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Methods of embedding real-world entities into a linear space for implementing the generalized linear approach to dependence estimation

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The task of dependences estimation on the sets of real-world objects

A set of real-world objects $\omega \in \Omega$.

A set of hidden characteristic values $y \in \mathbb{Y}$.

Real existing hidden function $y(\omega): \Omega \to \mathbb{Y}$.

Observer desire:

To have a tool for estimating the hidden characteristic for the real objects

$\hat{y}(\omega): \Omega \to \mathbb{Y}$; $\hat{y}(\omega) \neq y(\omega)$ – error.

Generalized linear approach

Observer (his/her computer) interpret a real-world objects as points in some linear space:
$\mathbf{x}(\omega): \Omega \to \mathbb{X}$ – linear space.

- addition is commutative $\mathbf{x}' + \mathbf{x}'' = \mathbf{x}'' + \mathbf{x}'$
  and associative $(\mathbf{x}' + \mathbf{x}'') + \mathbf{x}''' = \mathbf{x}' + (\mathbf{x}'' + \mathbf{x}''')$;
- identity element of addition $\mathbf{x} + \boldsymbol{\phi} = \mathbf{x}$, $c\boldsymbol{\phi} = \boldsymbol{\phi}$;
- inverse element of addition $(-\mathbf{x}) + \mathbf{x} = \boldsymbol{\phi}$;
- compatibility of scalar multiplication with field multiplication $c'(c''\mathbf{x}) = (c'c'')\mathbf{x}$;
- identity element of scalar multiplication $1\mathbf{x} = \mathbf{x}$;
- distributivity of multiplication and addition $(c' + c'')\mathbf{x} = c'\mathbf{x} + c''\mathbf{x}$, $c(\mathbf{x}' + \mathbf{x}'') = c\mathbf{x}' + c\mathbf{x}''$.

Generalized Linear Approach

Linear space of real-world object perception

Observer (his/her computer) interpret a real-world objects as points in some linear space:
$\boldsymbol{x}(\omega): \Omega \to \mathbb{X}$ – linear space.

– $\boldsymbol{x}' + \boldsymbol{x}'' = \boldsymbol{x}'' + \boldsymbol{x}'$, $(\boldsymbol{x}' + \boldsymbol{x}'') + \boldsymbol{x}''' = \boldsymbol{x}' + (\boldsymbol{x}'' + \boldsymbol{x}''')$;

– zero element $\boldsymbol{x} + \boldsymbol{\phi} = \boldsymbol{x}$, $c\boldsymbol{\phi} = \boldsymbol{\phi}$, $(-\boldsymbol{x}) + \boldsymbol{x} = \boldsymbol{\phi}$;

– $c'(c''\boldsymbol{x}) = (c'c'')\boldsymbol{x}$, $1\boldsymbol{x} = \boldsymbol{x}$, $0\boldsymbol{x} = \boldsymbol{\phi}$;

– $(c' + c'')\boldsymbol{x} = c'\boldsymbol{x} + c''\boldsymbol{x}$, $c(\boldsymbol{x}' + \boldsymbol{x}'') = c\boldsymbol{x}' + c\boldsymbol{x}''$.

Indefinite scalar product

$\boldsymbol{x}, \boldsymbol{\upsilon} \in \mathbb{X}$ – two arbitrary points in linear space,
$K(\boldsymbol{x}, \boldsymbol{\upsilon}): \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ – scalar function of two arguments,
(1) $K(\boldsymbol{x}, \boldsymbol{\upsilon}) = K(\boldsymbol{\upsilon}, \boldsymbol{x})$ – symmetry,
(2) $K(\boldsymbol{x},\ c'\boldsymbol{\upsilon}' + c''\boldsymbol{\upsilon}'') = c'K(\boldsymbol{x}, \boldsymbol{\upsilon}') + c''K(\boldsymbol{x}, \boldsymbol{\upsilon}'')$ – bilinearity.

If suggest
(3) $K(\boldsymbol{x}, \boldsymbol{x}) \geq 0$, then it will be common inner product, so $\sqrt{K(\boldsymbol{x}, \boldsymbol{x})} = \|\boldsymbol{x}\|$ – is a norm.

For our approach it is enough just (1) and (2).
It is so-called indefinite scalar product in the pseudo-Euclidean space. There is no norm.

General Linear Model of Dependency

The object in linear space $\boxed{\boldsymbol{x}(\omega)\in\mathbb{X} \text{ – just observer imagination}}$

Inner product (indefinite) $\boxed{K(\boldsymbol{x},\boldsymbol{v}):\mathbb{X}\times\mathbb{X}\rightarrow\mathbb{R} \text{ – just observer imagination}}$

Parameters of model of dependency $(\boldsymbol{v},b)$
$$\begin{cases}\boldsymbol{v}\in\mathbb{X} \text{ – directional point (vector) in the same space}\\ b\in\mathbb{R} \text{ – model shift (intercept)}\end{cases}$$

Generalized linear feature of object $z(\boldsymbol{x},\boldsymbol{v},b)=K(\boldsymbol{x},\boldsymbol{v})+b$: $\mathbb{X}\xrightarrow{(\boldsymbol{v},b)}\mathbb{R}$

Goal characteristic of object $y(\omega)\in\mathbb{Y}$ – given by nature.

Link function,
Usually convex on z $\boxed{q(y,z):\mathbb{Y}\times\mathbb{R}\rightarrow\mathbb{R}^{+} \text{ – just observer imagination}}$

Parametric Loss function $q(y,\boldsymbol{x},\boldsymbol{v},b)=q\big(y,z(\boldsymbol{x},\boldsymbol{v},b)\big):\mathbb{Y}\times\mathbb{X}\xrightarrow{(\boldsymbol{a},b)}\mathbb{R}^{+}$

Decision rule $\hat{y}(\boldsymbol{x}\,|\,\boldsymbol{v},b)=\operatorname*{argmin}_{y\in\mathbb{Y}} q(y,\boldsymbol{x},\boldsymbol{v},b)$: $\mathbb{X}\xrightarrow{(\boldsymbol{v},b)}\mathbb{Y}$

Training set $(\boldsymbol{X},\boldsymbol{Y})=\Big\{\big(\boldsymbol{x}(\omega_j),y(\omega_j)\big)=(\boldsymbol{x}_j,y_j),\ j=1,...,N\Big\}$

Family of convex regularized functions $\boxed{V(\boldsymbol{v}\,|\,\mu):\ \mathbb{X}\xrightarrow{\mu}\mathbb{R}^{+} \text{ – just observer imagination}}$

Training process – find $(\boldsymbol{v}\in\mathbb{X},b\in\mathbb{R})$:
Minimization of regularized empirical risk

$$V(\boldsymbol{v}\,|\,\mu)+c\sum_{j=1}^{N}q\big(y_j,z(\boldsymbol{x}_j,\boldsymbol{v},b)\big)\rightarrow\min(\boldsymbol{v},b),$$
$$z(\boldsymbol{x}_j,\boldsymbol{v},b)=K(\boldsymbol{x}_j,\boldsymbol{v})+b$$

The criterion is convex if regularized function $V(\boldsymbol{v}\,|\,\mu)$ and link function $q(y,z)$ are convex on $\boldsymbol{v}\in\mathbb{X}$ and $z\in\mathbb{R}$.

Objects Embedding into a Linear Space

1. Arbitrary pairwise function of objects comparing

Set of real-world objects $\omega \in \Omega$

The pairwise comparing function in symmetric $S(\omega', \omega'') = S(\omega'', \omega'):\ \Omega \times \Omega \to \mathbb{R}$

Let us choose an arbitrary element as its "center" $\phi \in \Omega$

Two-argument commonality symmetric function for the center $\phi \in \Omega$:

$$K_\phi(\omega', \omega'') = \frac{1}{2}\big[S(\omega', \phi) + S(\omega'', \phi) - S(\omega', \omega'')\big]:\quad \Omega \times \Omega \to \mathbb{R}$$

Let us assume, for simplicity sake, that the set of real-world objects finite $|\Omega| = M$ (probably, a VERY huge number!)

The symmetric commonality matrix for some center $\phi \in \Omega$:

$$\mathbf{K}_\phi = \begin{pmatrix} K_\phi(\omega_1, \omega_1) & \cdots & K_\phi(\omega_1, \omega_M) \\ \vdots & \ddots & \vdots \\ K_\phi(\omega_M, \omega_1) & \cdots & K_\phi(\omega_M, \omega_M) \end{pmatrix},$$

eigen values are real numbers $\xi_{\phi,i} \in \mathbb{R},\ i = 1, \dots, M$,

eigen vectors are orthonormal $\mathbf{z}_{\phi,i} \in \mathbb{R}^M,\ \mathbf{z}_{\phi,i}^T \mathbf{z}_{\phi,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Eigen values in decrease order $\underbrace{\xi_{\phi,1} \geq 0, \dots, \xi_{\phi,p_\phi} \geq 0}_{positive},\ \underbrace{\xi_{\phi,p_\phi+1} < 0, \dots, \xi_{\phi,M} < 0}_{negative}$

The pair of integers $p_\phi + q_\phi = M$ – signature of commonality matrix

Theorem: *The signature of matrix* $\mathbf{K}_\phi$ *does not depend on the choice of the center* $\phi \in \Omega$.

Matrix $\mathbf{K}_\phi = \sum_{i=1}^{p} \xi_{\phi,i} \mathbf{z}_{\phi,i} \mathbf{z}_{\phi,i}^T - \sum_{i=p+1}^{M} \bar{\xi}_{\phi,i} \mathbf{z}_{\phi,i} \mathbf{z}_{\phi,i}^T\ (M \times M)$ – VERY huge!

Objects Embedding into a Linear Space

1. Arbitrary pairwise function of objects comparing

Set of real-world objects $\omega \in \Omega$

The pairwise comparing function in symmetric $S(\omega', \omega'') = S(\omega'', \omega')$: $\Omega \times \Omega \to \mathbb{R}$

Let us choose an arbitrary element as its “center” $\phi \in \Omega$

Two-argument commonality symmetric function for the center $\phi \in \Omega$:

$$K_\phi(\omega', \omega'') = \frac{1}{2}\left[S(\omega', \phi) + S(\omega'', \phi) - S(\omega', \omega'')\right]: \quad \Omega \times \Omega \to \mathbb{R}$$

Let the set of real-world objects finite $|\Omega| = M$ (probably, a VERY huge number!)

The pair of integers $p_\phi + q_\phi = M$ – signature (*does not depend on center*)

Matrix $\mathbf{K}_\phi = \sum_{i=1}^{p} \xi_{\phi,i} \mathbf{z}_{\phi,i} \mathbf{z}_{\phi,i}^T - \sum_{i=p+1}^{M} \bar{\xi}_{\phi,i} \mathbf{z}_{\phi,i} \mathbf{z}_{\phi,i}^T$ $(M \times M)$ – is very huge**.**

Let assume $\mathbf{K}_\phi$ as a set of inner products

$$\mathbf{K}_\phi = \begin{pmatrix} \boldsymbol{x}_{\phi,1}^T \mathbf{J}_p \boldsymbol{x}_{\phi,1} & \cdots & \boldsymbol{x}_{\phi,1}^T \mathbf{J}_p \boldsymbol{x}_{\phi,M} \\ \vdots & \ddots & \vdots \\ \boldsymbol{x}_{\phi,M}^T \mathbf{J}_p \boldsymbol{x}_{\phi,1} & \cdots & \boldsymbol{x}_{\phi,M}^T \mathbf{J}_p \boldsymbol{x}_{\phi,M} \end{pmatrix}, \quad \mathbf{J}_p = \begin{pmatrix} \mathbf{I}_{p \times p} & \mathbf{0}_{p \times (M-p)} \\ \mathbf{0}_{(M-p) \times p} & -\mathbf{I}_{(M-p) \times (M-p)} \end{pmatrix}$$ – identity matrix of signature p

So, we associate the elements of arbitrary finite set $\Omega = \{\omega_1, \ldots, \omega_M\}$,

$\phi \in \Omega$ – center, with M-dimensional vectors of real features of objects

$\boldsymbol{x}_{\phi,1} = \boldsymbol{x}_{\phi,\omega_1} \in \mathbb{R}^M, \ldots, \boldsymbol{x}_{\phi,M} = \boldsymbol{x}_{\phi,\omega_M} \in \mathbb{R}^M$. Zero element – vector of $\boldsymbol{x}_\phi \in \mathbb{R}^M$.

Objects Embedding into a Linear Space

1. Arbitrary pairwise function of objects comparing

Set of real-world objects $\omega \in \Omega$

The pairwise comparing function in symmetric $S(\omega', \omega'') = S(\omega'', \omega')$: $\Omega \times \Omega \to \mathbb{R}$

Let us choose an arbitrary element as its "center" $\phi \in \Omega$

Two-argument commonality symmetric function for the center $\phi \in \Omega$:

$$K_\phi(\omega', \omega'') = \frac{1}{2}\left[S(\omega', \phi) + S(\omega'', \phi) - S(\omega', \omega'')\right]: \quad \Omega \times \Omega \to \mathbb{R}$$

Let the set of real-world objects finite $|\Omega| = M$ (probably, a VERY huge number!)

So, we associate the elements of arbitrary finite set $\Omega = \{\omega_1, ..., \omega_M\}$,

$\phi \in \Omega$ – center, with M -dimensional vectors of real features of objects

$\boldsymbol{x}_{\phi,1} = \boldsymbol{x}_{\phi,\omega_1} \in \mathbb{R}^M, ..., \boldsymbol{x}_{\phi,M} = \boldsymbol{x}_{\phi,\omega_M} \in \mathbb{R}^M$. Zero element – vector of $\boldsymbol{x}_\phi \in \mathbb{R}^M$.

Embedding in to linear space, with two-argument function

$K(\boldsymbol{x}', \boldsymbol{x}'') = \boldsymbol{x}'^T \mathbf{J}_p \boldsymbol{x}''$: $\mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ with properties:

symmetry $K(\boldsymbol{x}', \boldsymbol{x}'') = K(\boldsymbol{x}'', \boldsymbol{x}')$,

bilinearity $K(c'\boldsymbol{x}' + c''\boldsymbol{x}'', \boldsymbol{x}''') = c'K(\boldsymbol{x}', \boldsymbol{x}''') + c''K(\boldsymbol{x}'', \boldsymbol{x}''')$.

Not holds $K(\boldsymbol{x}, \boldsymbol{x}) \geq 0$.

So it is indefinite inner products.

Pseudo-Euclidean linear space $\mathbb{R}^M$ based on $\Omega = \{\omega_1, ..., \omega_M\}$. There is no norm.

Objects Embedding into a Linear Space

1. Arbitrary pairwise function of objects comparing

Set of real-world objects $\omega \in \Omega$

The pairwise comparing function in symmetric $S(\omega', \omega'') = S(\omega'', \omega'): \ \Omega \times \Omega \to \mathbb{R}$

Let us choose an arbitrary element as its “center” $\phi \in \Omega$

Two-argument commonality symmetric function for the center $\phi \in \Omega$:

$$K_\phi(\omega', \omega'') = \frac{1}{2}\big[S(\omega', \phi) + S(\omega'', \phi) - S(\omega', \omega'')\big]: \quad \Omega \times \Omega \to \mathbb{R}$$

Let the set of real-world objects finite $|\Omega| = M$ (probably, a VERY huge number!)

Inner product (indefinite) $K(\boldsymbol{x}, \boldsymbol{v}): \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ – just observer imagination

Parameters of model of dependency $(\boldsymbol{v}, b)$ $\begin{cases} \boldsymbol{v} \in \mathbb{X} & \text{– directional point (vector) in the same space} \\ b \in \mathbb{R} & \text{– model shift (intercept)} \end{cases}$

Generalized linear characteristic of object $z(\boldsymbol{x}, \boldsymbol{v}, b) = K(\boldsymbol{x}, \boldsymbol{v}) + b: \ \mathbb{X} \xrightarrow{(\boldsymbol{v}, b)} \mathbb{R}$

How we can search the directional vector in imaginary space $\boldsymbol{v} \in \mathbb{X}$?

Basic set of the objects $\{\omega_1^0, ..., \omega_n^0\} \subset \Omega$, and their mapping image $\{\boldsymbol{x}_\phi(\omega_1^0), ..., \boldsymbol{x}_\phi(\omega_n^0)\}$.

Directional vector – linear combination of images (fancies) of basic objects

$$\boldsymbol{v}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i \boldsymbol{x}_\phi(\omega_i^0), \ \sum_{i=1}^{n} a_i = 0.$$

Theorem: Generalized linear characteristic of object

$$z(\omega, \boldsymbol{a}, b) = -\frac{1}{2}\sum_{i=1}^{n} a_i S(\omega, \omega_i^0) + b, \ \sum_{i=1}^{n} a_i = 0.$$

So, the embedding into linear space is just our fantasy!

Objects Embedding into a Linear Space

2. Distance pairwise function of objects comparing

Set of real-world objects $\omega \in \Omega$.

Now as before, the comparison function $S(\omega', \omega'') = S(\omega'', \omega'): \ \Omega \times \Omega \to \mathbb{R}$ **symmetric.**

Additional requirements:

non-negativity $S(\omega', \omega'') \geq 0$, zero value for the same argument $S(\omega, \omega) = 0$.

Let us note the distance as $d(\omega', \omega'') = \sqrt{S(\omega', \omega'')}$.

Central element $\phi \in \Omega$.

Two-argument commonality symmetric function for the center $\phi \in \Omega$:

$$K_\phi(\omega', \omega'') = \frac{1}{2}\left[d^2(\omega', \phi) + d^2(\omega'', \phi) - d^2(\omega', \omega'') \right]: \ \ \Omega \times \Omega \to \mathbb{R}$$

Let the set of real-world objects finite $|\Omega| = M$ (probably, a VERY huge number!)

There is no change in theoretical speculations.

Inner product (indefinite) $K(\boldsymbol{x}, \boldsymbol{v}): \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ – observer fantasy

Parameters of model of dependency $(\boldsymbol{v}, b)$ $\begin{cases} \boldsymbol{v} \in \mathbb{X} \text{ – directional point (vector) in the same space} \\ b \in \mathbb{R} \text{ – model shift (intercept)} \end{cases}$

Basic set of the objects $\{\omega_1^0, \ldots, \omega_n^0\} \subset \Omega$, and their mapping image $\{\boldsymbol{x}_\phi(\omega_1^0), \ldots, \boldsymbol{x}_\phi(\omega_n^0)\}$.

Theorem: Generalized linear characteristic of object $z(\omega, \boldsymbol{a}, b) = -\frac{1}{2}\sum_{i=1}^{n} a_i d^2(\omega, \omega_i^0) + b, \ \sum_{i=1}^{n} a_i = 0.$

So, the embedding into linear space is just our fantasy!

Objects Embedding into a Linear Space

3. Pairwise function of objects comparing is a metric

Set of real-world objects $\omega \in \Omega$.

Now as before, the comparison function $S(\omega', \omega'') = S(\omega'', \omega'):\ \Omega \times \Omega \to \mathbb{R}$

Additional requirements:

$S(\omega, \omega) = 0$, triangle inequality $S(\omega', \omega'') + S(\omega'', \omega''') \geq S(\omega', \omega''')$.

Let us note the distance as $d(\omega', \omega'') = \sqrt{S(\omega', \omega'')}$.

Central element $\phi \in \Omega$.

Two-argument commonality symmetric function for the center $\phi \in \Omega$:

$$K_\phi(\omega', \omega'') = \frac{1}{2}\left[d^2(\omega', \phi) + d^2(\omega'', \phi) - d^2(\omega', \omega'') \right]:\ \ \Omega \times \Omega \to \mathbb{R}$$

Let the set of real-world objects finite $|\Omega| = M$ (probably, a VERY huge number!)

There is no change in theoretical speculations.

Inner product (indefinite) $K(\boldsymbol{x}, \boldsymbol{v}): \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ – observer fantasy

Parameters of model of dependency $(\boldsymbol{v}, b)$ $\begin{cases} \boldsymbol{v} \in \mathbb{X} \text{ – directional point (vector) in the same space} \\ b \in \mathbb{R} \text{ – model shift (intercept)} \end{cases}$

Basic set of the objects $\{\omega_1^0, \dots, \omega_n^0\} \subset \Omega$, and their mapping image $\{\boldsymbol{x}_\phi(\omega_1^0), \dots, \boldsymbol{x}_\phi(\omega_n^0)\}$.

Theorem: Generalized linear characteristic of object $z(\omega, \boldsymbol{a}, b) = -\frac{1}{2}\sum_{i=1}^{n} a_i d^2(\omega, \omega_i^0) + b,\ \sum_{i=1}^{n} a_i = 0.$

So, the embedding into linear space is just our fantasy!

Training Criterion: Minimum of Empirical Regularized Risk

Set of objects $\omega \in \Omega$, pairwise comparison function $S(\omega', \omega''): \Omega \times \Omega \to \mathbb{R}$.

Basic set $\{\omega_i^0, i=1,\dots,n\} \subset \Omega$	There is no information about goal characteristic. Just matrix of pairwise comparisons $\left[S(\omega_i, \omega_k),\ i,k=1,\dots,n\right]$
Training set (a part of basic set) $\{\omega_j, j=1,\dots,N\} \subset \Omega$	The known values of goal characteristic $y(\omega_j) \in \mathbb{Y}$, and matrix of pairwise comparisons $\left[S(\omega_j, \omega_l),\ j,l=1,\dots,N\right]$

Embedding (just mental) of objects into linear imaginary space with inner product, in general with indefinite inner product:

$\boldsymbol{x}(\omega): \Omega \to \mathbb{X}$, $K(\boldsymbol{x}', \boldsymbol{x}''): \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ – directly following from $S(\omega', \omega'')$.

Generalized linear characteristic of object $z(\boldsymbol{x}_j, \boldsymbol{v}, b) = K(\boldsymbol{x}_j, \boldsymbol{v}) + b$,

$\boldsymbol{v} \in \mathbb{X}$ – sought for directional vector, $b \in \mathbb{R}$ – sought for shift.

Link function defines by observer $q(y, z): \mathbb{Y} \times \mathbb{R} \to \mathbb{R}^+$, convex on z.

Regularized function defines by observer $V(\boldsymbol{v} \mid \mu): \mathbb{X} \to \mathbb{R}^+$

Training – finding $(\boldsymbol{v} \in \mathbb{X}, b \in \mathbb{R})$ by the criterion of minimum regularized empirical risk	$V(\boldsymbol{v} \mid \mu) + c\sum_{j=1}^{N} q\left(y_j, z(\boldsymbol{x}_j, \boldsymbol{v}, b)\right) \to \min(\boldsymbol{v}, b)$

Parametric representations of directional vector $\boldsymbol{v}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i \boldsymbol{x}_\phi(\omega_i^0)$, $\sum_{i=1}^{n} a_i = 0$.

Training Criterion: Minimum of Empirical Regularized Risk

Set of objects $\omega \in \Omega$, pairwise comparison function $S(\omega', \omega''): \Omega \times \Omega \to \mathbb{R}$.

Basic set $\{\omega_i^0, i = 1, \ldots, n\} \subset \Omega$	There is no information about goal characteristic. Just matrix of pairwise comparisons $\left[S(\omega_i, \omega_k),\ i, k = 1, \ldots, n\right]$
Training set (a part of basic set) $\{\omega_j, j = 1, \ldots, N\} \subset \Omega$	The known values of goal characteristic $y(\omega_j) \in \mathbb{Y}$, and matrix of pairwise comparisons $\left[S(\omega_j, \omega_l),\ j, l = 1, \ldots, N\right]$

Link function defines by observer $q(y, z): \mathbb{Y} \times \mathbb{R} \to \mathbb{R}^+$, convex on z.

Regularized function defines by observer $V(\boldsymbol{v} \mid \mu): \mathbb{X} \to \mathbb{R}^+$

Training – finding $(\boldsymbol{v} \in \mathbb{X}, b \in \mathbb{R})$ by the criterion of minimum regularized empirical risk	$V(\boldsymbol{v} \mid \mu) + c \sum_{j=1}^{N} q\big(y_j, z(\boldsymbol{x}_j, \boldsymbol{v}, b)\big) \to \min(\boldsymbol{v}, b)$

Parametric representations of directional vector $\boldsymbol{v}(\boldsymbol{a}) = \sum_{i=1}^{n} a_i \boldsymbol{x}_\phi(\omega_i^0),\ \sum_{i=1}^{n} a_i = 0$.

Convex parametric regularized function $V(\boldsymbol{a} \mid \mu): \mathbb{R}^n \to \mathbb{R}^+$

Parametric training: Search $(\boldsymbol{a} \in \mathbb{R}^n, b \in \mathbb{R})$,	$V(\boldsymbol{a} \mid \mu) + c \sum_{j=1}^{N} q\big(y_j, z(\omega_j, \boldsymbol{a}, b)\big) + b \to \min(\boldsymbol{a}, b)$, $z(\omega_j, \boldsymbol{a}, b) = \sum_{i=1}^{n} \big(S(\omega, \omega_i^0)\big) a_i + b$, convex criterion.

Selection of subset of basic objects: L_1 and L_2 regularization combining

Elastic Net regularization: $V(\boldsymbol{a} \mid \mu) = \sum_{i=1}^{n} a_i^2 + \mu \sum_{i=1}^{n} |a_i|,\ \mu \geq 0$ – selectivity parameter.

Training criterion: $\sum_{i=1}^{n} a_i^2 + \mu \sum_{i=1}^{n} |a_i| + \sum_{j=1}^{N} q\Big(y_j, z(\omega_j, \boldsymbol{a}, b)\Big),\ \ z(\omega_j, \boldsymbol{a}, b) = \sum_{i=1}^{n} \Big(S(\omega_j, \omega_i^0)\Big) a_i + b.$

Theorem. Let's values of $(\hat{\lambda}_1, \ldots, \hat{\lambda}_N)$ are decision of dual task of convex programming:

$$\begin{cases} \frac{1}{2\beta} \sum_{i=1}^{n} \left\{ \min\left[\mu/2 + \sum_{j=1}^{N} \lambda_j x_{ij},\ 0,\ \mu/2 - \sum_{j=1}^{N} \lambda_j x_{ij} \right] \right\}^2 - \sum_{j=1}^{N} \underbrace{\min_{z \in \mathbb{R}} \Big(q(y_j, z) + \lambda_j z \Big)}_{\text{concave on } \lambda} \to \min(\lambda_1, \ldots, \lambda_N), \\ \sum_{j=1}^{N} \lambda_j = 0. \end{cases}$$

Then

$$\hat{a}_i = \begin{cases} \Big(\sum_{j=1}^{N} \hat{\lambda}_j S(\omega_j, \omega_i^0) + \mu/2 \Big) < 0,\ \sum_{j=1}^{N} \hat{\lambda}_j S(\omega_j, \omega_i^0) < -\mu/2, \\ 0, \quad -\mu/2 \leq \sum_{j=1}^{N} \hat{\lambda}_j S(\omega_j, \omega_i^0) \leq \mu/2, \\ \Big(\sum_{j=1}^{N} \hat{\lambda}_j S(\omega_j, \omega_i^0) - \mu/2 \Big) > 0,\ \sum_{j=1}^{N} \hat{\lambda}_j S(\omega_j, \omega_i^0) > \mu/2, \end{cases} \qquad \begin{cases} \hat{b} = \frac{1}{N} \sum_{j=1}^{N} \Big(\hat{z}_j - \sum_{i=1}^{n} \hat{a}_i x_{ij} \Big), \text{ where} \\ \hat{z}_j = \arg\min_{z \in \mathbb{R}} \Big(q(y_j, z) + \hat{\lambda}_j z \Big). \end{cases}$$

As larger $\mu \geq 0$, then more the selectivity degree of basic objects.

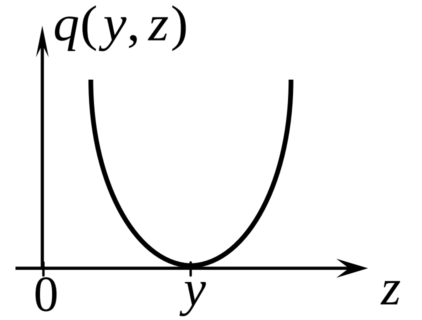
Link Functions for Some Particular Types of Goal Characteristic

Link function $q(y,z): \mathbb{Y}\times\mathbb{R}\to\mathbb{R}^{+}$ – observer fantasy

1) Regression analysis.

Goal characteristic $y\in\mathbb{Y}=\mathbb{R}$ – real number.

$q(y,z)=(y-z)^2$.

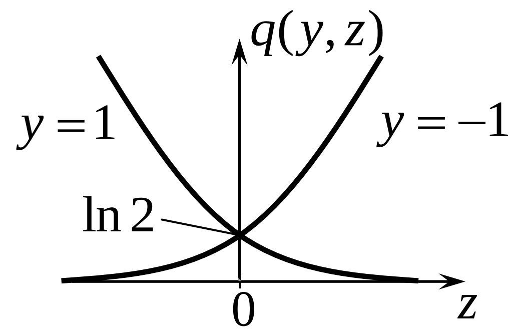


2) Two class pattern recognition task.

Goal characteristic $y\in\mathbb{Y}=\{-1,1\}$ – class index.

Logistic regression:

$q(y,z)=\ln\left[1+\exp(-y\,z)\right]$.

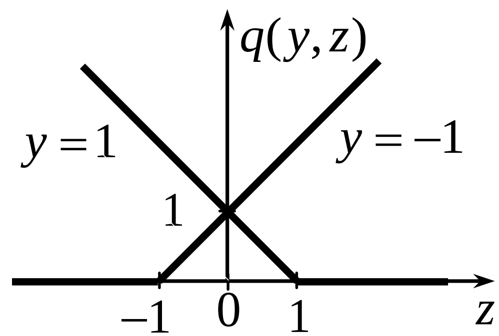


3) Two class pattern recognition task.

Goal characteristic $y\in\mathbb{Y}=\{-1,1\}$ – class index.

Support vector machine:

$$q(y,z)=\max\left[0,\ 1-yz\right]=\begin{cases}1-yz, & 1-yz>0,\\ 0, & 1-yz\le 0.\end{cases}$$



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Thank you for your attention!