On the role of the Fejér summation in terrain modeling

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The method is intended for the processing of digital elevation models (DEMs) within a single framework, including DEM global approximation, denoising, generalization, as well as calculating the partial derivatives of elevation and morphometric variables.

The method is based on high-order orthogonal expansions using the Chebyshev polynomials of the first kind with the subsequent Fejér summation.

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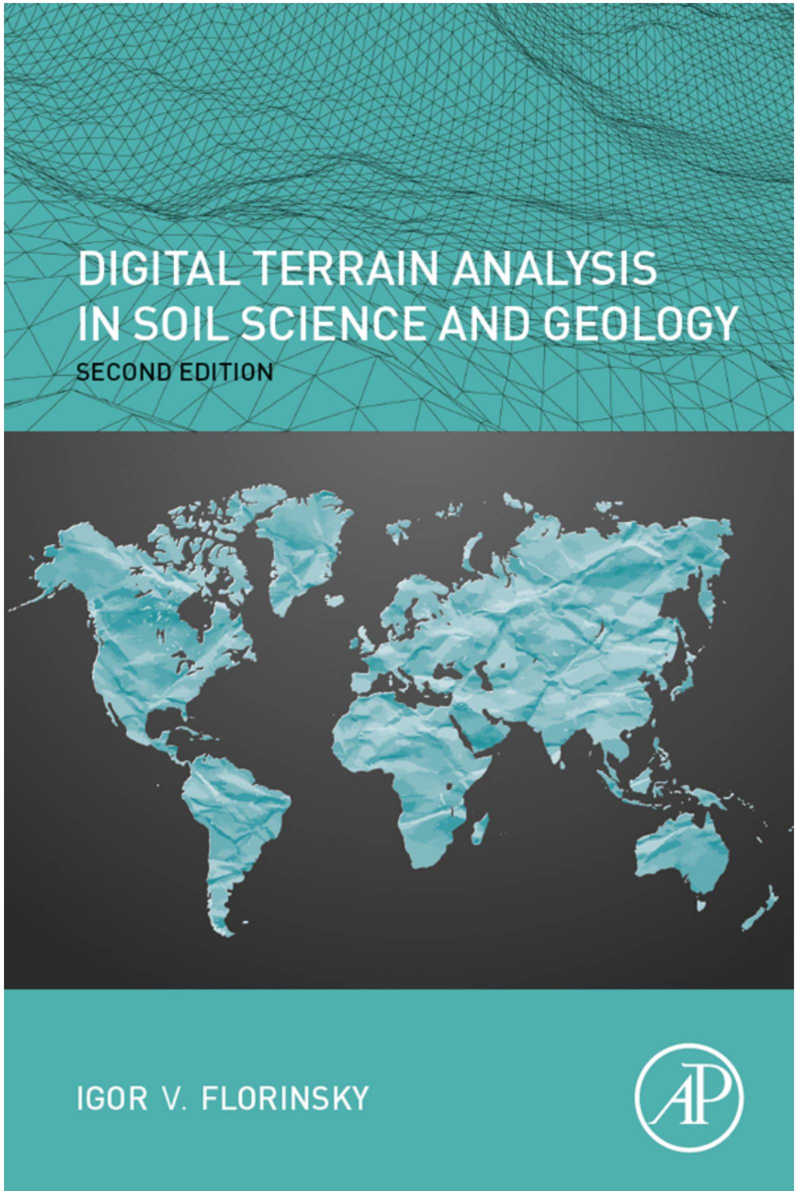
Digital terrain modeling (geomorphometry) is a science of quantitative modeling and analysis of the topographic surface and relationships between topography and other natural and artificial components of geosystems.

Digital terrain modeling is widely used to solve various multiscale problems of geomorphology, hydrology, remote sensing, soil science, geology, geophysics, geobotany, glaciology, oceanology, climatology, planetology, and other disciplines.



DIGITAL TERRAIN ANALYSIS IN SOIL SCIENCE AND GEOLOGY

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Method

Original function: $z = f(x, y)$

Approximation using the bivariate Chebyshev series expansion:

$$z(x, y) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} c_{ij} T_i(x) T_j(y)$$

<u>4 main stages</u>:

- Calculation of expansion coefficients
- The Fejér summation
- Reconstruction of the approximated function
- Calculation of derivatives

The 2D approximation of the original function can be computationally performed as a superposition of two one-dimensional approximations by the variables x and y.

1. Calculation of expansion coefficients

Univariate Chebyshev series expansion:

$$u(x) = \sum_{i=0}^{l-1} c_i T_i(x)$$

The Chebyshev polynomials:

$$T_i(x) = \cos(i \arccos x)$$

Orthogonality condition :

- continuous form

$$(T_i, T_j) = \int_{-1}^{1} \frac{T_i(x) T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{if } i \neq j \\ \pi/2 & \text{if } i = j \end{cases}$$

- discrete form

$$(T_n, T_m) = \frac{2}{k} \sum_{i=1}^{k} T_n(\xi_i) T_m(\xi_i) = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

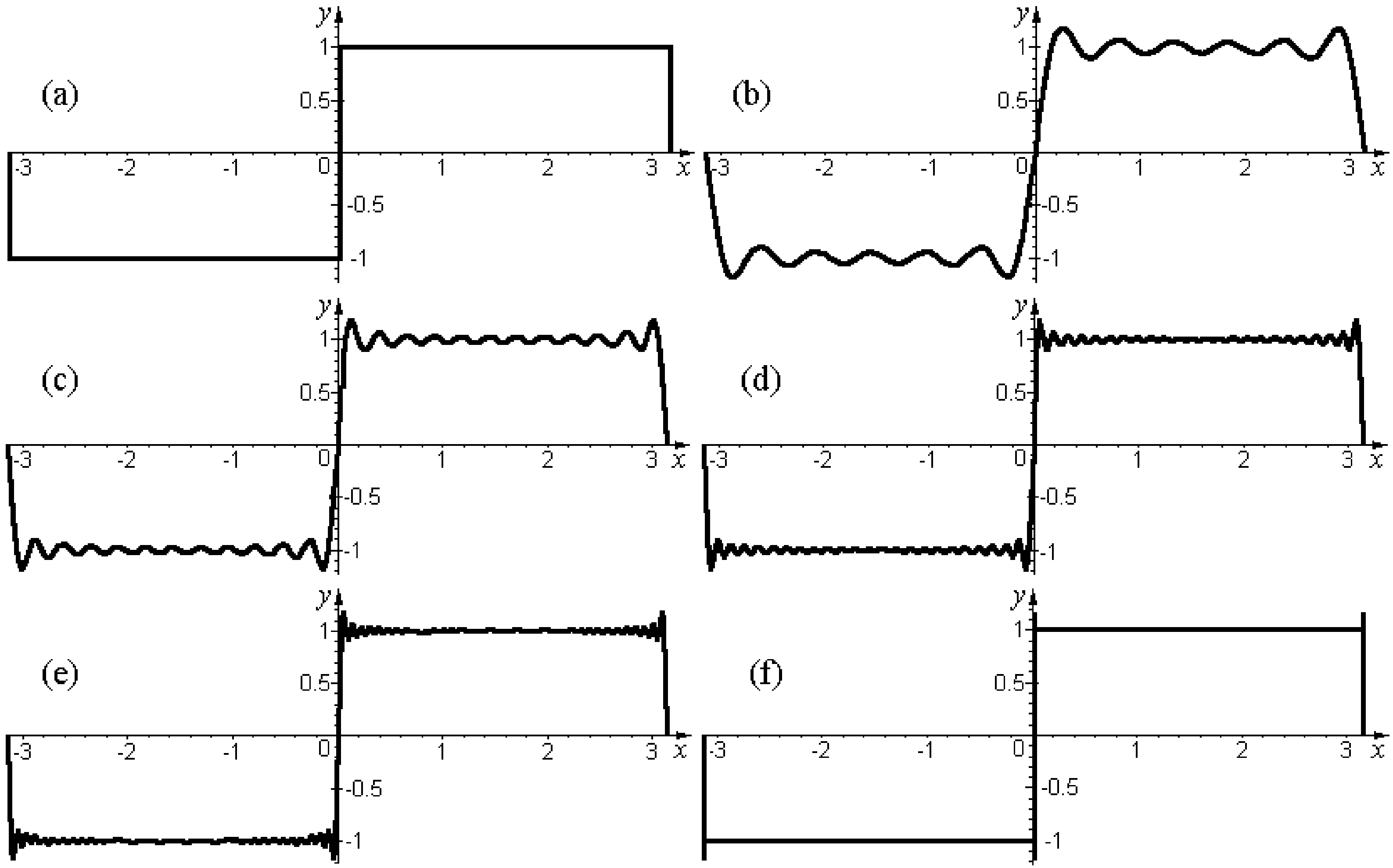
Nodes of the Gaussian quadrature grid :

$$\xi_i = \cos\left(\frac{\pi(i - 1/2)}{k}\right), \quad i = 1, 2, \ldots, k$$

Expansion coefficients :

$$c_j = (u, T_j) = \frac{2}{k} \sum_{i=1}^{k} u(\xi_i) T_j(\xi_i)$$

Approximations based on orthogonal polynomials always lead to oscillatory artifacts due to the Gibbs phenomenon :



2. The Fejér summation

The Fejér theorem: the arithmetic mean of the partial sums of the Fourier series of a continuous function converges uniformly to the function.

The Fejér theorem is also true for other orthogonal series.

A uniform convergence of the approximation can be achieved replacing the initial expansion

$$u(x) = \sum_{i=0}^{l-1} c_i T_i(x)$$

by the arithmetic mean of the partial sums of the Chebyshev series:

$$\tilde{u}_n(x) = \frac{1}{n} \sum_{l=1}^{n} u_l(x)$$

Expansion coefficients: $\tilde{c}_j = \frac{l-j}{l} c_j$

Jerri A.J. *The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet Approximations*. Boston: Kluwer Academic Publishers, 1998, 336 p.

3. Reconstruction of the approximated function

A simple summation of the orthogonal series.

Generalization and/or denoising of the function are performed decreasing the number of expansion coefficients $\tilde{c}_j$ used for the reconstruction.

4. Calculation of derivatives

Expansion coefficients of partial derivatives

$$
\begin{cases}
c'_{l-1} = 0 \\
c'_{l-2} = 2(l-2)\tilde{c}_{l-1} \\
\ldots \\
c'_j = c'_{j+2} + 2j\tilde{c}_{j+1}\ , \quad j = l-3,\ldots,0 \\
\ldots \\
c'_0 = \dfrac{c'_0}{\sqrt{2}}
\end{cases}
$$

Repeating the stages 1–4 for the variable y.

Obtaining expressions for the partial derivatives of elevation p, q, r, s, t in the form of the bivariate orthogonal series:

$$p(x, y) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} c_{ij}^{p} T_i(x) T_j(y)$$

$$q(x, y) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} c_{ij}^{q} T_i(x) T_j(y)$$

$$r(x, y) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} c_{ij}^{r} T_i(x) T_j(y)$$

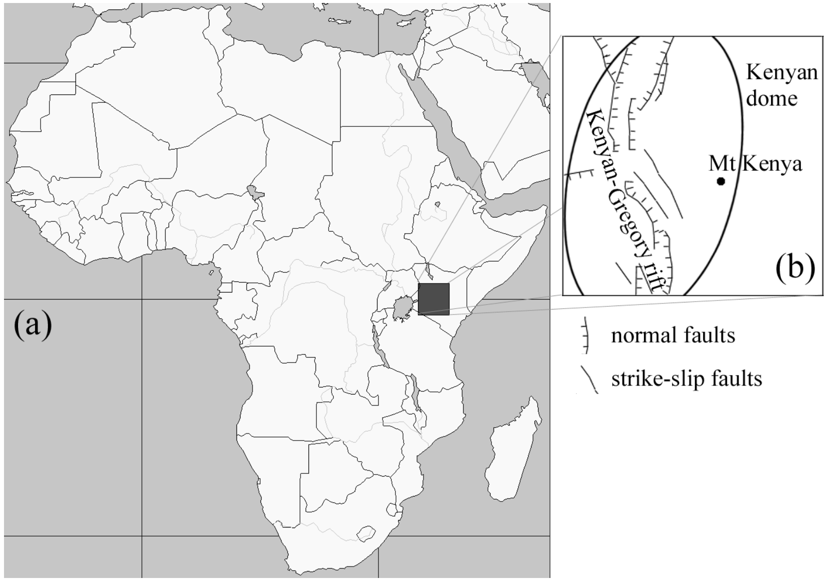
$$s(x, y) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} c_{ij}^{s} T_i(x) T_j(y)$$

$$t(x, y) = \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} c_{ij}^{t} T_i(x) T_j(y)$$

Objective

To evaluate a role of the Fejér summation in the treatment of digital terrain models.

Study area



Kenya

Initial data: Global digital elevation model (DEM) SRTM30_PLUS

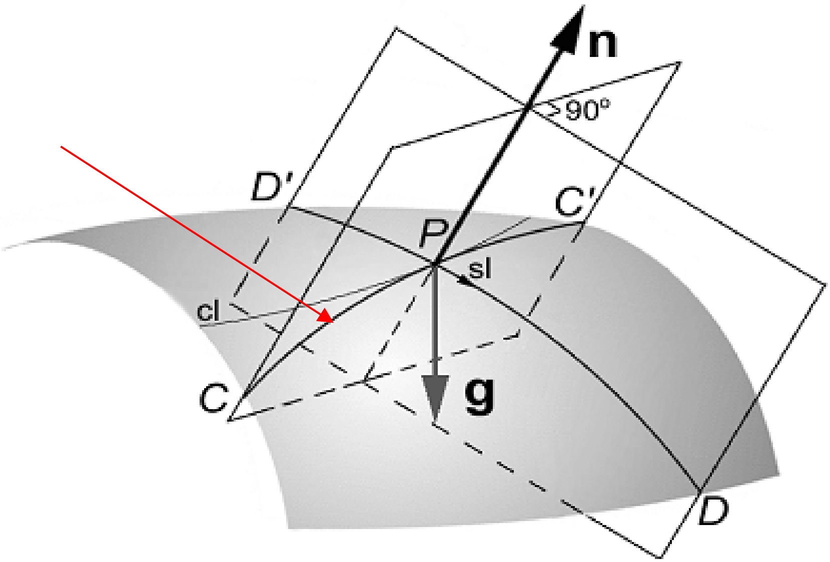
DEM: 230,880 points (elevation matrix 480 × 481)

$$w = 30''$$

Definitions of some morphometric variables

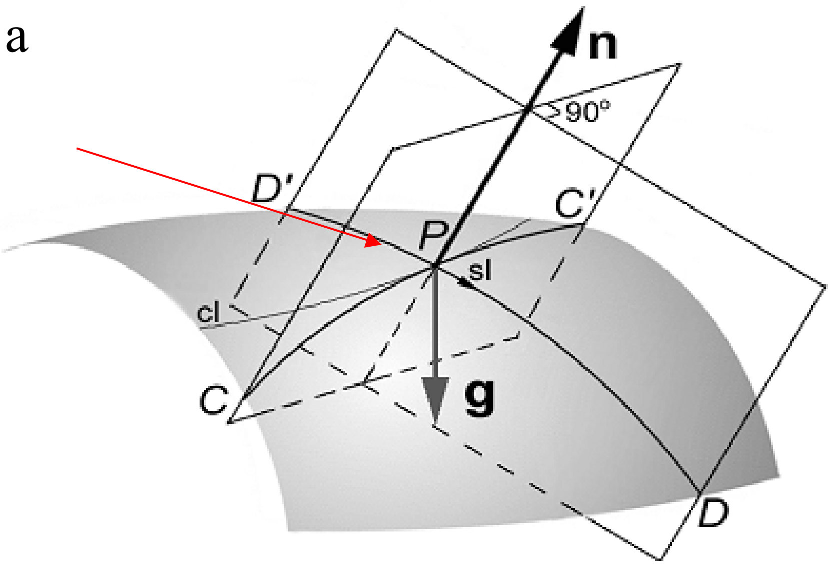
Horizontal curvature is a curvature of a normal section tangential to a contour line at a given point of the topographic surface.

$$k_h = -\frac{q^2 r - 2pqs + p^2 t}{\left(p^2 + q^2\right)\sqrt{1 + p^2 + q^2}}$$



Vertical curvature is a curvature of a normal section having a common tangent line with a slope line at a given point of the topographic surface.

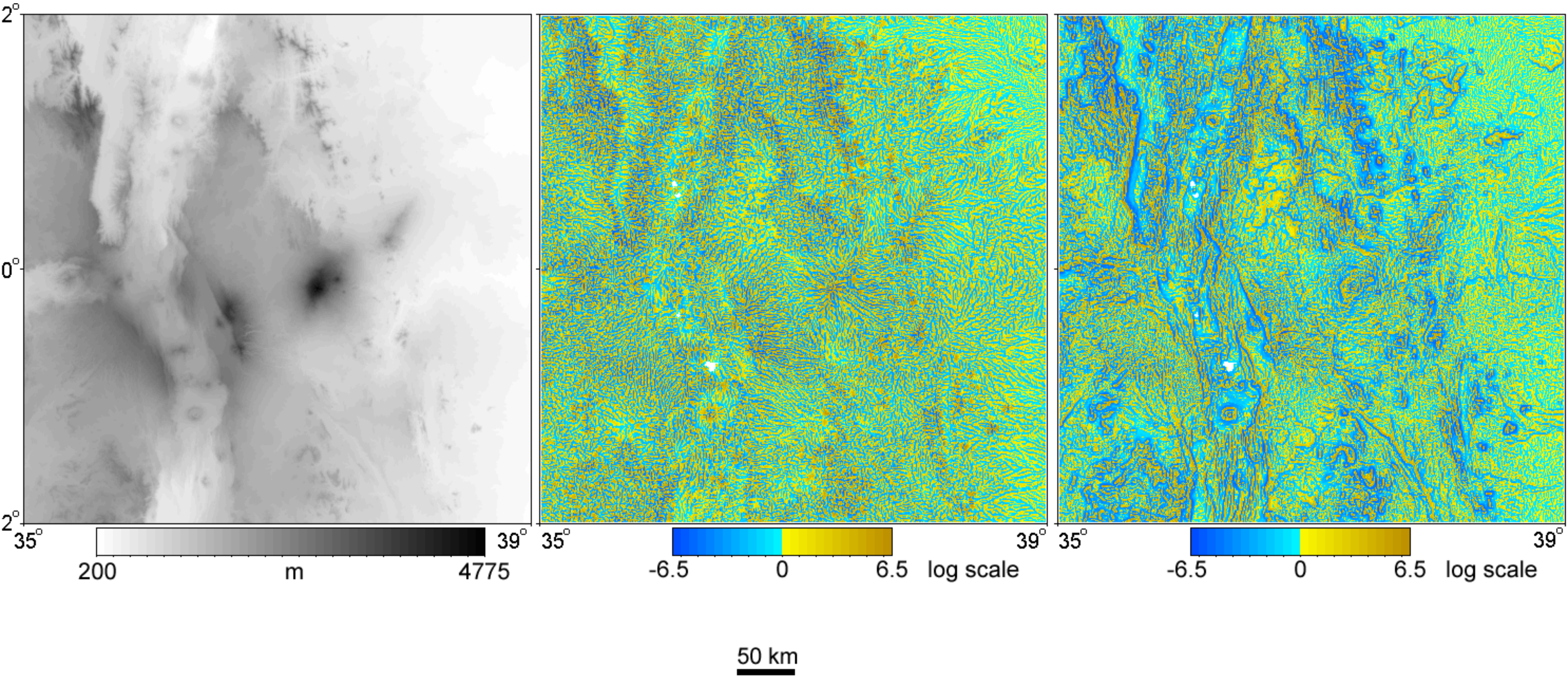
$$k_v = -\frac{p^2 r + 2pqs + q^2 t}{\left(p^2 + q^2\right)\sqrt{\left(1 + p^2 + q^2\right)^3}}$$



Approximation of elevation and calculation of curvatures

Initial models

Elevation | Horizontal curvature | Vertical curvature

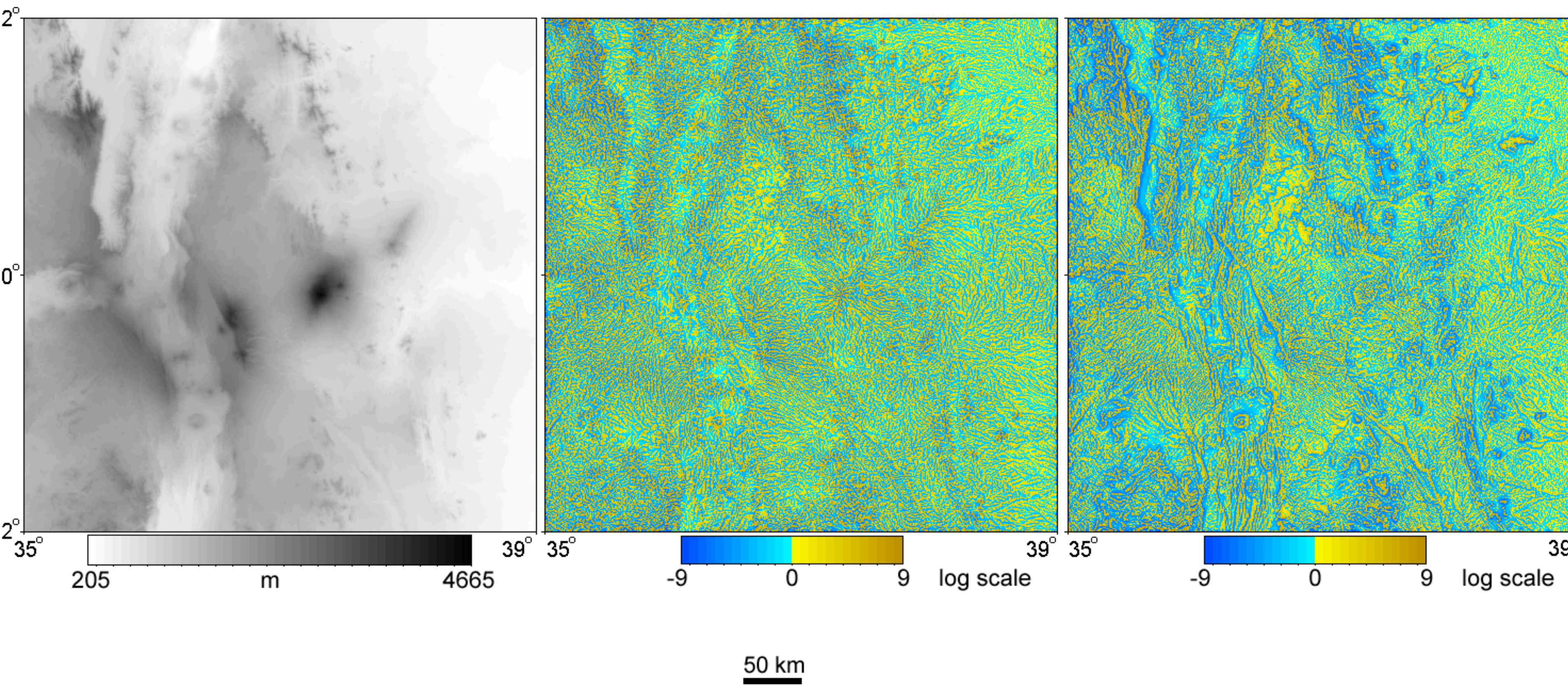


Reconstruction with 2880 expansion coefficients

Elevation

Horizontal curvature

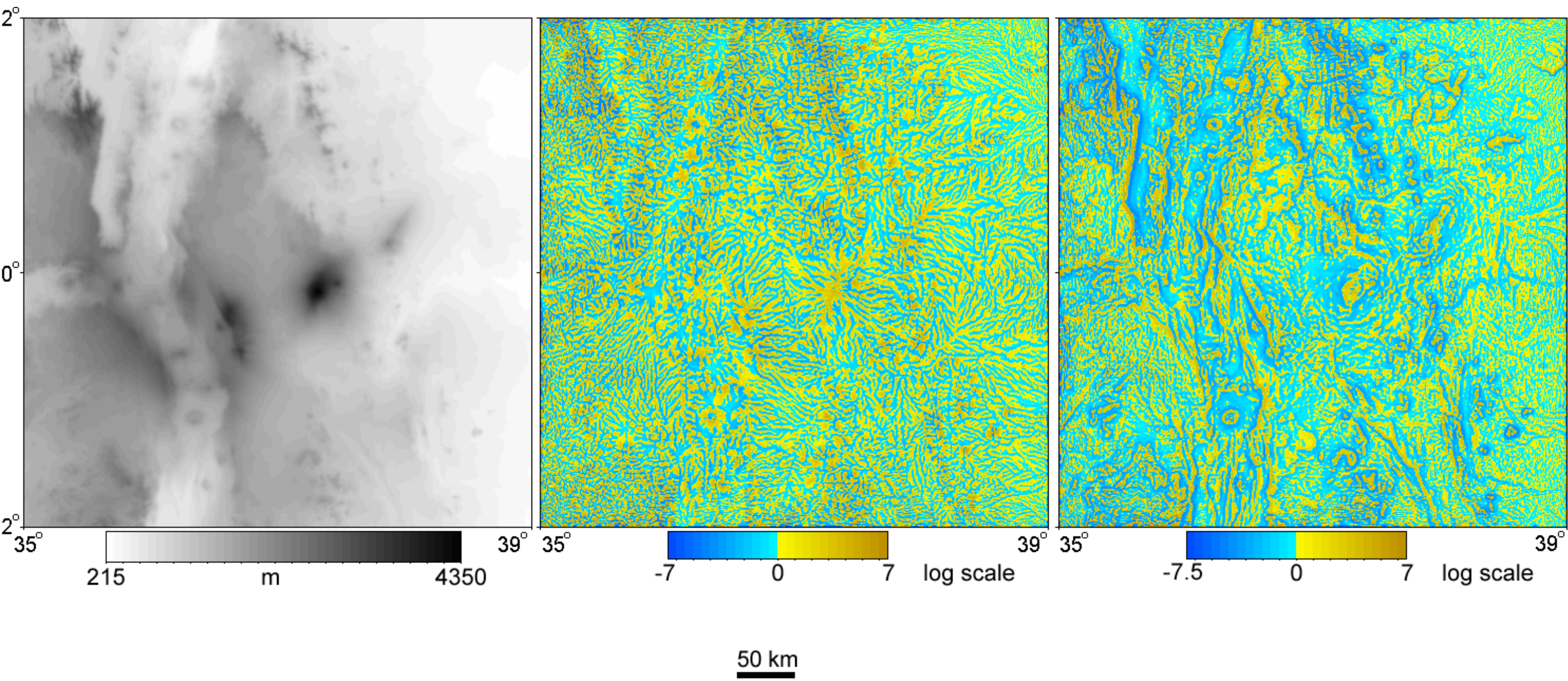
Vertical curvature



Approximation of elevation and calculation of curvatures

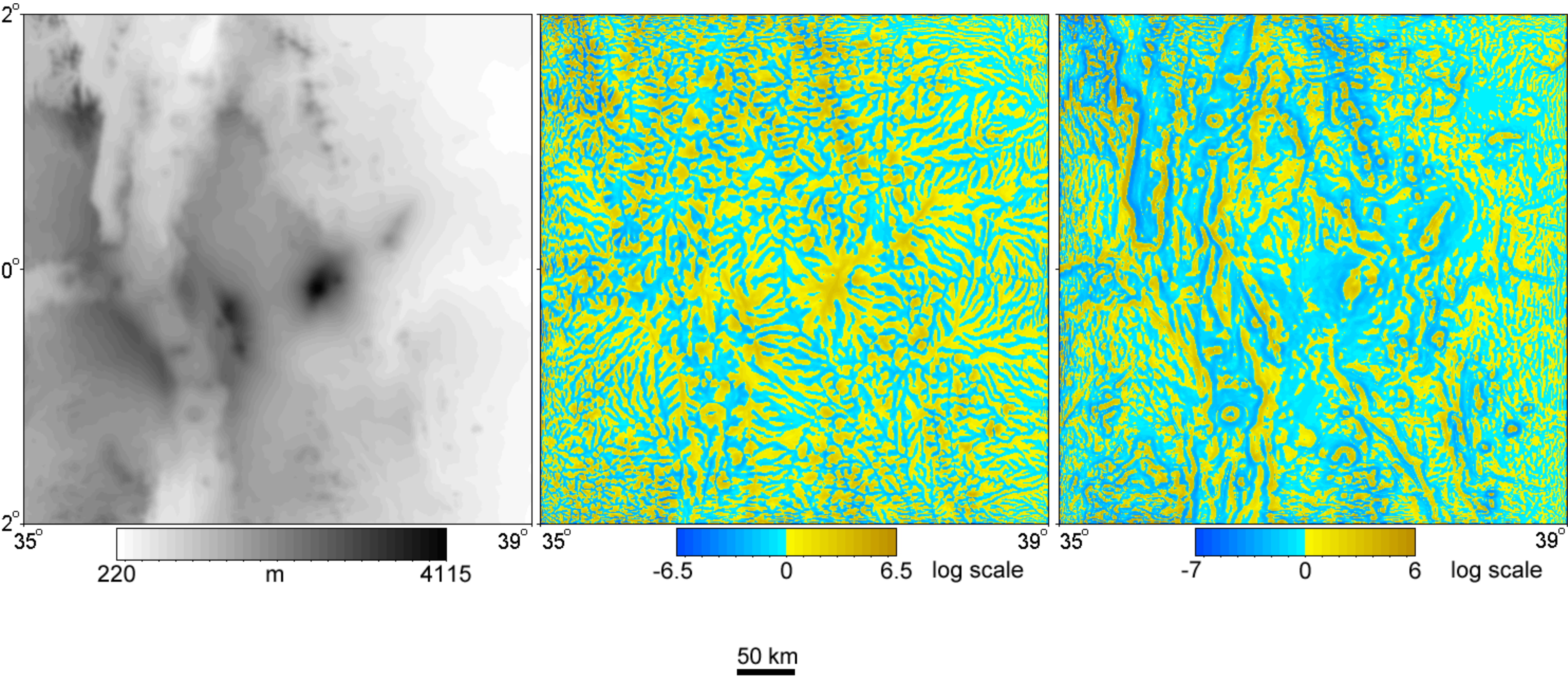
Reconstruction with 480 expansion coefficients

Elevation | Horizontal curvature | Vertical curvature



Reconstruction with 240 expansion coefficients

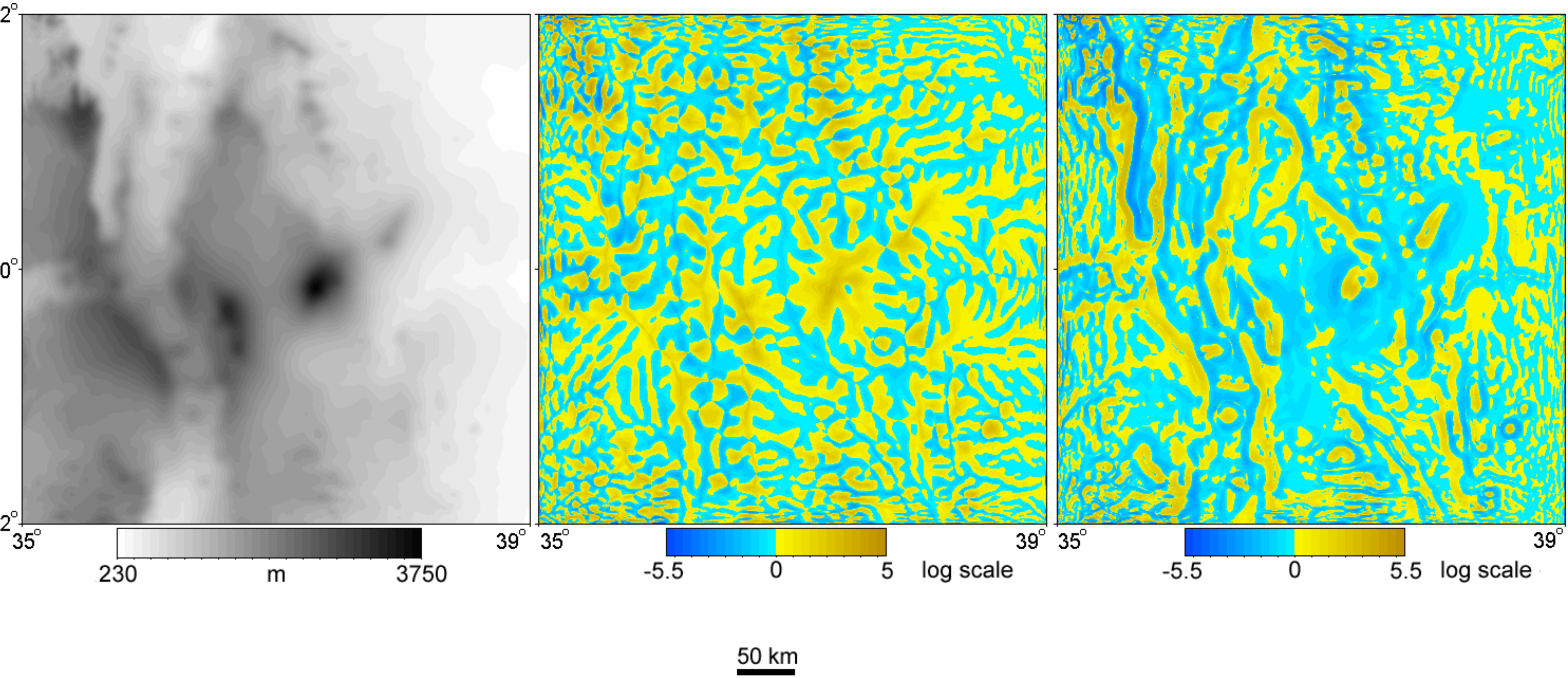
Elevation | Horizontal curvature | Vertical curvature



Approximation of elevation and calculation of curvatures

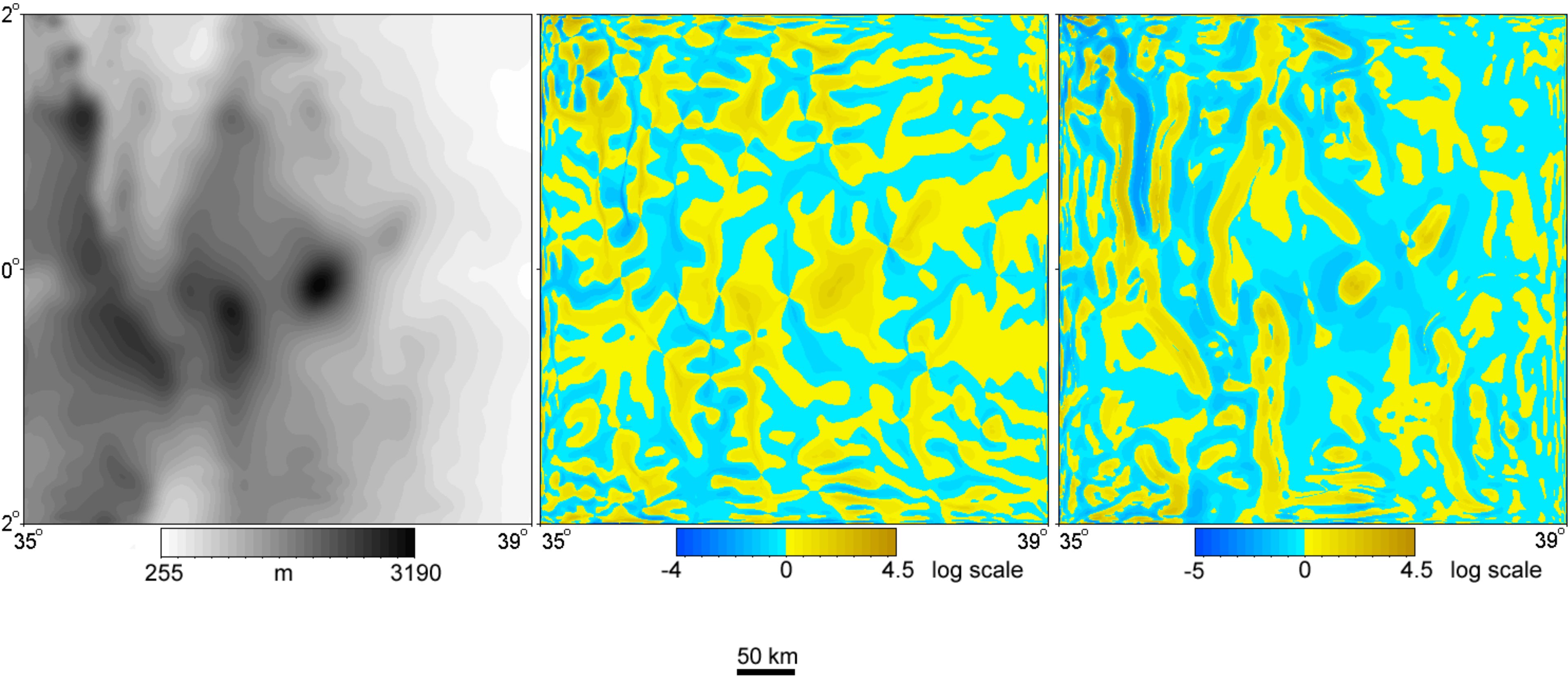
Reconstruction with 120 expansion coefficients

Elevation | Horizontal curvature | Vertical curvature



Reconstruction with 60 expansion coefficients

Elevation | Horizontal curvature | Vertical curvature

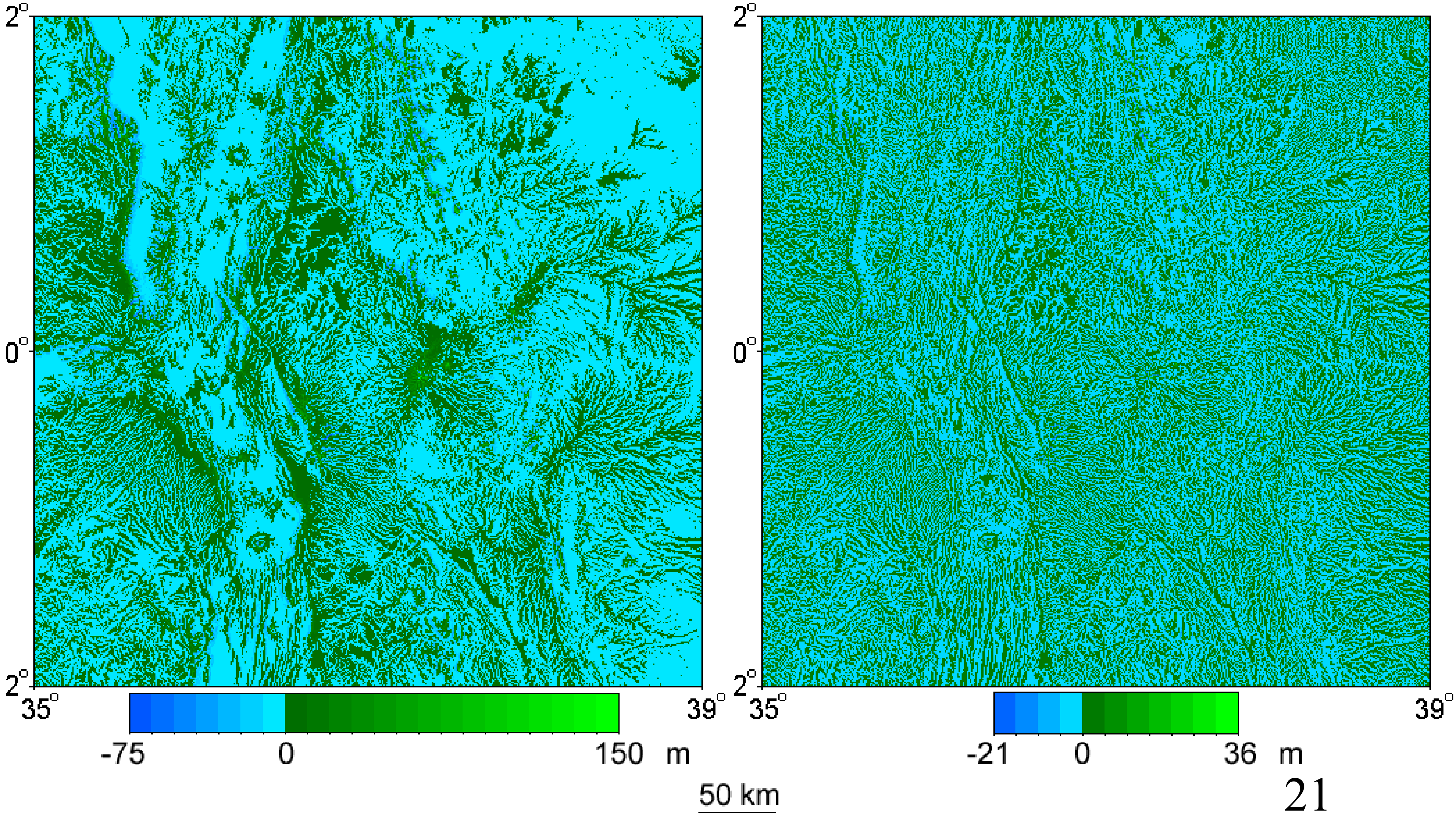


Evaluation of the convergence rate of approximation and the role of the Fejér summation

Residual models for the reconstructed DEMs

2880 expansion coefficients

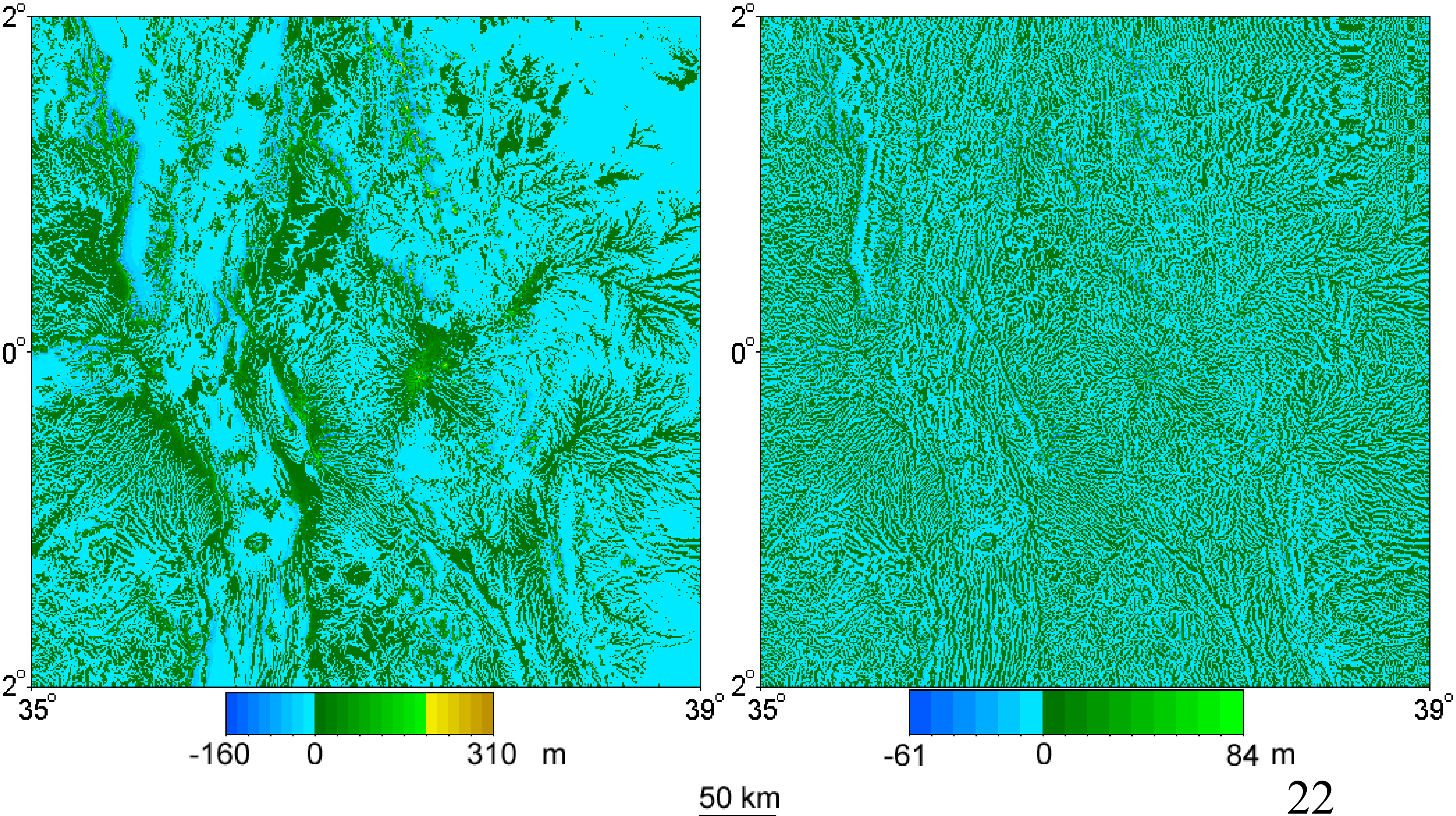
with the Fejér summation | without the Fejér summation



Residuals for the reconstructed DEMs

960 expansion coefficients

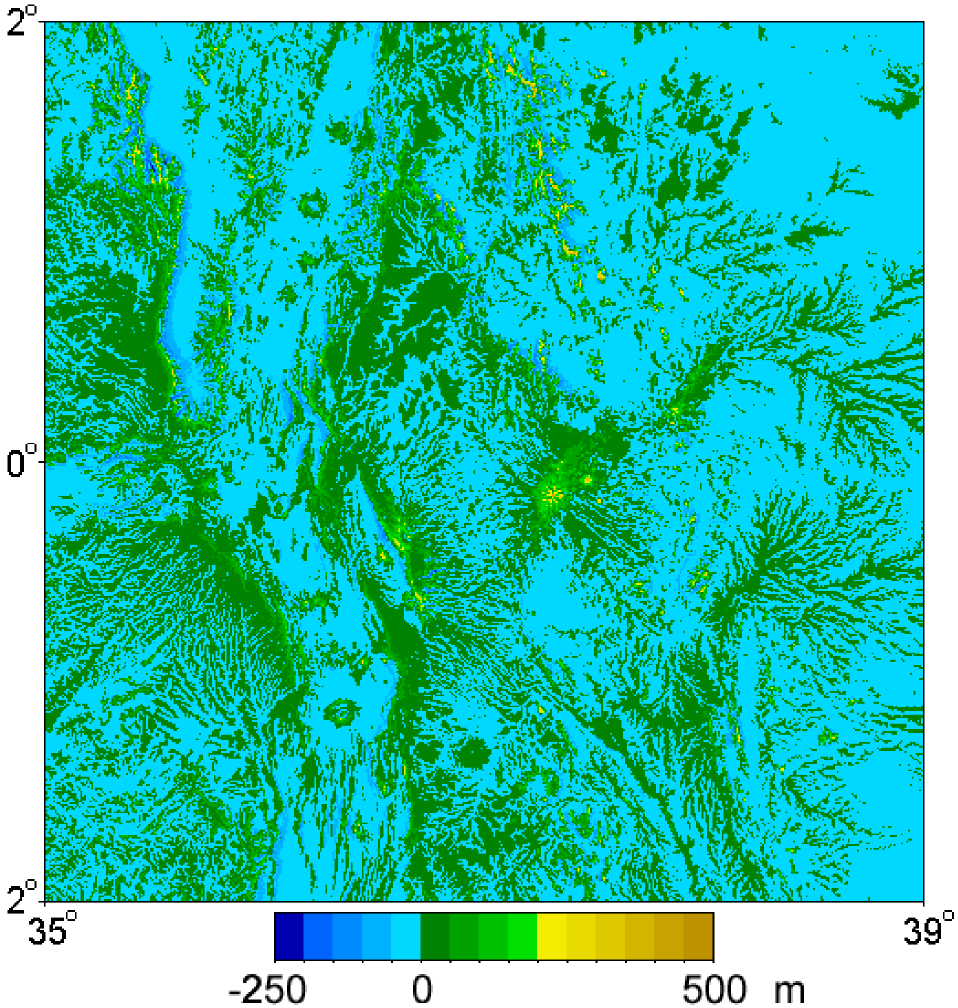
with the Fejér summation | without the Fejér summation



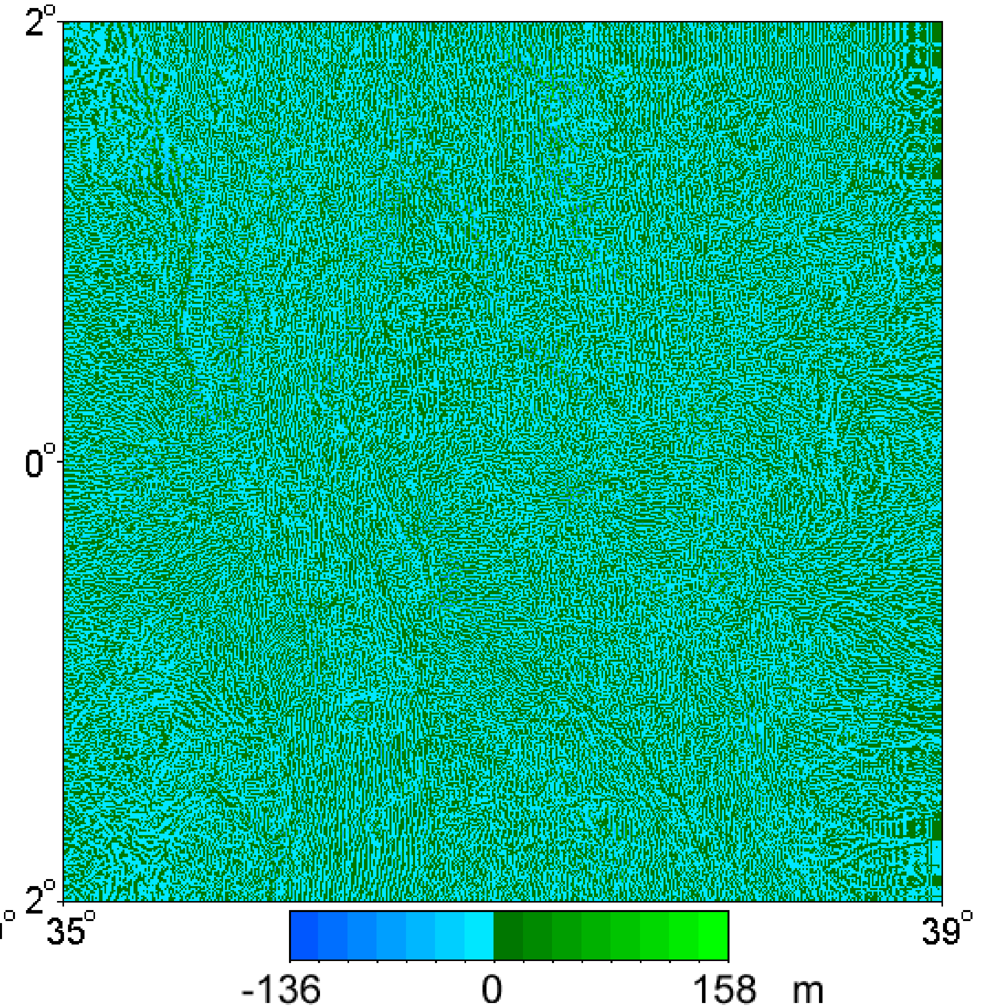
Residuals for the reconstructed DEMs

480 expansion coefficients

with the Fejér summation



without the Fejér summation

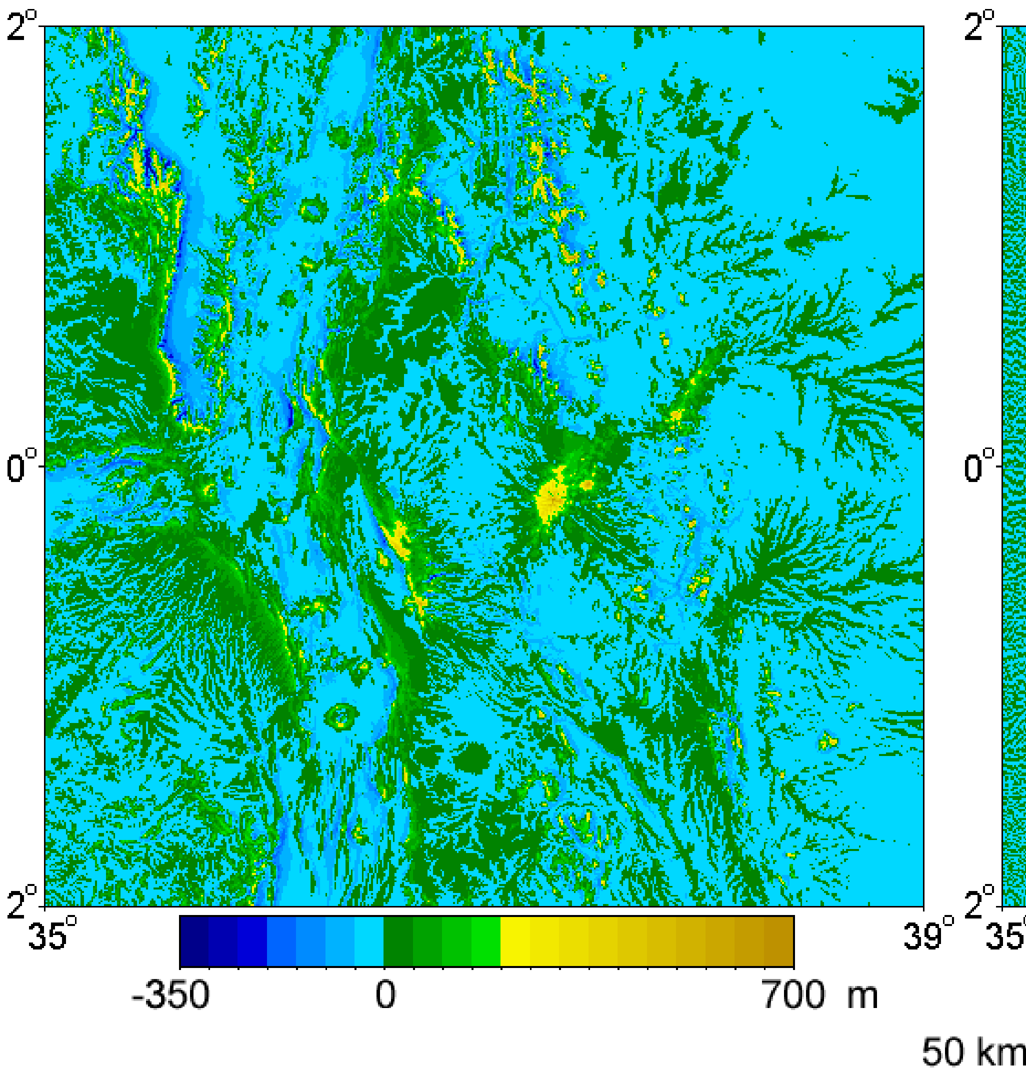


50 km

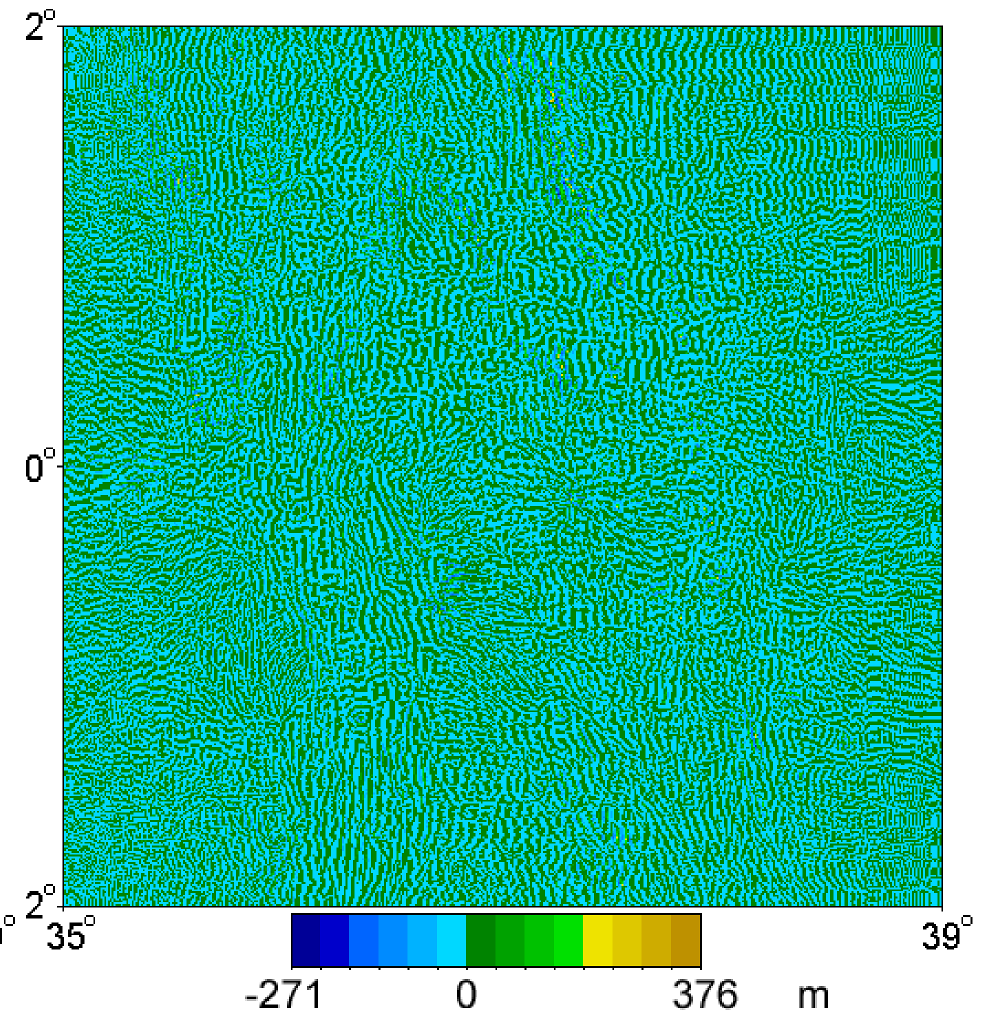
Residuals for the reconstructed DEMs

240 expansion coefficients

with the Fejér summation



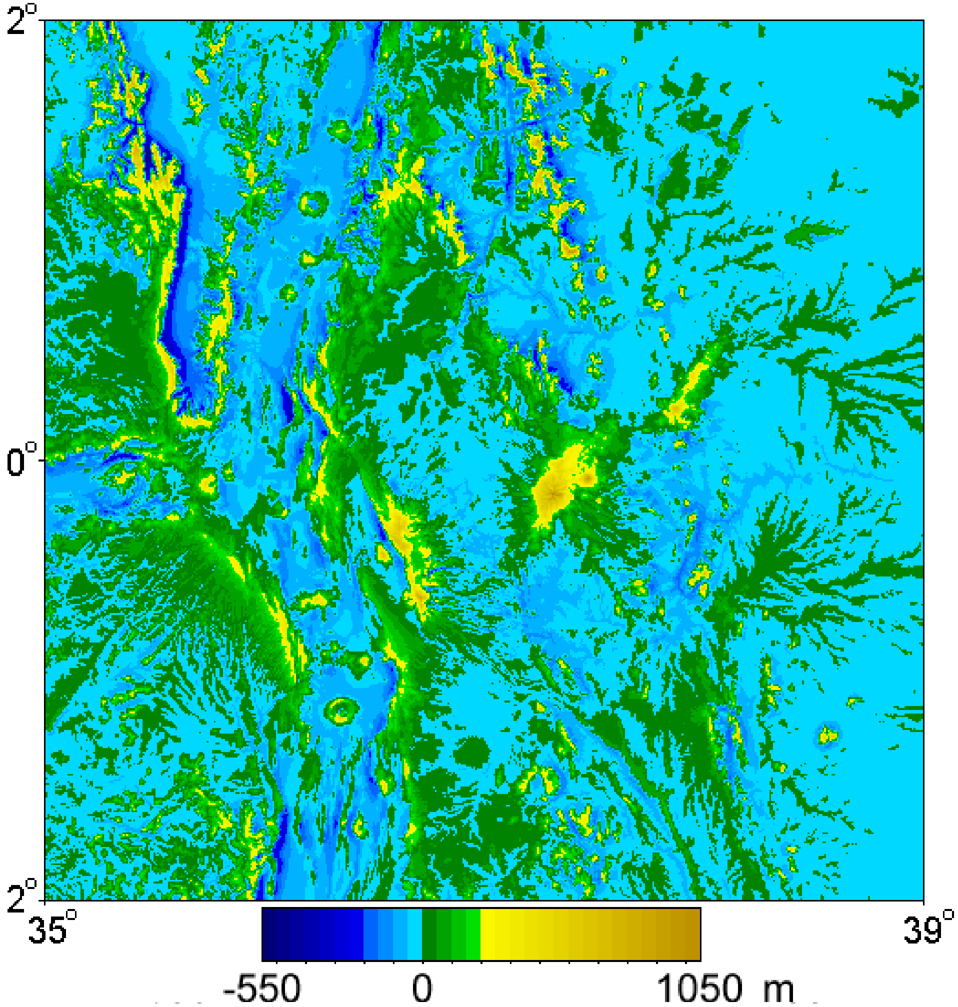
without the Fejér summation



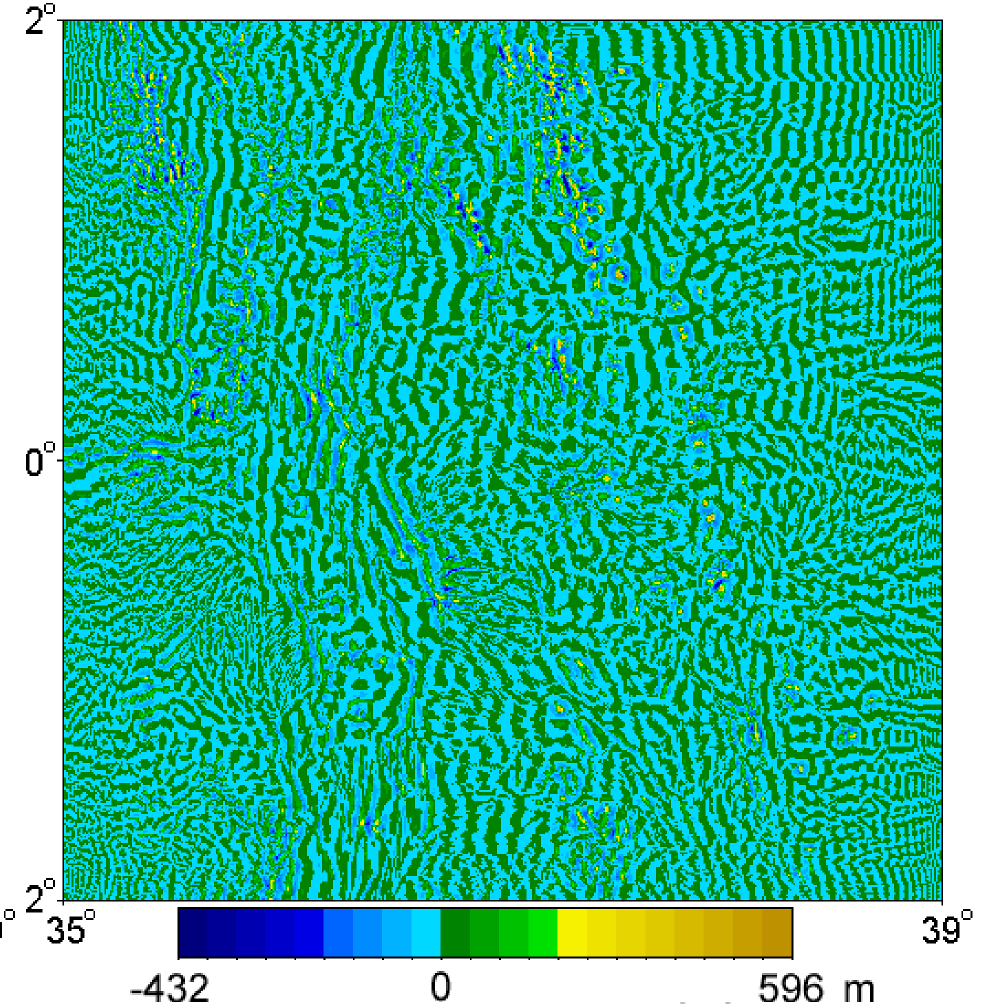
Residuals for the reconstructed DEMs

120 expansion coefficients

with the Fejér summation



without the Fejér summation

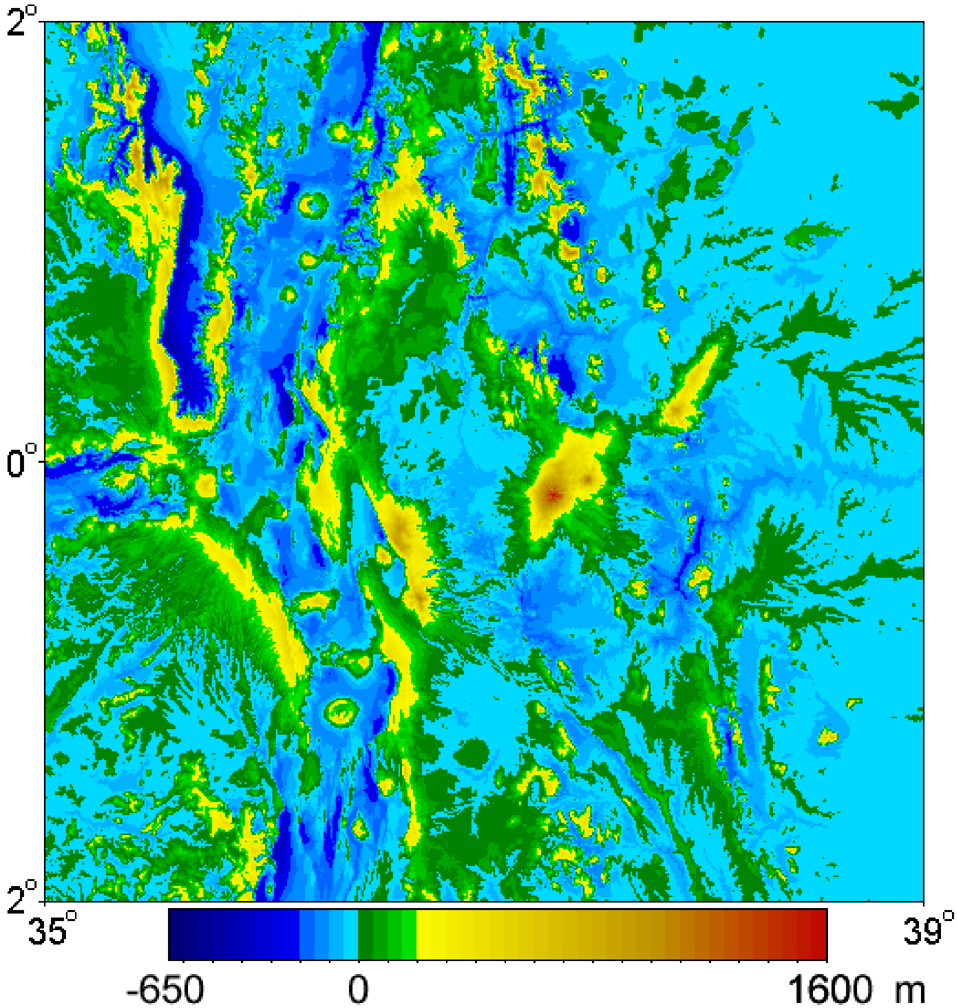


50 km

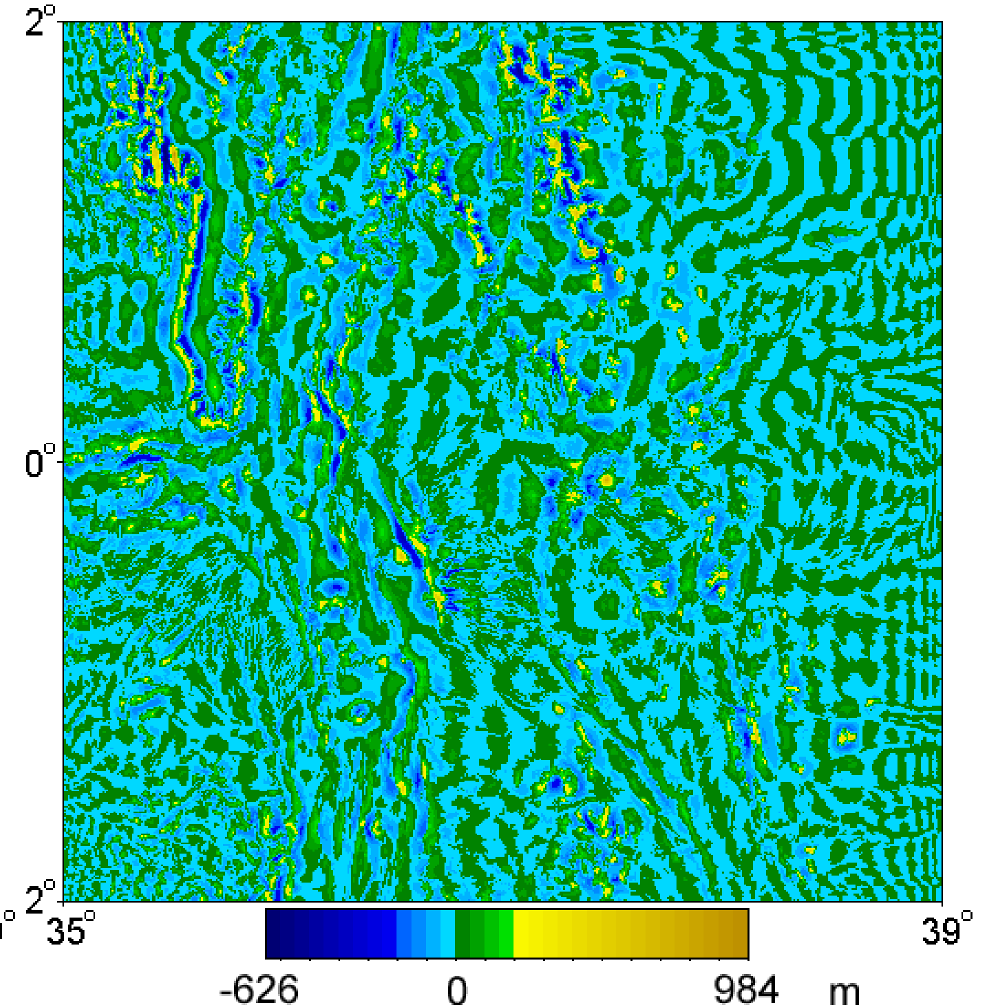
Residuals for the reconstructed DEMs

60 expansion coefficients

with the Fejér summation

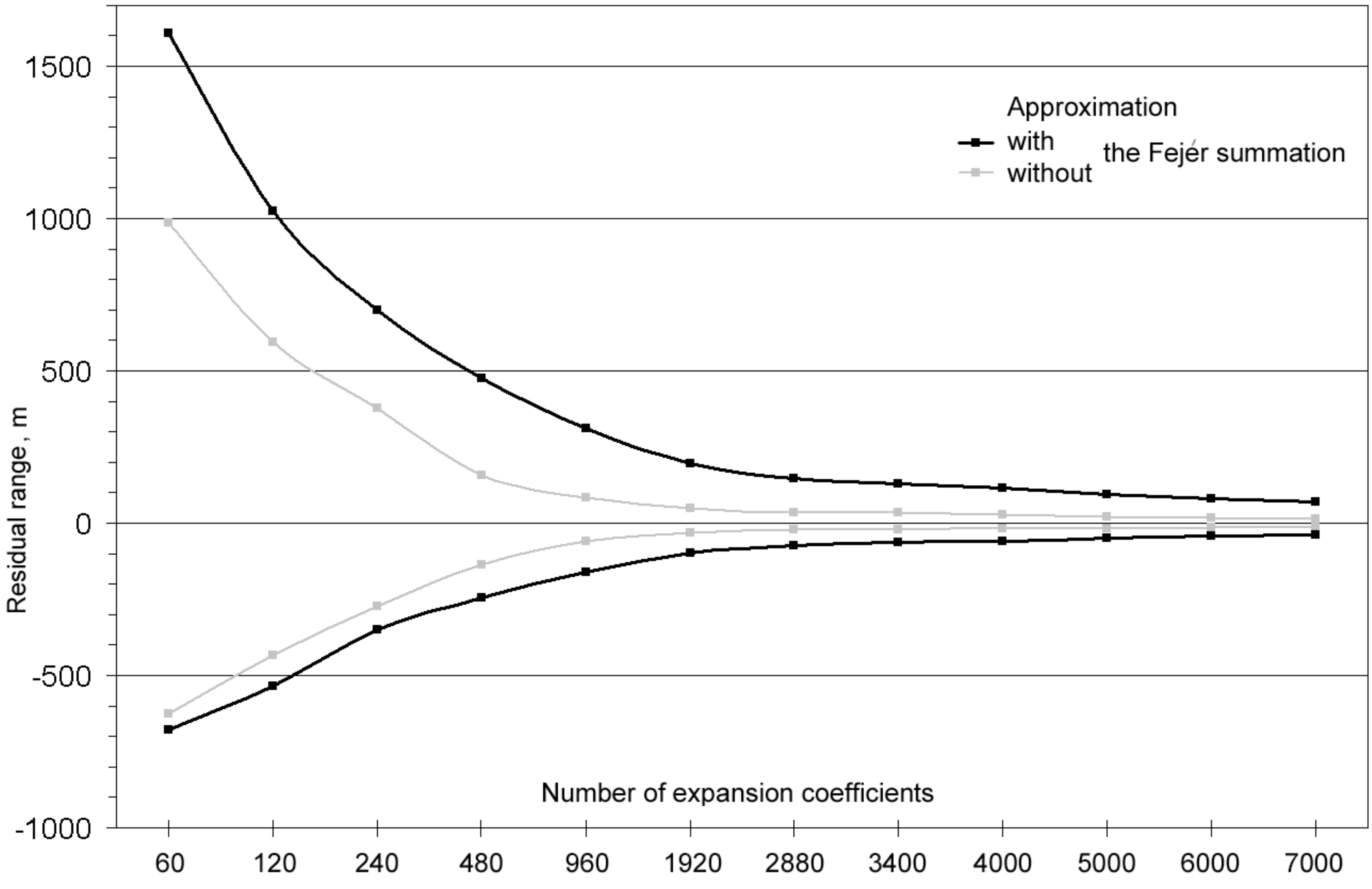


without the Fejér summation



50 km

Convergence rate of the approximation with and without the Fejér summation



Statistics for residual models

Statistics	Number of expansion coefficients											
	60	120	240	480	960	1920	2880	3400	4000	5000	6000	7000
	With the Fejér summation											
Minimum	–418.27	–287.46	–198.30	–146.08	–94.71	–57.51	–44.35	–39.97	–36.06	–31.54	–28.84	–28.20
Maximum	1163.49	669.29	427.11	265.23	158.20	90.72	64.26	55.43	48.49	41.61	37.50	36.09
Average	6.57	3.40	1.41	0.66	0.33	0.17	0.12	0.10	0.09	0.08	0.07	0.08
Standard deviation	131.69	84.94	53.50	32.94	20.19	12.07	8.83	7.75	6.82	5.72	4.97	4.47
	Without the Fejér summation											
Minimum	–369.48	–251.56	–178.15	–80.92	–37.86	–20.73	–13.06	–13.41	–7.32	–8.47	–7.96	–5.38
Maximum	679.34	322.94	219.00	70.32	40.10	18.14	13.65	12.98	11.26	8.48	8.15	7.12
Average	0.45	–0.66	–0.37	0.03	–0.01	0.00	0.01	0.02	0.01	–0.02	–0.01	0.01
Standard deviation	69.58	42.46	24.12	11.48	5.29	2.74	1.83	1.63	1.31	1.05	0.89	0.80

The sample size is 2209.

Range of the original function	206—4774 m (4568 m)
Range of residuals (with the Fejér summation)	–73—46 м (219 m → 4.8 %)
Range of residuals (without the Fejér summation)	–21—36 м (57 m → 1.3 %)
Average residual (with the Fejér summation)	±8.8 m
Average residual (without the Fejér summation)	±1.8 m
Accuracy of the original data (SRTM1)	±5.0—9.0 m

Conclusions on the convergence rate and ‘accuracy’ of the DEM approximation

DEM approximations with and without the Fejér summation are marked by a distinct monotonic convergence.

The convergence rate of the Fejér-free approximation is higher.

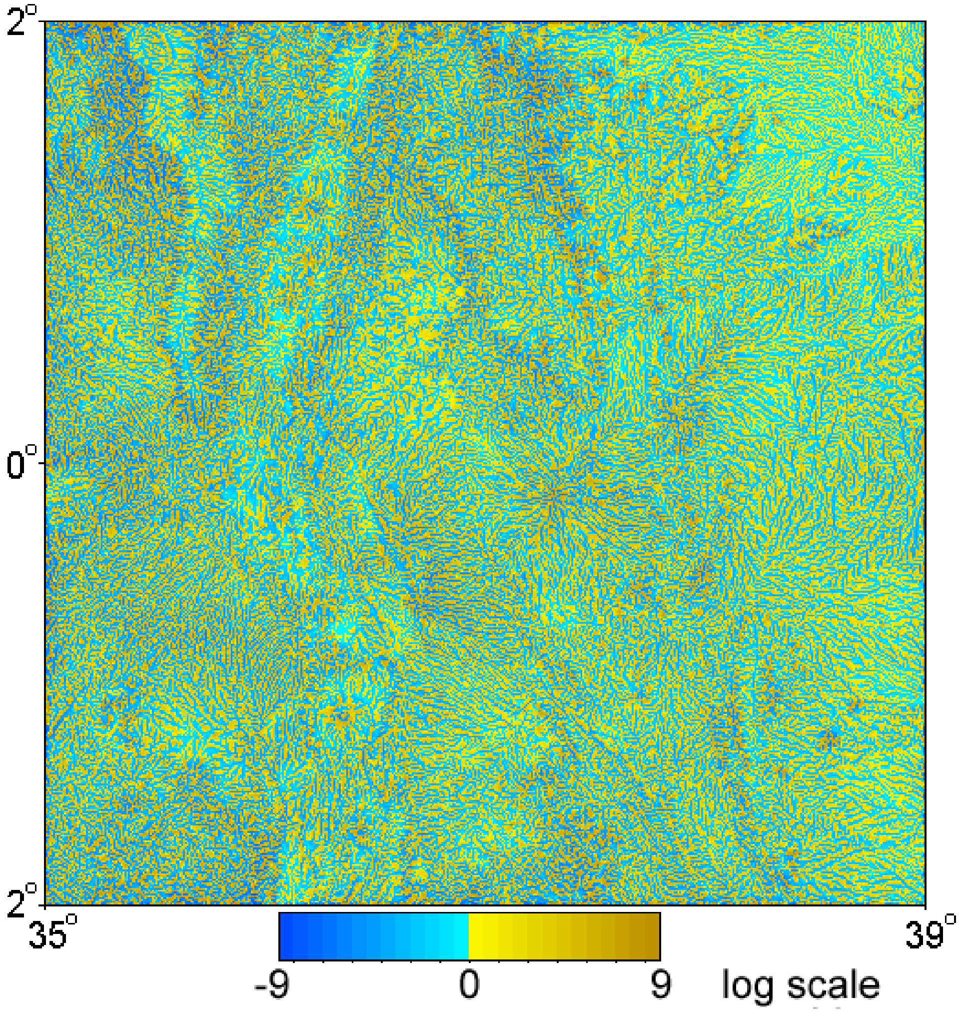
The residuals for the DEM reconstructed with, e.g., 2880 expansion coefficients without the Fejér summation does not exceed 1.3 % of the elevation range in the initial DEM.

Evaluation of the role of the Fejér summation in calculation of curvatures

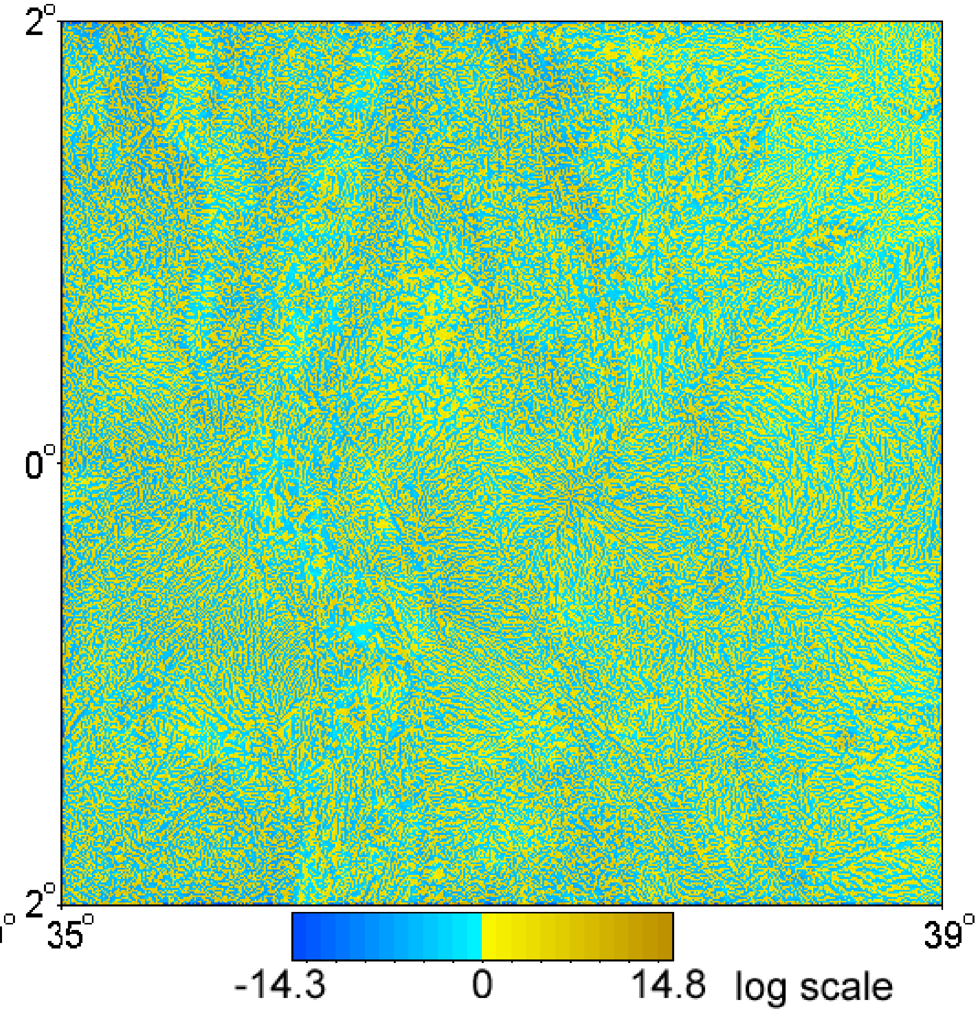
Horizontal curvature

2880 expansion coefficients

with the Fejér summation



without the Fejér summation

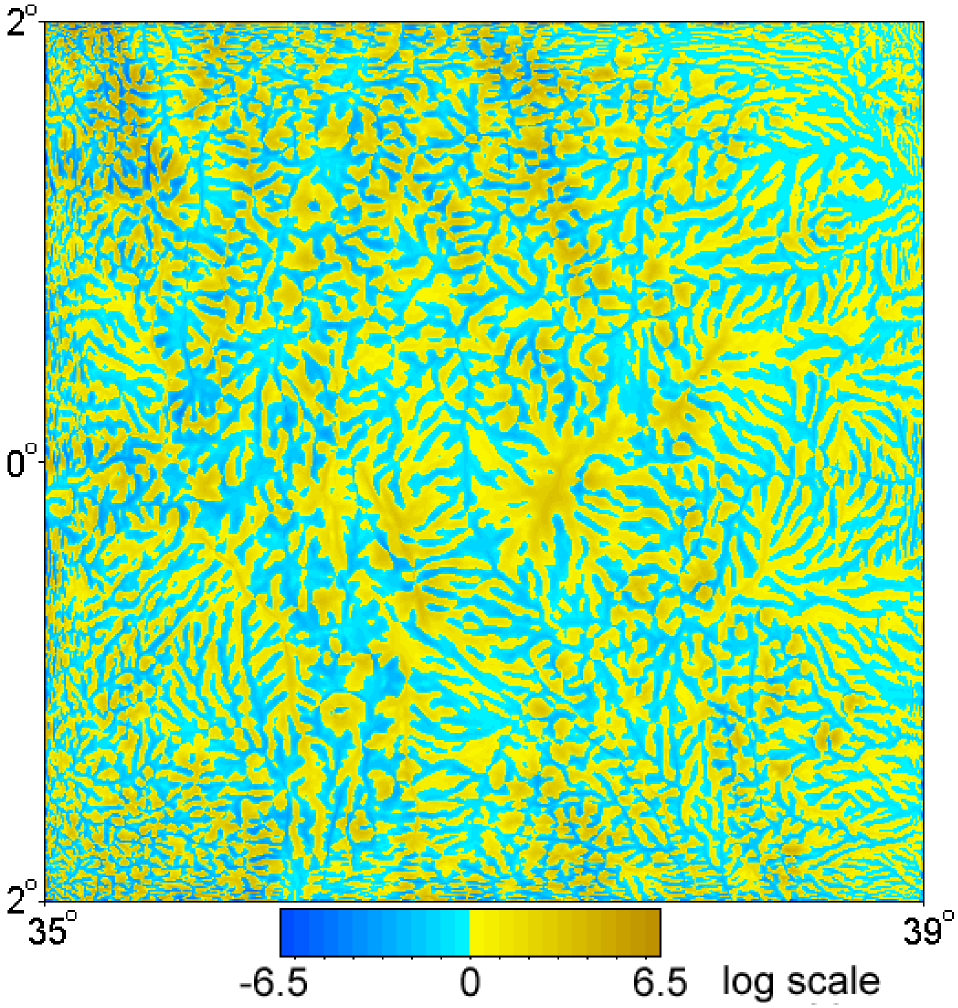


50 km

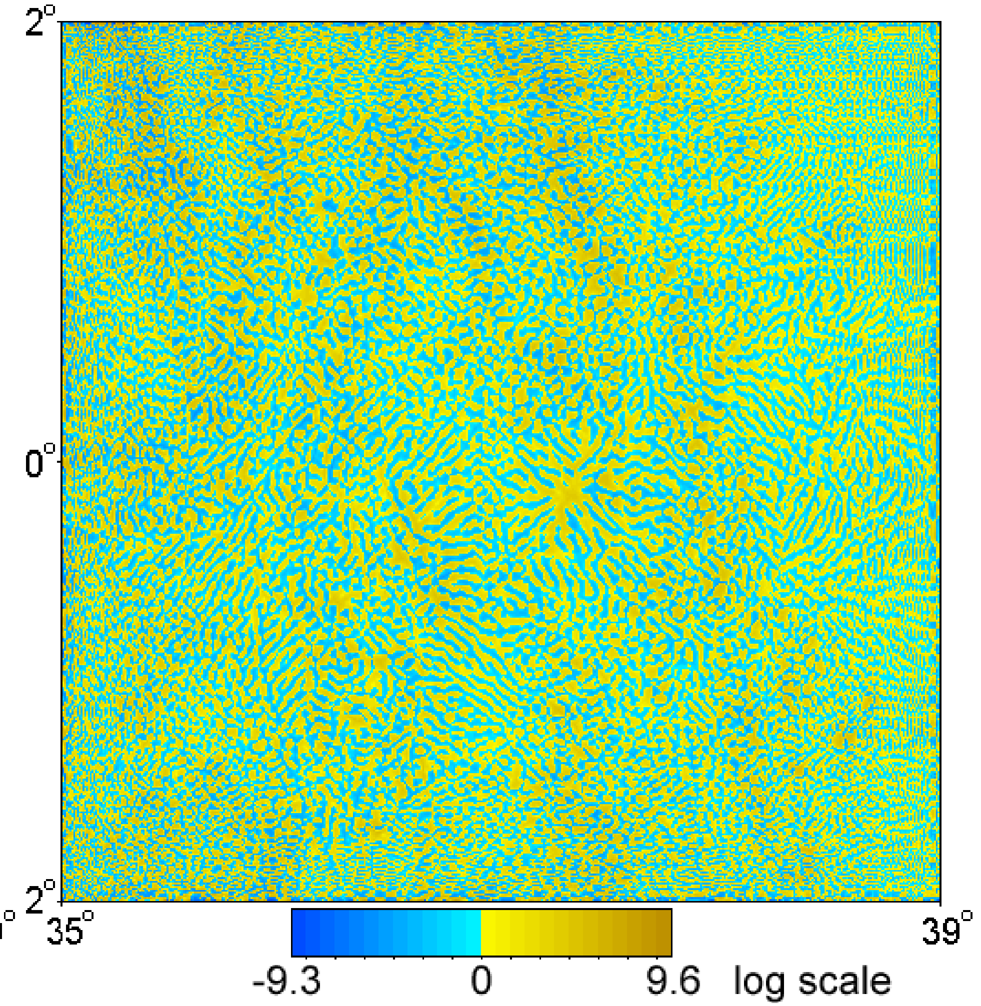
Horizontal curvature

240 expansion coefficients

with the Fejér summation



without the Fejér summation

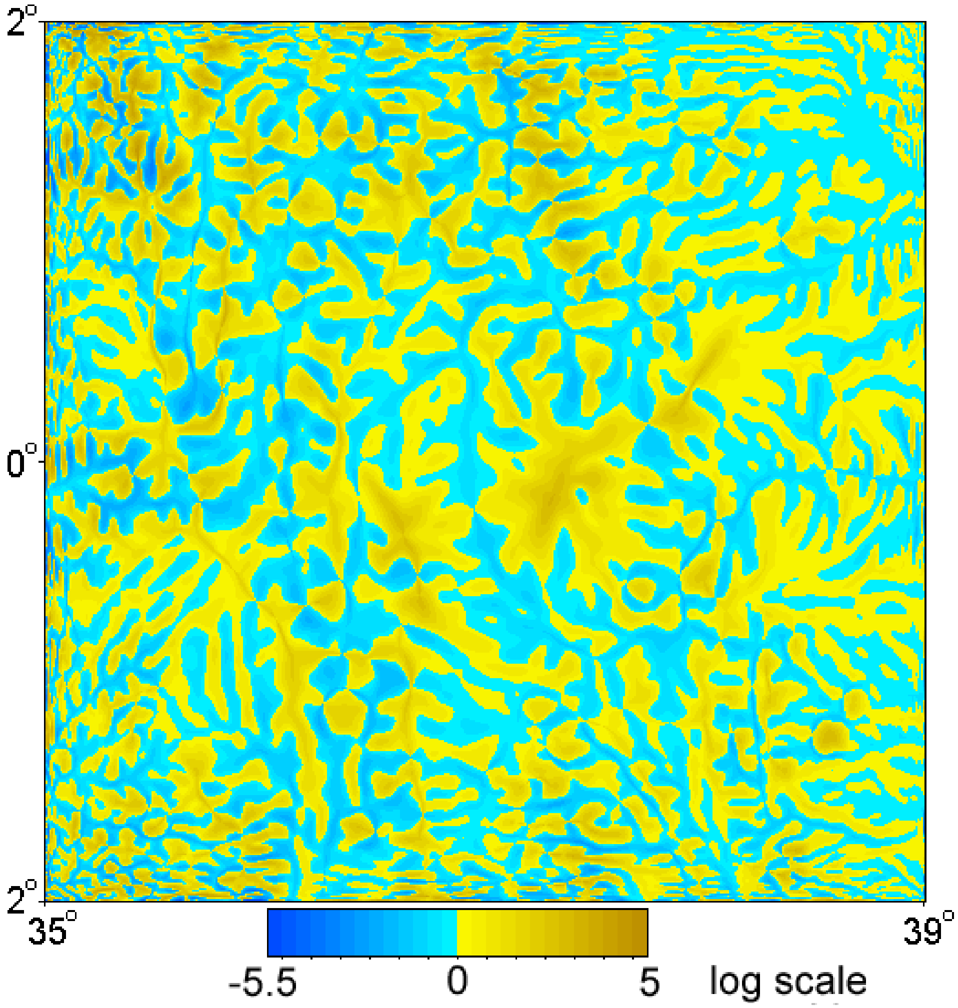


50 km

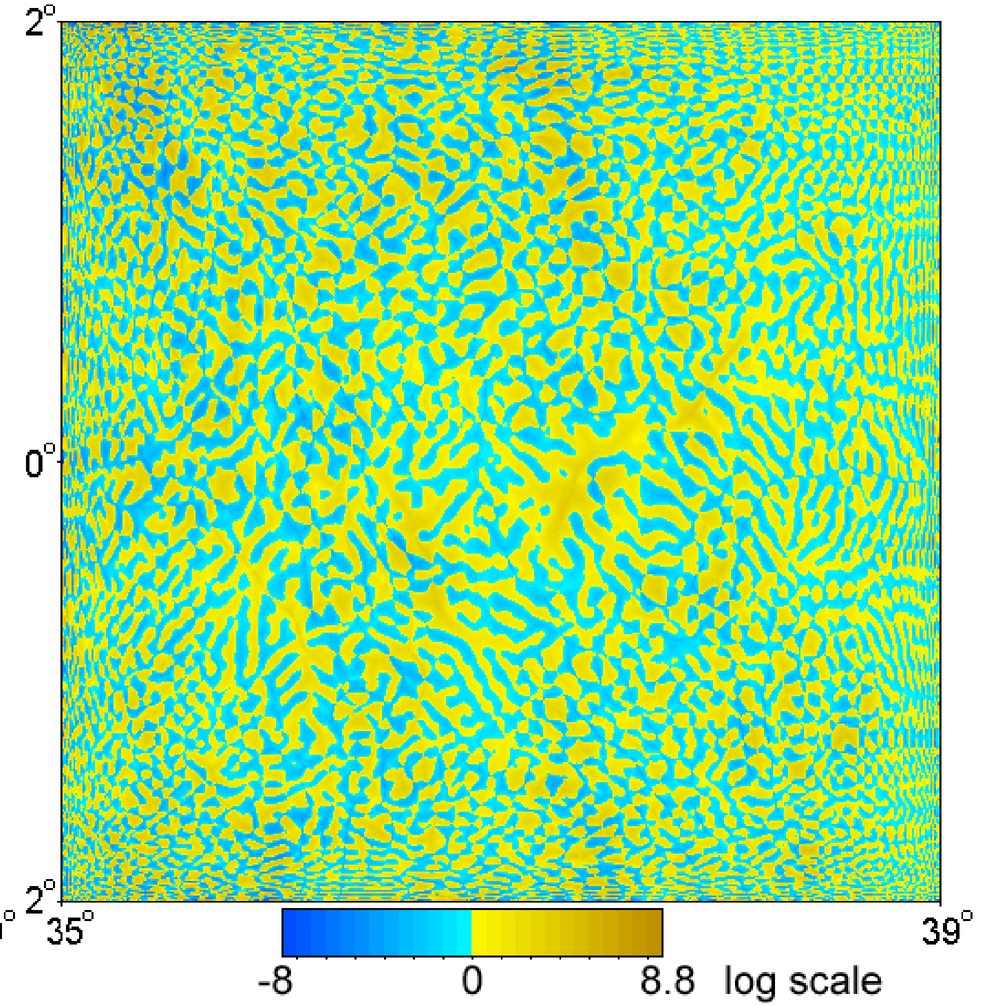
Horizontal curvature

120 expansion coefficients

with the Fejér summation



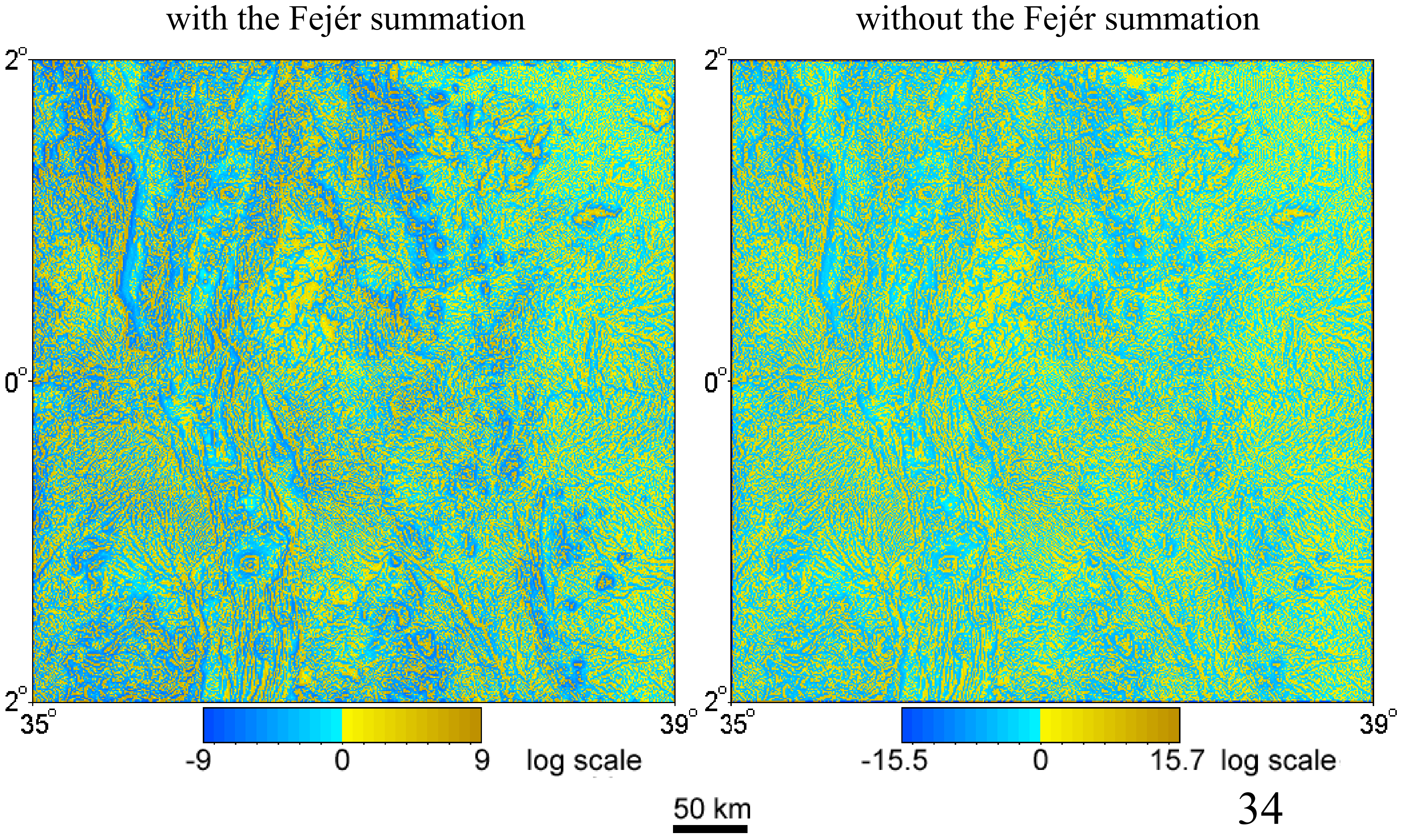
without the Fejér summation



50 km

Vertical curvature

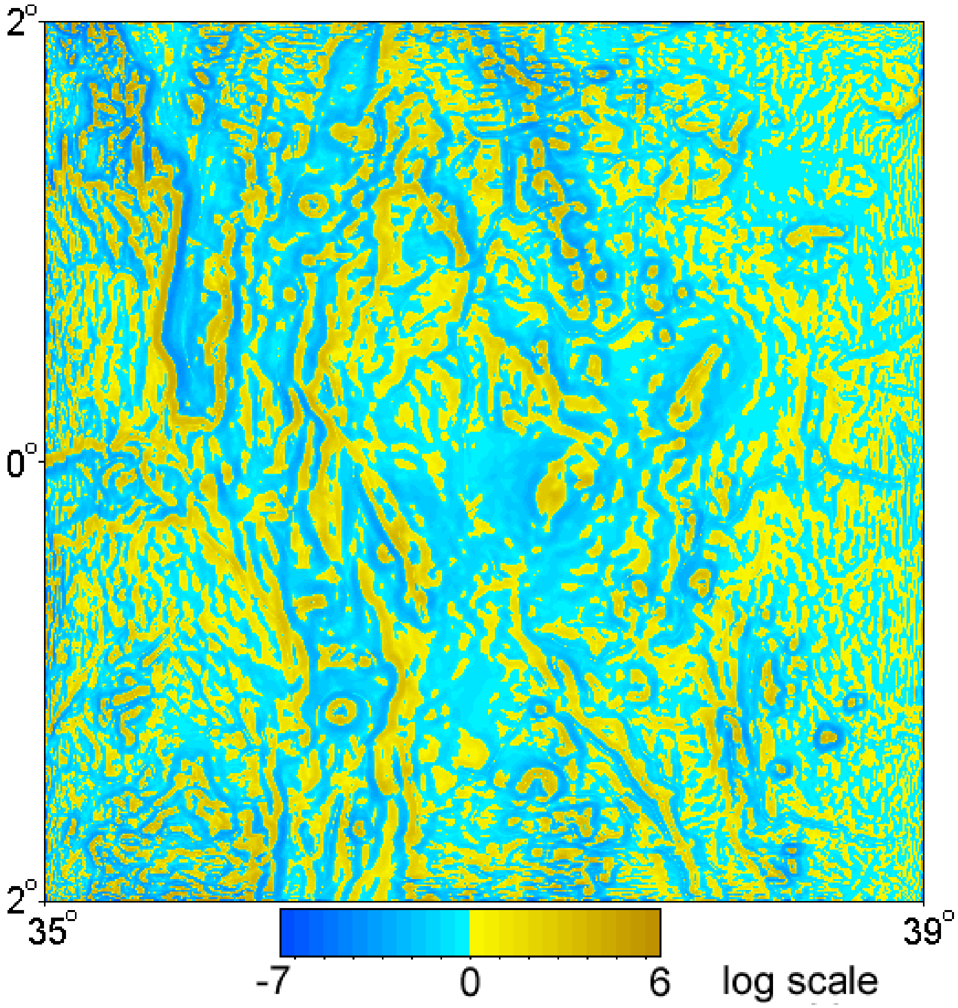
2880 expansion coefficients



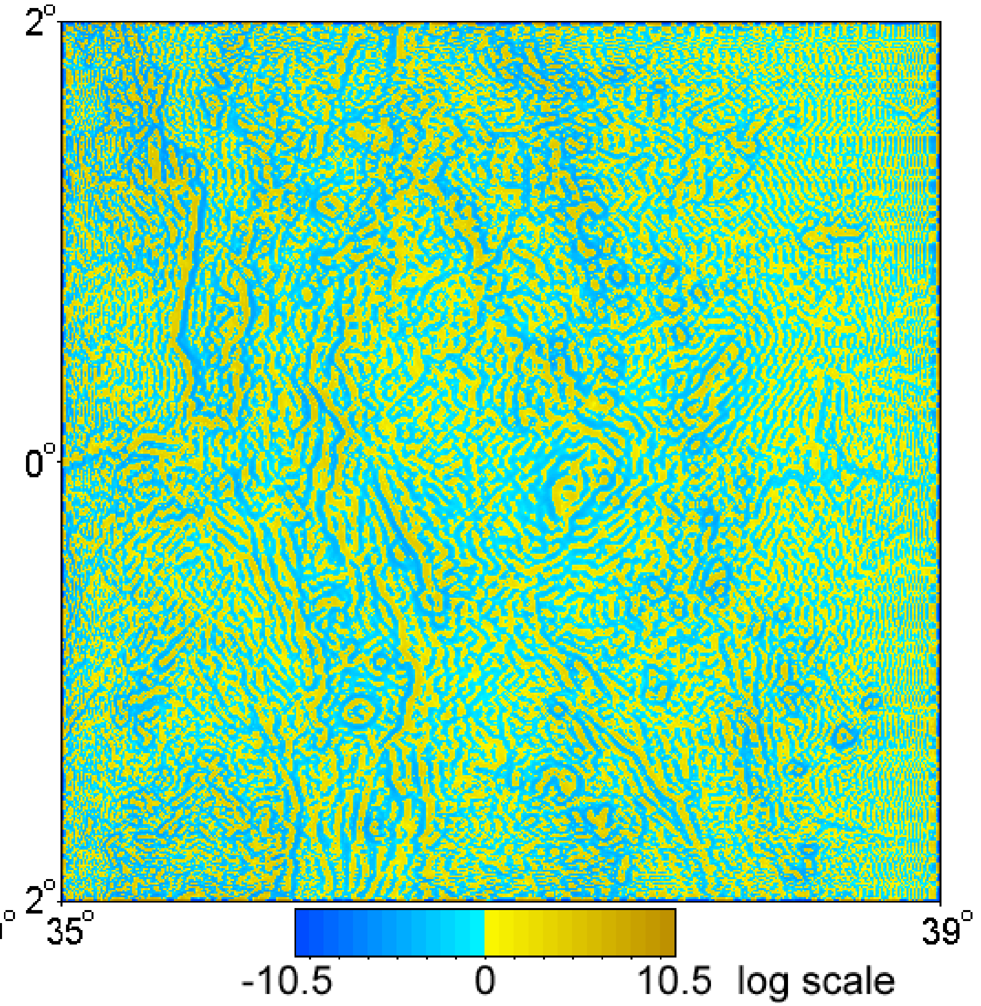
Vertical curvature

240 expansion coefficients

with the Fejér summation



without the Fejér summation

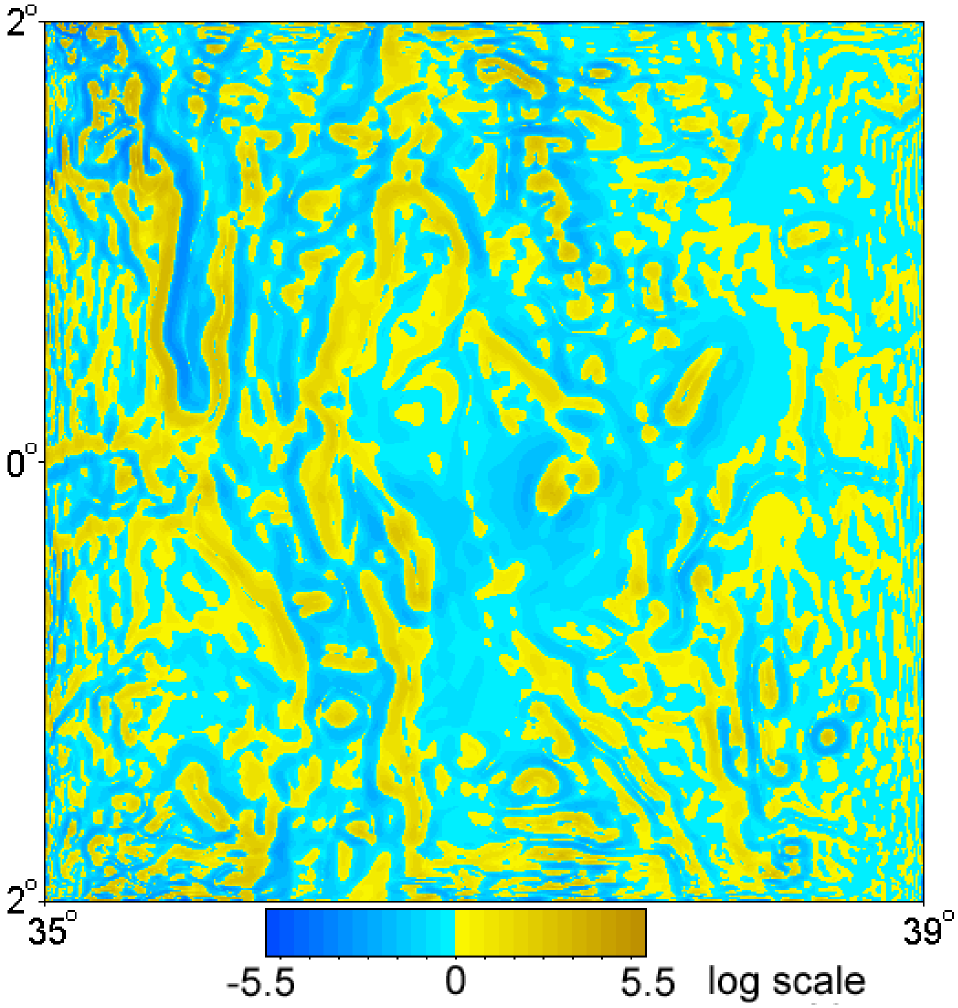


50 km

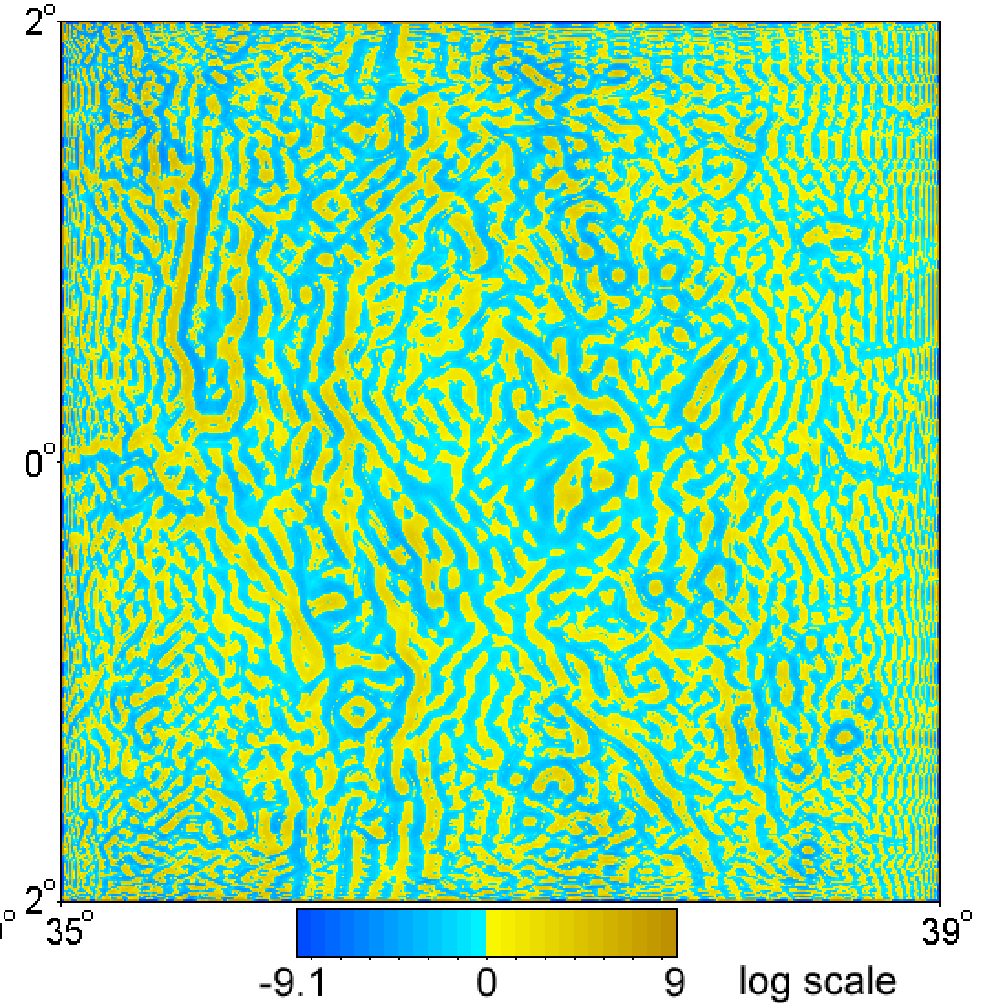
Vertical curvature

120 expansion coefficients

with the Fejér summation



without the Fejér summation



50 km

Conclusions on ‘accuracy’ of the derivation of morphometric variables

Pronounced oscillatory artifacts (traces of the Gibbs phenomenon) emerged on the maps of curvatures derived from the Fejér -free approximated DEMs.

The reconstructed DEMs include these oscillations but they are weakly expressed and cannot be seen on the elevation maps. Differentiation amplifies their manifestation on the curvature maps.

Such artifacts do not appear on the curvature maps if the DEM approximation included the Fejér summation.

Thus the Fejér summation, suppressing the Gibbs phenomenon, is the necessary stage of the terrain modeling.

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