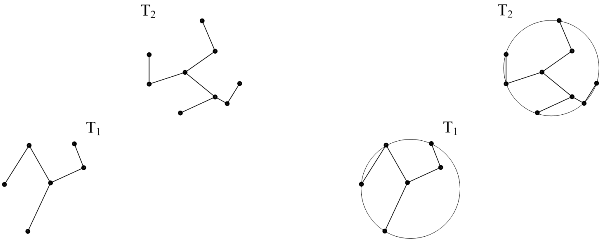

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Problem decomposition

Define $\rho(T_1, T_2)$ as a distance between centers of circles $B(T_1)$ and $B(T_2)$.

Assertion

If $\rho(T_1, T_2) > 5R$ then the considered instance Min-2-SCCP can be decomposed into two TSP instances for $G(T_1)$ and $G(T_2)$.

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Proof sketch

Suppose, on the contrary, that there is an optimal 2-SCC $\mathcal{C} = \{C_1, C_2\}$ such that $C_1 \cap T_1 \neq \varnothing$ and $C_1 \cap T_2 \neq \varnothing$.

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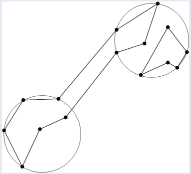

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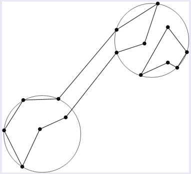
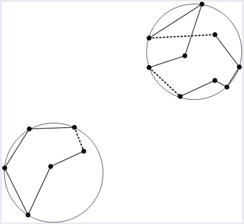


Then C_1 contains at least two edges, spanning T_1 and T_2

Problem decomposition

Proof (ctd.)

- By the condition, the weight of each of them is greater than $3R$
- Remove them and close the cycles inside $B(T_1)$ and $B(T_2)$



- Obtain the lighter 2-SCC

Problem decomposition

Statement

If $\rho(T_1, T_2) \leqslant 5R$ then the maximum inter-node distance $D(G)$ for the graph G is no more than $\frac{7\sqrt{3}}{3}OPT$.

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- In our case $D(G) \leqslant 7R$
- Due to Young's inequality and $D \leqslant MSF \leqslant OPT$ we have

$$R \leqslant \frac{\sqrt{3}}{3} D \leqslant \frac{\sqrt{3}}{3} \cdot OPT,$$

- i.e. $D(G) \leqslant \frac{7\sqrt{3}}{3} \cdot OPT$.

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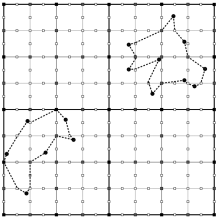
In this case Min-2-SCCP instance can be enclosed into some axis-aligned square $\mathcal{S}$ of size $7/\sqrt{3} \cdot OPT$

Main idea

Randomized partitioning of the square $\mathcal{S}$ into smaller subsquares and subsequent search for minimum 2-SCC of special kind

1) every inter-node segment of its cycles is piece-wise linear and intersects all squares' borders at special points (*portals*) only;
2) portals number and locations together with maximum number of intersections (for each border) are defined in advance and depend on accuracy parameter c;



Rounded Min-2-SCCP

Definition

Instance of Min-2-SCCP is called *rounded* if

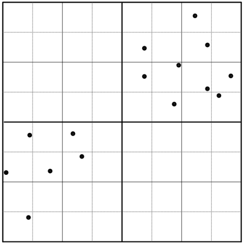
- every vertex of the graph G has integral coordinates $x_i, y_i \in \mathbb{N}^0_{O(n)}$
- for any edge e, $w(e) \geqslant 4$

Lemma 3

PTAS for rounded Min-2-SCCP implies PTAS for Min-2-SCCP (in the general case)

Quad-trees for rounded Min-2-SCCP

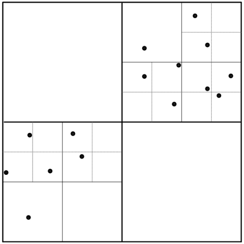
Set up a regular 1-step axis-aligned grid on the square $\mathcal{S}$ with side-length of $L = O(n)$.



We are using the concept of *quad-tree*

Quad-trees for rounded Min-2-SCCP

Root is the square $\mathcal{S}$. For every square (including the root), make a partition of the square into 4 child subsquares. Repeat it until all child squares will contain no more than 1 node of the instance.



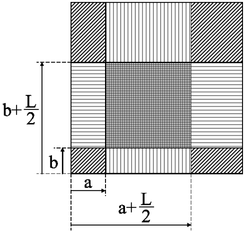
Shifted Quad-tree

Definition

Suppose, $a, b \in \mathbb{N}_L^0$, we call the Quad-tree $T(a,b)$ *shifted Quad-tree*, if coordinates of its center is

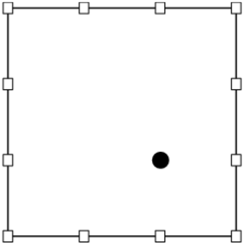
$$((L/2 + a) \mod L, (L/2 + b) \mod L).$$

Child squares of $T(a,b)$, as its center, is considered *modulo* L



Definition

- Consider fixed values $m, r \in \mathbb{N}$.
- For any square S, assign regular partition of its border, including vertices of the square and consisting of $4(m+1)$ points.
- Such a partition is called *m-regular partition*, and all its elements — *portals*.



Definitions

m-regular portal set

Union of m-regular partitions for all borders of not-a-leaf nodes of Quadro-tree $T(a, b)$ is called m-regular portal set. Denote it $P(a, b, m)$.

(m, r)-approximation

Suppose, π is a simple cycle in the Min-2-SCCP instance graph G (on the plane), $V(\pi)$ is its node-set. Closed piece-wise linear route $l(\pi)$ is called (m, r)-approximation (of the cycle π) if

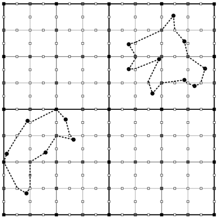
1) node-set of the route $l(\pi)$ is a some subset of $V(\pi) \cup P(a, b, m)$,
2) π and $l(\pi)$ visit the nodes from $V(\pi)$ in the same order,
3) for any square (being a node of $T(a, b)$), $l(\pi)$ intersects its arbitrary edge no more than r times, and exclusively in the points of $P(a, b, m)$.

Once more definition

(k, m, r)-cycle cover

k-scc consisting of (m, r)-approximations is called (k, m, r)-cycle cover

Obviously, an arbitrary $(1, m, r)$-cycle cover contains the only (m, r)-approximation which is a Hamiltonian cycle.
Let us consider $(2, m, r)$-cycle covers...



Structure Theorem for Euclidean Min-2-SCCP

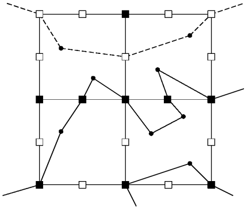
Theorem 4

- *Suppose* $c > 0$ *is fixed,*
- L *is size of square* $\mathcal{S}$ *for a given instance of rounded 2-MHC.*
- *Suppose discrete stochastic variables* a, b *are distributed uniformly on the set* $\mathbb{N}_L^0$.
- *Then for* $m = O(c \log L)$ *and* $r = O(c)$ *with probability at least* $\frac{1}{2}$ *there is* $(2, m, r)$*-cycle cover which weight is no more than* $(1 + \frac{1}{c})OPT$.

Dynamic Programming

$(2, m, r, S)$-segment

Let some $(2, m, r)$-cycle cover C and some node S of the tree $T(a, b)$ be chosen. A family of partial routes $C \cap S$
is called $(2, m, r, S)$-segment (of the cover C).



Bellman equation

Task (S, R_1, R_2, κ)

Input.

- Node S of the tree $T(a, b)$.
- Cortege $R_i : \mathbb{N}_{q_i} \to (P(a, b, m) \cap \partial S)^2$ defines a sequence of the start-finish pairs of portals (s^i_j, t^i_j) which are crossing-points of ∂S by (m, r)-approximation l_i.
- Number κ is equal to the number of cycles of the building $(2, m, r)$-cycle cover, intersecting the interior of S.

Output minimum-cost $(2, m, r, S)$-segment.

Denote by $W(S, R_1, R_2, \kappa)$ value of the task (S, R_1, R_2, κ).

$$W(S, R_1, R_2, \kappa) = \min_{\tau} \sum_{i=I}^{IV} W(S^i, R^i_1(\tau), R^i_2(\tau), \kappa^i(\tau)),$$

Derandomization

Denote by $APP(a,b)$ a weight of the approximate solution constructed by DP for the tree $T(a,b)$.

$$P\left(APP(a,b) \leqslant (1+\frac{1}{c})OPT\right) \geqslant 1/2,$$

Hence, there is a pair $(a^*,b^*) \in \mathbb{N}_L^0$, for which the equation

$$OPT \leqslant APP(a^*,b^*) \leqslant (1+1/c)OPT$$

is valid.

Theorem 5

Euclidean Min-2-SCCP has a Polynomial-Time Approximation Scheme with complexity bound $O(n^3(\log n)^{O(c)})$.

Conclusion and Open Problems

- The proposed PTAS seems to be easily extendable onto Min-k-SCCP in d-dimensional Euclidean space
- Due to well-known PCP theorem there is no PTAS for Metric Min-k-SCCP. But, what about approximation threshold value for this problem?

Thank you for your attention!