BONFERRONI-TYPE INEQUALITIES AND BINOMIALLY BOUNDED FUNCTIONS

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Abstract. We present a unified approach to an important subclass of Bonferroni-type inequalities by considering so-called binomially bounded functions. Our main result associates with each binomially bounded function a Bonferroni-type inequality. By appropriately choosing this function, several well-known and new results are deduced.

1. Introduction

Let A_v, $v \in V$, be finitely many events in some probability space $(\Omega, \mathcal{A}, P)$. The classical Bonferroni inequalities state that for any $r \in \mathbb{N}_0$,

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \geq (-1)^r \sum_{\substack{I\subseteq V\\ 0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) \tag{1.1}$$

where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. There are a lot of improvements and applications of these inequalities, see e.g., [3] for a detailed survey and [1] for some recent developments.

In this paper, we establish a new improvement of the classical Bonferroni inequalities by introducing an additional term on the right-hand side of (1.1), which involves the $(r+1)$-subsets of V and a so-called binomially bounded function. By choosing this function appropriately, several well-known and new results are obtained in a unified way.

2. Binomially bounded functions

The concept of a binomially bounded function arose from the proof of our main result and its consequences in Section 3.

Definition 2.1. For any finite set V and any $k \in \mathbb{N}$ we use $[V]^k$ to denote the set of k-subsets of V. A function $f : [V]^k \to \mathbb{R}$ is called *binomially bounded* if

$$\sum_{\substack{I\subseteq W\\ |I|=k}} f(I) \leq \binom{|W|-1}{k-1} \tag{2.1}$$

for any non-empty subset W of V.

Remark 2.1. As an immediate consequence of the preceding definition we observe that any binomially bounded function $f : [V]^k \to \mathbb{R}$ satisfies $f(I) \leq 1$ for any $I \in [V]^k$.

A huge class of binomially bounded functions is identified in the following proposition.

Proposition 2.1. *Let V be a finite set, and let p_v, $v \in V$, be non-negative reals such that $\sum_{v\in V} p_v \leq 1$. Then, for any $k \in \mathbb{N}$ the function $f : [V]^k \to \mathbb{R}$ which is defined by*

$$f(I) := \sum_{i\in I} p_i$$

is binomially bounded.

Proof. For any non-empty subset W of V we find that

$$\sum_{\substack{I\subseteq W\\|I|=k}} f(I) \;=\; \sum_{\substack{I\subseteq W\\|I|=k}}\sum_{i\in I} p_i \;=\; \sum_{i\in W} p_i \sum_{\substack{I\subseteq W\\|I|=k\\i\in I}} 1 \;=\; \sum_{i\in W} p_i \binom{|W|-1}{k-1} \;\leq\; \binom{|W|-1}{k-1},$$

which proves the statement. □

Example 2.1. Let V be a non-empty finite set and $k \in \mathbb{N}$. By putting $p_v := 1/|V|$ for any $v \in V$ in the preceding proposition we observe that the function $f : [V]^k \to \mathbb{R}$, which is defined by $f(I) := |I|/|V|$ for any $I \in [V]^k$, is binomially bounded.

Example 2.2. Let $G = (V, E)$ be a tree. For any two-element subset I of V define $f(I) := 1$ if I is an edge of G, and $f(I) := 0$ otherwise. Then, for any non-empty subset W of V,

$$\sum_{\substack{I\subseteq W\\|I|=2}} f(I) \;=\; m(G[W]) \;\leq\; |W|-1\,, \tag{2.2}$$

where $m(G[W])$ denotes the number of edges in the vertex-induced subgraph $G[W]$. The inequality in (2.2) holds since $G[W]$ is a tree, and since the number of edges in any tree equals the number of its vertices minus 1. By (2.2), $f : [V]^2 \to \mathbb{R}$ is binomially bounded.

3. Main result and consequences

We are now ready to state our main result.

Theorem 3.1. *Let A_v, $v \in V$, be finitely many events in some probability space $(\Omega, \mathcal{A}, P)$. Then, for any $r \in \mathbb{N}_0$ and any binomially bounded function $f : [V]^{r+1} \to \mathbb{R}$ we have*

$$(-1)^r P\left(\bigcup_{v\in V} A_v\right) \;\geq\; (-1)^r \sum_{\substack{I\subseteq V\\0<|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\subseteq V\\|I|=r+1}} P\left(\bigcap_{i\in I} A_i\right) f(I)\,. \tag{3.1}$$

The proof of Theorem 3.1 makes use of the following simple lemma.

Lemma 3.2. *For any non-empty finite set V and any $r \in \mathbb{N}_0$,*

$$\sum_{\substack{I\subseteq V\\|I|\leq r}} (-1)^{|I|} \;=\; (-1)^r \binom{|V|-1}{r}.$$

Proof. This follows from the well-known combinatorial identity

$$\sum_{k=0}^{r} (-1)^k \binom{m}{k} \;=\; (-1)^r \binom{m-1}{r} \quad (m \in \mathbb{N}, r \in \mathbb{N}_0),$$

which can easily be proved by the WZ method. □

Proof of Theorem 3.1. By the method of indicators [3] it suffices to prove that

$$(-1)^r 1_{\bigcup_{v\in V} A_v} \;\geq\; (-1)^r \sum_{\substack{I\subseteq V\\0<|I|\leq r}} (-1)^{|I|-1} 1_{\bigcap_{i\in I} A_i} + \sum_{\substack{I\subseteq V\\|I|=r+1}} 1_{\bigcap_{i\in I} A_i} f(I) \tag{3.2}$$

where 1_A denotes the indicator function of A. In order to prove (3.2) it suffices to show that for any $\omega \in \bigcup_{v\in V} A_v$,

$$(-1)^r \;\geq\; (-1)^r \sum_{\substack{I\subseteq V_\omega\\0<|I|\leq r}} (-1)^{|I|-1} + \sum_{\substack{I\subseteq V_\omega\\|I|=r+1}} f(I) \tag{3.3}$$

where $V_\omega := \{v \in V \mid \omega \in A_v\}$. By the requirement on f and Lemma 3.2 we find that

$$\sum_{\substack{I\subseteq V_\omega\\|I|=r+1}} f(I) \le \binom{|V_\omega|-1}{r} = (-1)^r \sum_{\substack{I\subseteq V_\omega\\|I|\le r}} (-1)^{|I|} = (-1)^r - (-1)^r \sum_{\substack{I\subseteq V_\omega\\0<|I|\le r}} (-1)^{|I|-1}.$$

which establishes (3.3). Thus, the proof of the theorem is complete. □

Remark 3.1. In view of the preceding proof it should be obvious that Theorem 3.1 (as well as its subsequent corollaries) remains valid if P is replaced by some arbitrary finite measure μ (e.g., the counting measure) on the algebra generated by the sets A_v, $v \in V$.

Theorem 3.1 offers some freedom in choosing the function $f : [V]^{r+1} \to \mathbb{R}$. As shown below, by choosing this function appropriately, some known and new results on Bonferroni-type inequalities can be obtained in a concise and unified way. We start with deducing Galambos' inequality [2], which for $r = 1$ specializes to Kwerel's inequality [7].

Corollary 3.3. [2] *Let A_v, $v \in V$, be events in some probability space $(\Omega, \mathcal{A}, P)$, where V is assumed to be finite and non-empty. Then, for any $r \in \mathbb{N}_0$ we have*

$$(3.4)\quad (-1)^r P\left(\bigcup_{v\in V} A_v\right) \ge (-1)^r \sum_{\substack{I\subseteq V\\0<|I|\le r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \frac{r+1}{|V|} \sum_{\substack{I\subseteq V\\|I|=r+1}} P\left(\bigcap_{i\in I} A_i\right).$$

Proof. Corollary 3.3 follows from Theorem 3.1 by considering the binomially bounded function of Example 2.1. □

The following inequality seems to be new even in the particular case $r = 1$. It agrees with Galambos' inequality of Corollary 3.3 if all probabilities $P(A_v)$, $v \in V$, are equal, or if all probabilities $P\left(\bigcap_{i\in I} A_i\right)$ are equal for all subsets $I \subseteq V$ satisfying $|I| = r + 1$.

Corollary 3.4. *If, in addition to the requirements of Corollary 3.3, $P(A_v) > 0$ for at least one $v \in V$, then*

$$(3.5)\quad (-1)^r P\left(\bigcup_{v\in V} A_v\right) \ge (-1)^r \sum_{\substack{I\subseteq V\\0<|I|\le r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\subseteq V\\|I|=r+1}} P\left(\bigcap_{i\in I} A_i\right) \sum_{i\in I} P(A_i) \Big/ \sum_{v\in V} P(A_v).$$

Proof. Define $p_i := P(A_i)/\sum_{v\in V} P(A_v)$ for any $i \in V$. Then, by Proposition 2.1 it follows that the function $f : [V]^{r+1} \to \mathbb{R}$, which is defined by $f(I) := \sum_{i\in I} P(A_i)/\sum_{v\in V} P(A_v)$ for any $I \in [V]^{r+1}$, is binomially bounded. The result now follows from Theorem 3.1. □

Our next corollary generalizes a result due to Kounias [6], which is obtained by considering the particular case $r = 1$.

Corollary 3.5. *Under the requirements of Corollary 3.3,*

$$(3.6)\quad (-1)^r P\left(\bigcup_{v\in V} A_v\right) \ge (-1)^r \sum_{\substack{I\subseteq V\\0<|I|\le r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \max_{j\in V} \sum_{\substack{I\subseteq V\\|I|=r+1\\j\in I}} P\left(\bigcap_{i\in I} A_i\right).$$

Proof. Fix some $j \in V$ and define $p_i := \delta_{ij}$ for any $i \in V$, where δ is the usual Kronecker delta. Then, by applying Proposition 2.1 it follows that the function $f : [V]^{r+1} \to \mathbb{R}$, which is defined by $f(I) := \sum_{i \in I} p_i$ for any $I \in [V]^{r+1}$, is binomially bounded. Note that $f(I) = 1$ if $j \in I$, and $f(I) = 0$ if $j \notin I$. Thus, by applying Theorem 3.1 we obtain

$$(-1)^r P\left(\bigcup_{v \in V} A_v\right) \geq (-1)^r \sum_{\substack{I \subseteq V \\ 0 < |I| \leq r}} (-1)^{|I|-1} P\left(\bigcap_{i \in I} A_i\right) + \sum_{\substack{I \subseteq V \\ |I| = r+1 \\ j \in I}} P\left(\bigcap_{i \in I} A_i\right).$$

The result now follows by taking the maximum of the right-hand side over all $j \in V$. $\square$

In our next corollary, we rediscover a prominent result due to Hunter [5] and Worsley [9].

Corollary 3.6. [5, 9] *Let A_v, $v \in V$, be events in some probability space $(\Omega, \mathcal{A}, P)$. Then, for any tree $G = (V, E)$ on the index-set of these events the following inequality holds:*

$$P\left(\bigcup_{v \in V} A_v\right) \leq \sum_{v \in V} P(A_v) - \sum_{\{i,j\} \in E} P(A_i \cap A_j). \tag{3.7}$$

Proof. Corollary 3.6 follows from Theorem 3.1 by considering the binomially bounded function of Example 2.2. $\square$

The Hunter-Worsley bound of Corollary 3.6 has been generalized from trees to hypertrees by Tomescu [8], and then from hypertrees to sparse uniform hypergraphs by Grable [4]. A quite different generalization to chordal graphs can be found in [1].

Recall that a hypergraph $H = (V, \mathcal{E})$ is called *k-uniform* if each edge $E \in \mathcal{E}$ consists of exactly k vertices. A k-uniform hypergraph $H = (V, \mathcal{E})$ is called *sparse* if for any non-empty subset W of V the induced subhypergraph $H[W] := (W, \mathcal{E} \cap 2^W)$ has at most $\binom{|W|-1}{k-1}$ edges. As a final consequence of our main result, we now deduce Grable's inequality.

Corollary 3.7. [4] *Let A_v, $v \in V$, be events in some probability space $(\Omega, \mathcal{A}, P)$. Then, for any $r \in \mathbb{N}_0$ and any sparse $(r+1)$-uniform hypergraph $H = (V, \mathcal{E})$ on the index-set of these events the following inequality holds:*

$$(-1)^r P\left(\bigcup_{v \in V} A_v\right) \geq (-1)^r \sum_{\substack{I \subseteq V \\ 0 < |I| \leq r}} (-1)^{|I|-1} P\left(\bigcap_{i \in I} A_i\right) + \sum_{I \in \mathcal{E}} P\left(\bigcap_{i \in I} A_i\right). \tag{3.8}$$

Proof. For any $(r+1)$-element subset I of V define $f(I) := 1$ if I is an edge of H, and $f(I) := 0$ otherwise. Then, since H is a sparse $(r+1)$-uniform hypergraph,

$$\sum_{\substack{I \subseteq W \\ |I| = r+1}} f(I) = m(H[W]) \leq \binom{|W|-1}{r} \tag{3.9}$$

for any non-empty subset W of V, where $m(H[W])$ denotes the number of edges in $H[W]$. By (3.9), f is binomially bounded. The result now follows by applying Theorem 3.1. $\square$

Remark 3.2. Although, as we saw in the proof of the preceding corollary, any sparse uniform hypergraph gives rise to a binomially bounded function, the two concepts are not equivalent. There are binomially bounded functions (e.g., those in the proofs of Corollaries 3.3 and 3.4) to which no sparse uniform hypergraph corresponds. In view of Grable's inequality and our main result it should be clear, however, that sparse uniform hypergraphs are in fact equivalent to *0,1-valued* binomially bounded functions.

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