Temporal Difference Learning

Reinforcement Learning

September 29, 2020

MSU

Reminder: Value functions

State value function (**V-function**): $V^{\pi}(s) := \mathbb{E}_{\mathcal{T}\sim\pi} \sum_{t\geqslant 0} \gamma^t r_t \mid s_0 = s$

State-action value function (**Q-function**): $Q^{\pi}(s,a) := \mathbb{E}_{\mathcal{T}\sim\pi} \sum_{t\geqslant 0} \gamma^t r_t \mid s_0 = s, a_0 = a$

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Optimal V-function: $V^*(s) := \max_{\pi} V^{\pi}(s)$

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Optimal Q-function: $Q^*(s,a) := \max_\pi Q^\pi(s,a)$

Bellman equations: $\forall \pi, s, a$:

$$Q^\pi(s,a) := r(s,a) + \gamma \mathbb{E}_{s'\sim p(s'|s,a)} \mathbb{E}_{a'\sim\pi(a'|s')} Q^\pi(s',a')$$

$$Q^*(s,a) := r(s,a) + \gamma \mathbb{E}_{s'\sim p(s'|s,a)} \max_{a'} Q^*(s',a')$$

Reminder: Dynamic Programming

Setup: $|\mathcal{S}| \ll \infty, |\mathcal{A}| \ll \infty$
Assumption: model is known

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- **Policy Evaluation**:
  compute V^{π_k} for current policy π_k;
- **Policy Improvement**:
  $\pi_{k+1}(s) \leftarrow \operatorname*{argmax}_a Q^{\pi_k}(s, a)$;
- repeat;

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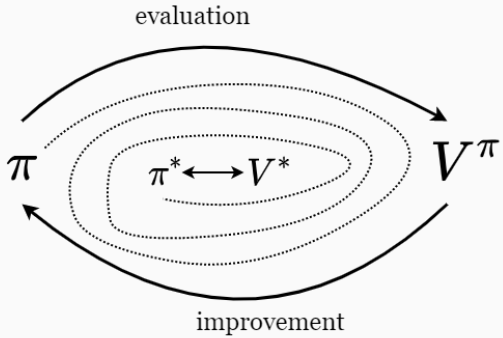

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(e. g. for V^*):

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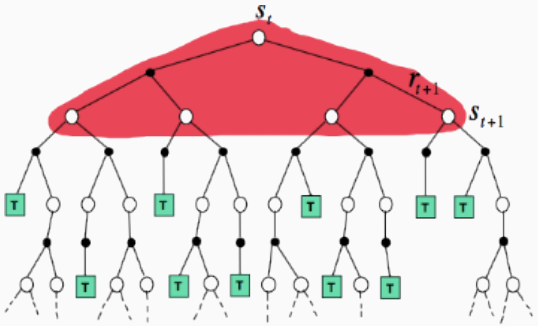

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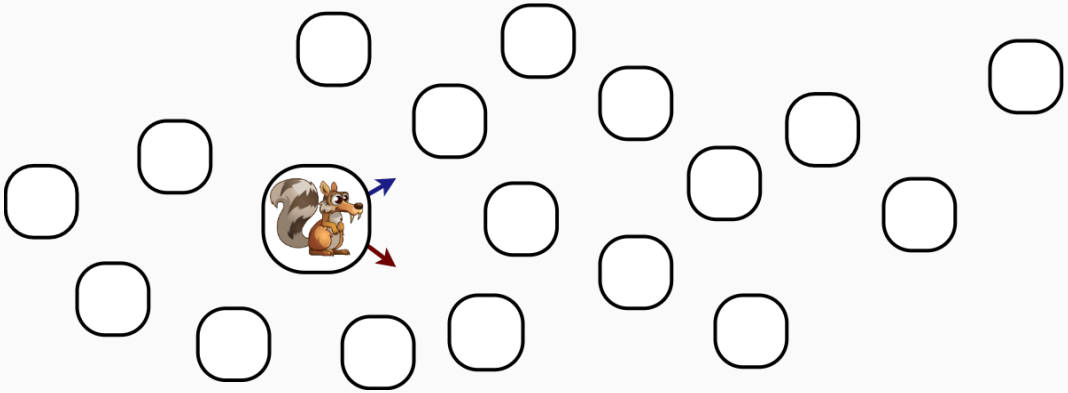

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Learn model through experience

Use VI / PI with learned model to find optimal policy

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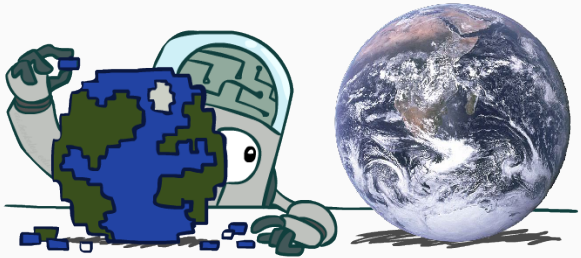

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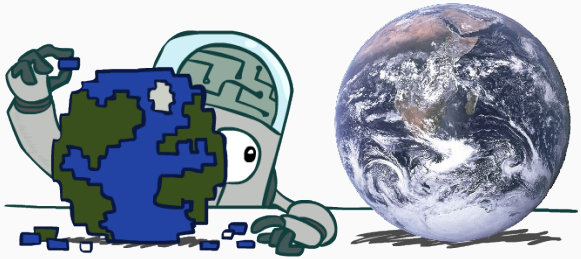

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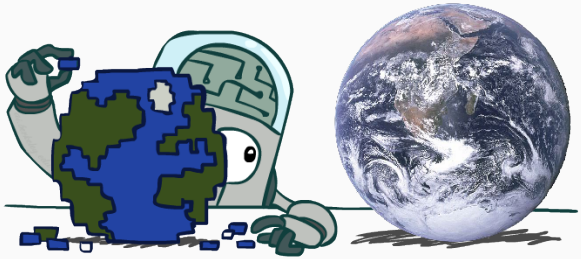

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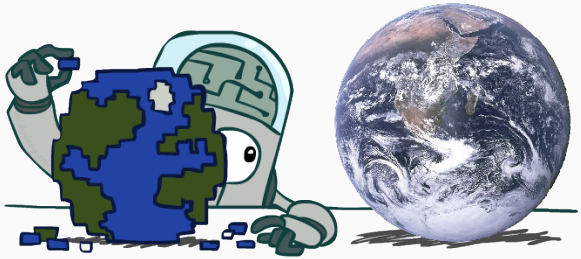

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Can we derive *some* model-free PI / VI?

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where $\mathcal{T}_i \sim \pi \mid s_0 = s, a_0 = a$

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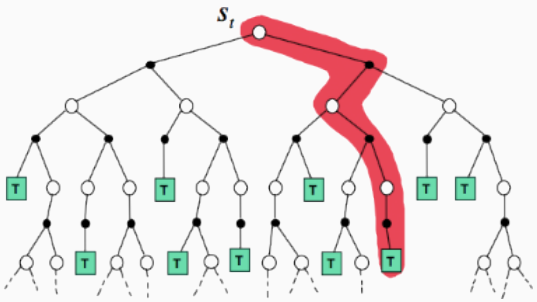
Can we have i. i. d. samples?



Monte Carlo Algorithm

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Initialize $Q^\pi(s, a)$ and policy $\pi(s)$ arbitrarily.

for $k = 0, 1, 2 \ldots$

- play several games using π;
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× wastes information about state connections;

× high variance of collected samples;

× no clear theoretical guarantees;

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Reuse samples from previous policy
(for example, with smaller weight)

Closer look at Monte Carlo

Suppose we want to compute online mean m of i. i. d. samples $x_1, x_2 \ldots$. On step k:

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Convergence:

$m_k \xrightarrow[k\to\infty]{\text{w. p. 1}} \mathbb{E}x$ if learning rate satisfies **Robbins–Monro conditions**:

$$\sum_{k=1}^{+\infty} \alpha_k = +\infty, \quad \sum_{k=1}^{+\infty} \alpha_k^2 < +\infty$$

Solving equations

Equation	Method	Update
$x = f(x)$	Point Iteration	$x_{k+1} := f(x_k)$
$x = \mathbb{E}_{s'} f(s')$	Exponential smoothing	$x_{k+1} := x_k + \alpha_k \left(f(s') - x_k\right)$ $s' \sim p(s')$
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Temporal Difference: intuition

View 1: solving Bellman expectation equation using Robbins-Monro scheme:

$$Q^{\pi}(s,a) = \mathbb{E}_{s'}\underbrace{\left[r(s,a) + \gamma \mathbb{E}_{a'} Q^{\pi}(s',a')\right]}_{f(s',x)}$$

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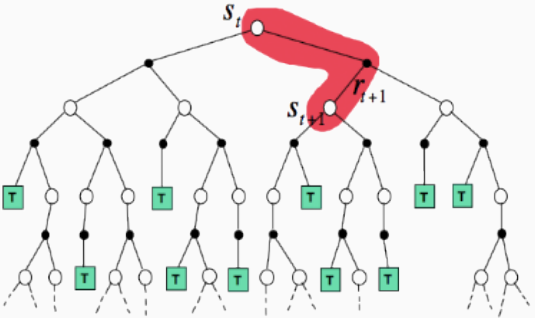

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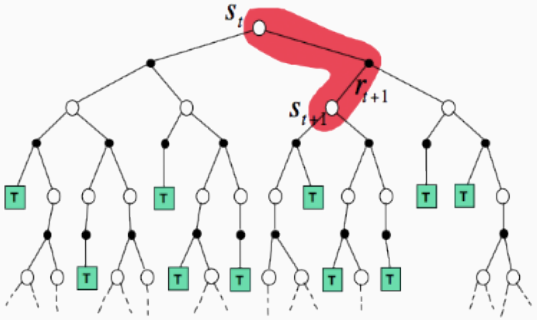
View 2: one-step bootstrapping: approximating future rewards using current approximation.

$$r + \gamma \underbrace{\left(r' + \gamma r'' + \gamma^2 r''' + \dots\right)}_{\approx Q_k^\pi(s',a')}$$



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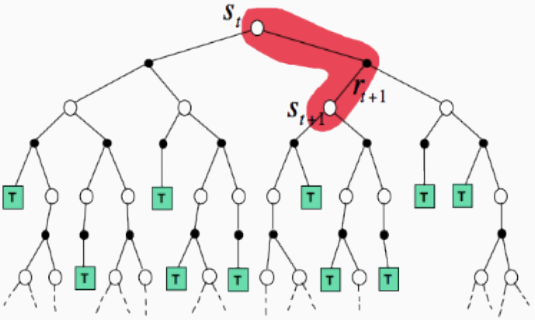
our own predictions used as targets!



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For transition (s, a, r, s', a'):

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$a' \sim \pi(a' \mid s')$ — **random variable** from algorithm.

Temporal Difference vs Monte Carlo

$$Q^{\pi}_{k+1}(s,a) \leftarrow Q^{\pi}_{k}(s,a) + \alpha_k \left(y(s,a) - Q^{\pi}_{k}(s,a) \right)$$

Which is better?

Temporal Difference

$y(s,a) := r + \gamma Q^{\pi}_{k}(s',a')$

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$y(s,a) := r + \gamma r' + \gamma^2 r'' + \dots$

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✓ updates after each step

× slow reward propagation

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× updates at the end of the episode

✓ fast reward propagation

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- × introduces bias

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- ✓ fast reward propagation
- × high variance
- ✓ unbiased

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$y(s,a) := r + \gamma r' + \gamma^2 r'' + \dots$

- × updates at the end of the episode
- ✓ fast reward propagation
- × high variance
- ✓ unbiased

Best estimation is somewhere in between (see TD(λ)).

Value Iteration and Policy Iteration connection

What if we improve our policy after every step?

$$\pi_k(s) \leftarrow \underset{a}{\operatorname{argmax}}\, Q_k^{\pi}(s, a)$$

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\end{aligned}$$

View 1: Policy Iteration with policy improvement after *each* policy evaluation update;
View 2: solving Bellman optimality equation (model-free Value Iteration);

Convergence of Q-learning updates

Let $Q_0^*(s, a)$ be initialized arbitrary, and for every s, a the following update is used:

$$Q_{k+1}^*(s, a) \leftarrow Q_k^*(s, a) + \alpha_k(s, a) \left(r(s, a) + \gamma \max_{a'} Q_k^*(s'_{k,s,a}, a') - Q_k^*(s, a) \right)$$

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Then, if:

- $s'_{k,s,a} \sim p(s' \mid s, a)$;
- with probability 1 for every s, a learning rate $\alpha_k(s, a) \in [0, 1]$ satisfies Robbins-Monro conditions:

$$\sum_{k=0}^{\infty} \alpha_k(s, a) = +\infty \quad \sum_{k=0}^{\infty} \alpha_k^2(s, a) < +\infty$$

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then $Q_k^*(s, a) \xrightarrow[k \to \infty]{\text{w. p. } 1} Q^*(s, a)$

Exploration vs Exploitation

Analogy 1

Global optimization

Exploration vs Exploitation

Analogy 1

Global optimization

Exploration:

random search

Exploration vs Exploitation

Analogy 1

Global optimization

Exploration:

random search

Exploitation:

local optimization

Exploration vs Exploitation

Analogy 1
Global optimization

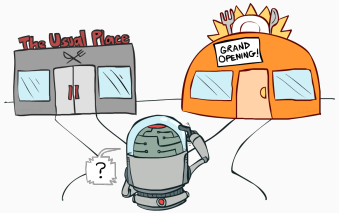
Exploration:
random search

Exploitation:
local optimization

Analogy 2
Multi-armed bandits
(regret minimization)

Exploration:
try something new

Exploitation:
try your favourite



Exploration vs Exploitation

Analogy 1
Global optimization

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Analogy 2
Multi-armed bandits
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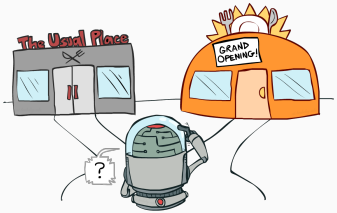

Analogy 3
General MDP

Exploration:
explore the environment

Exploitation:
solve the task

ε-greedy exploration

Do something random *sometimes*

Let's call policy ε-greedy with respect to some Q-function, if it behaves:

randomly with probability ε

greedy with probability $1 - \varepsilon$

ε-greedy exploration

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randomly with probability ε

greedy with probability $1 - \varepsilon$

$$\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|\mathcal{A}|} & \text{when } a = \operatorname*{argmax}_a Q(s, a) \\ \frac{\varepsilon}{|\mathcal{A}|} & \text{otherwise} \end{cases}$$

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$$\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|\mathcal{A}|} & \text{when } a = \operatorname*{argmax}_a Q(s, a) \\ \frac{\varepsilon}{|\mathcal{A}|} & \text{otherwise} \end{cases}$$

Decaying exploration:

$$\varepsilon_k \to 0, \quad \varepsilon_k > 0$$

Persistent exploration:

$$\varepsilon_k = \text{const}(k)$$

Q-learning

Q-learning (online)

Initialize $Q^*(s, a)$ arbitrarily;

observe s_0;

for $k = 0, 1, 2 \ldots$

- take action $a_k \sim \varepsilon$-greedy$(Q^*(s_k, a))$;
- observe r_k, s_{k+1};

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Q-learning (with replay buffer)

Initialize $Q^*(s, a)$ arbitrarily, $\mathcal{D} = \varnothing$;

Q-learning

Q-learning (online)

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- observe r_k, s_{k+1};
- store (s_k, a_k, r_k, s_{k+1}) in $\mathcal{D}$;

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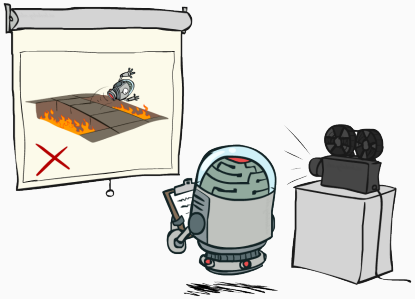
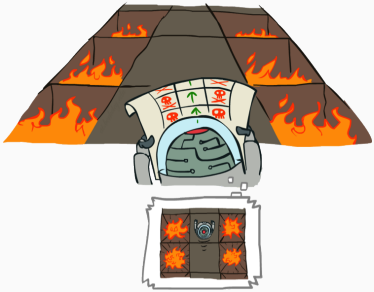
- observe r_k, s_{k+1};
- store (s_k, a_k, r_k, s_{k+1}) in $\mathcal{D}$;
- sample (s, a, r, s') from $\mathcal{D}$;
- $y := r + \gamma \max_{a'} Q^*(s', a')$;
- $Q^*(s, a) \leftarrow (1 - \alpha_k) Q^*(s, a) + \alpha_k y$

Q-learning is off-policy

Data source	Interaction policy	Convergence
Online experience	Infinite visitation	Q^*
Expert data	-	Q^*
Experience replay	Infinite visitation	Q^*

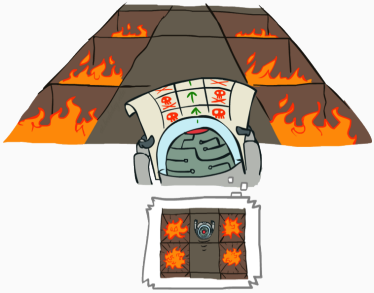


Expectation

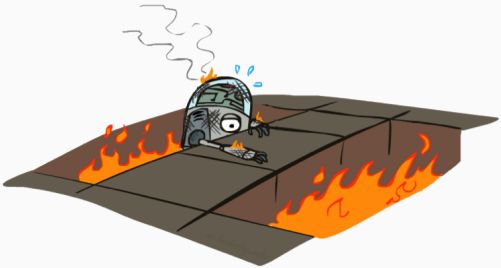


Safe RL

Expectation

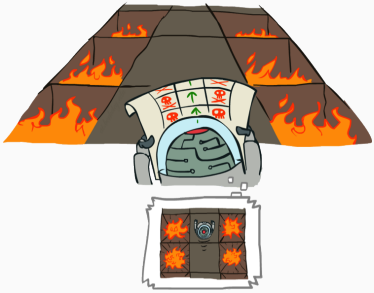


Reality

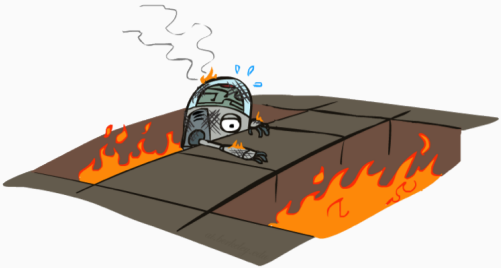


Safe RL

Expectation



Reality



Consider **the family of «ε-soft policies»**:

$$\forall s, a\colon \pi(a \mid s) \geqslant \frac{\varepsilon}{|\mathcal{A}|}$$

Q-learning vs SARSA

ε-soft Policy Improvement

In the family of «ε-soft policies» ε-greedy is policy improvement:

$$\tilde{\pi} := \varepsilon\text{-greedy}(Q^{\pi}(s,a)) \quad \Rightarrow \quad \tilde{\pi} \geq \pi$$

Q-learning vs SARSA

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Q-learning

a' taken from **target** policy

$$\pi_k(s) := \operatorname*{argmax}_a Q_k(s,a)$$

SARSA

a' taken from **behavior** policy

$$\pi_k := \varepsilon\text{-greedy}(Q_k(s,a))$$

Q-learning vs SARSA

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SARSA

a' taken from **behavior** policy

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SARSA convergence properties

SARSA converges to optimal ε-soft policy, satisfying

$$\pi \equiv \varepsilon\text{-greedy}(Q^{\pi}(s,a))$$

SARSA

SARSA

Initialize $Q^\pi(s, a)$ arbitrarily.

observe s_0, select a_0 randomly;
for $k = 0, 1, 2 \ldots$

- take action a_k, observe r_k, s_{k+1};
- sample $a_{k+1} \sim \varepsilon$-greedy$(Q^\pi(s_{k+1}, a))$
- $y := r_k + \gamma Q^\pi(s_{k+1}, a_{k+1})$;
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SARSA

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Expected-SARSA

Initialize $Q^\pi(s, a)$ arbitrarily.

observe s_0;
for $k = 0, 1, 2 \ldots$

- sample $a_k \sim \varepsilon$-greedy$(Q^\pi(s_k, a))$;
- take action a_k, observe r_k, s_{k+1};
- $y := r_k + \gamma \mathbb{E}_{a_{k+1}} Q^\pi(s_{k+1}, a_{k+1})$;
- $Q^\pi(s_k, a_k) \leftarrow (1 - \alpha_k) Q^\pi(s_k, a_k) + \alpha_k y$;

SARSA is on-policy

For transition (s, a, r, s', a'):

$$Q^{\pi}_{k+1}(s, a) \leftarrow Q^{\pi}_{k}(s, a) + \alpha_k \left(r + \gamma Q^{\pi}_{k}(s', a') - Q^{\pi}_{k}(s, a)\right)$$

Data source	**Interaction policy**	**Convergence**
Online experience	Decaying exploration	Q^*
	Persistent exploration	$Q^*_{\varepsilon\text{-greedy}}$
Expert data (a' taken from buffer)	-	

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Online experience	Decaying exploration	Q^*
	Persistent exploration	$Q^*_{\varepsilon\text{-greedy}}$
Expert data (a' taken from buffer)	-	$Q^{\pi_{\text{expert}}}$
Experience replay (a' taken from buffer)		

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Data source	**Interaction policy**	**Convergence**
Online experience	Decaying exploration	Q^*
	Persistent exploration	$Q^*_{\varepsilon\text{-greedy}}$
Expert data (a' taken from buffer)	-	$Q^{\pi_{\text{expert}}}$
Experience replay (a' taken from buffer)	(arbitrary)	divergence

«Fixing» SARSA (not really sarsa now)

For transition (s, a, r, s') generate $a' \sim \pi(a' \mid s')$:

$$Q^{\pi}_{k+1}(s, a) \leftarrow Q^{\pi}_{k}(s, a) + \alpha_k \left(r + \gamma Q^{\pi}_{k}(s', a') - Q^{\pi}_{k}(s, a)\right)$$

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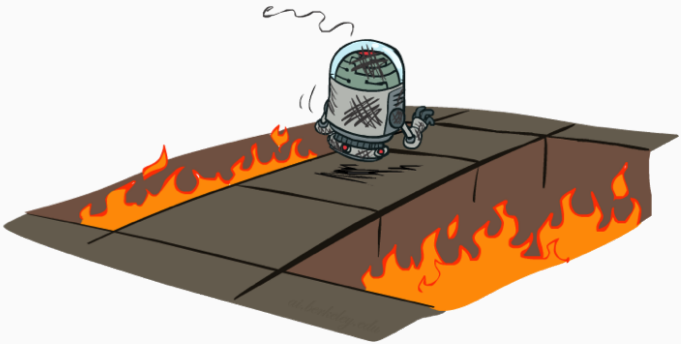
Data source	Interaction policy	Convergence
(any, but a' generated online)	Decaying exploration	Q^*
	Persistent exploration	$Q^*_{\varepsilon\text{-greedy}}$



Literature

- Sutton, Barto — Reinforcement Learning, an Introduction, ch. 5-6;
- Watkins, Dayan — Technical Note, Q-learning;

Pictures from Berkeley CS 188 | Introduction to Artificial Intelligence;