Регрессионные модели с ограничениями и регуляризацией при больших наборах регрессоров. Задача восстановления состава инвестиционного портфеля.

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Выпускная квалификационная работа на соискание степени бакалавра.

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Задача восстановления состава инвестиционного портфеля

Активы — ценные бумаги (акции, облигации, векселя и пр.), $i = 1, \ldots, n$, торгуемые на биржевом рынке и характеризумые в каждый момент времени своей биржевой ценой.

Доходность на некотором периоде владения t:

$$x_{t,i} = \frac{z_{t,i}^{end} - z_{t,i}^{beg}}{z_{t,i}^{beg}}, \text{относительная скорость изменения цены}$$

Активов на биржевом рынке n — тысячи.

Инвестиционный портфель — долевое распределение капитала некоторой компании среди части ценных бумаг $\hat{n} \ll n$ (как правило, глубокая тайна)

$\boldsymbol{\beta} = (\beta_1, \cdots \beta_n) \in \mathbb{R}^n$; $\quad \beta_i \geqslant 0, \quad \sum_{i=1}^{n} \beta_i = 1$

Задача — угадать распределение капитала по совокупности наблюдений за доходностями активов и портфеля в целом.

Returns Based Style Analysis (Уильям Шарп)

Returns Based Style Analysis по Шарпу

Линейная модель Шарпа: $y \cong \sum_{i=1}^{n} \beta_i x_i$

Совокупность наблюдений: $(y_t, x_{t,i}, i = 1, \dots, n), \quad t = 1, \dots T$
— доходности портфеля, активов.
Задача квадратичного программирования по Шарпу (RBSA)
в преположении $n < T$:

$$\begin{cases} (\hat{\beta}_1, \dots, \hat{\beta}_n) = \arg\min \sum_{t=1}^{T} (y_t - \sum_{i=1}^{n} \beta_i x_{t,i})^2 \\ \sum_{i=1}^{n} \beta_i = 1 \\ \beta_i \geqslant 0, i = 1, \dots, n \end{cases} \qquad (1)$$

Однако в реальности $n \gg T$ (тысячи против сотен).
В этой ситуации невозможно найти малое подмножество активов
$\hat{n} \ll n$, в действительности составляющих портфель.
Необходимо учитывать априорные предположения о том, как
администратор мог бы составить инвестиционный портфель.

Factor Search

В магистэрских диссертациях Ильи Пугача и Алексея Морозова, эта проблема была названа Factor Search: $|\hat{\mathbb{I}}| = \hat{n} \ll n = |\mathbb{I}|$
Основная идея — при оценивании долевого распределения капитала учитывать статистическую информацию о доходности исследумеого портфеля и доходностях всех биржевых активов.

$$\begin{cases} \boldsymbol{\beta}^T \boldsymbol{\Sigma} \boldsymbol{\beta} - \mu \bar{\mathbf{x}}^T \boldsymbol{\beta} + c \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta}\right)^T \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta}\right) \to \min(\boldsymbol{\beta}) \\ \mathbf{1}^T \boldsymbol{\beta} = 1; \qquad \boldsymbol{\beta} \geqslant \mathbf{0} \end{cases} \tag{2}$$

$$\mathbf{y} = (y_1, \cdots y_T) \in \mathbb{R}^T \qquad \mathbf{X} = (\mathbf{x}_1, \cdots \mathbf{x}_T) \in \mathbb{R}^{n \times T}$$

Здесь $\boldsymbol{\Sigma}$ $(n \times n)$ и $\bar{\mathbf{x}}$ (n) — результат технического анализа; ковариационная матрица и вектор математических ожиданий колебаний доходностей активов в прошлом;
Идея данной работы — дополнить технический анализ прошлых доходностей активов экспертным мнением о предпочтительном скрытом составе портфеля.

Математическая формализация экспертных суждений об ожидаемом скрытом составе портфеля

Предлагается — дополнить оценки коварицонной матрицы и среднего вектора доходностей активов $(\mathbf{\Sigma}, \bar{\mathbf{x}})$ матрицей и вектором той же стуктуры $(\mathbf{B}; \mathbf{z})$, формализующих мнение эксперта о разумном составе портфеля.
Пусть эксперт выразил свое мнение относительно:

Волатильности доходностей активов

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \in \mathbb{R}^n$$

Несовместности пар активов

$$\mathbf{R} = \begin{pmatrix} 1 & r_{12} & \dots & r_{1n} \\ & 1 & \dots & r_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

Предпочтительном составе портфеля

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

Агрегированная матрица: $\mathbf{B} = [\mathrm{Diag}(\mathbf{d})]^T \mathbf{R} [\mathrm{Diag}(\mathbf{d})]$

Математическая формализация экспертных суждений об ожидаемом скрытом составе портфеля

Агрегированная матрица: $\mathbf{B} = [\text{Diag}(\mathbf{d})]^T \mathbf{R} [\text{Diag}(\mathbf{d})]$
Структура этой матрицы аналогична структуре коварицонной матрицы:

$$\mathbf{B} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} = \begin{pmatrix} d_1 d_1 r_{11} & \dots & d_1 d_n r_{1n} \\ \vdots & \ddots & \vdots \\ d_n d_1 r_{n1} & \dots & d_n d_n r_{nn} \end{pmatrix} \quad (3)$$

$$\text{Сравним:} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 \sigma_1 \rho_{11} & \dots & \sigma_1 \sigma_n \rho_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_n \sigma_1 \rho_{n1} & \dots & \sigma_n \sigma_n \rho_{nn} \end{pmatrix} \quad (4)$$

Математическая формализация экспертных суждений об ожидаемом скрытом составе портфеля

Пусть эскперт выразил свое мнение относительно:

Волатильности доходностей активов

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \in \mathbb{R}^n$$

Несовместности пар активов

$$\mathbf{R} = \begin{pmatrix} 1 & r_{12} & \dots & r_{1n} \\ & 1 & \dots & r_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

Предпочтительном составе портфеля

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

Агрегированная матрица: $\mathbf{B} = [\mathrm{Diag}(\mathbf{d})]^T \mathbf{R} [\mathrm{Diag}(\mathbf{d})]$

Фактически эти матрицы определяют суждение эксперта о скрытом портфеле: $0 \leqslant \mu < \infty$ степень недоверия к доходностям активов

$$\begin{cases} \boldsymbol{\beta}^T \mathbf{B} \boldsymbol{\beta} - \mu \mathbf{z}^T \boldsymbol{\beta} \to \min(\boldsymbol{\beta}) \\ \mathbf{1}^T \boldsymbol{\beta} = 1; \qquad \boldsymbol{\beta} \geqslant \mathbf{0} \end{cases} \qquad (5)$$

Пример выбора матрицы B

Всего биржевых активов $\mathbb{I} = \{i = 1, \dots, n\}$, $n = 650$

Предположим, что эксперт разбил это множество на $m = 15$ групп, объединяющих активы, сходные по некоторым социально-деловым представлениям: $\mathbb{I} = \mathbb{I}_1 \cup \dots \cup \mathbb{I}_m$, $\mathbb{I}_k \cap \mathbb{I}_l = \varnothing, k \neq l$.

Эксперт предполагает, что в портфель вряд ли входят более одного представителя каждой группы.

В терминах матрицы **R** несовместимости пар активов это означает, что штраф $r_{i,j}$ для i, j из одной группы должен быть очень большим, например 0.9999, поскольку $r_{i,i} = 1$

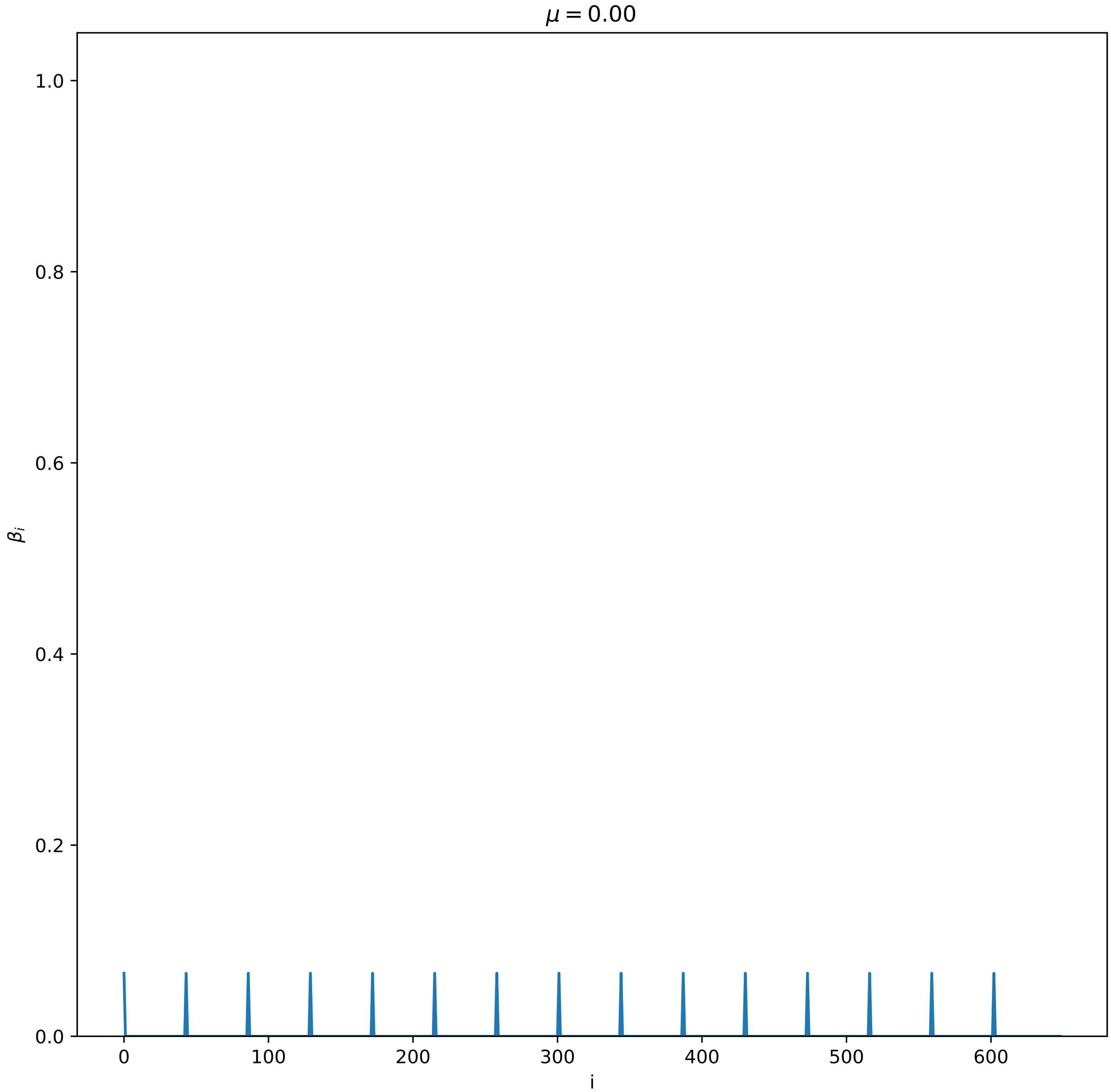
Соответственно, диагональные (a) и недиагональные (b) блоки **R** имеют вид:

1	0.99	$\cdots$	0.99
0.99	1	$\cdots$	0.99
$\vdots$	$\vdots$	$\ddots$	$\vdots$
0.99	0.99	$\cdots$	1

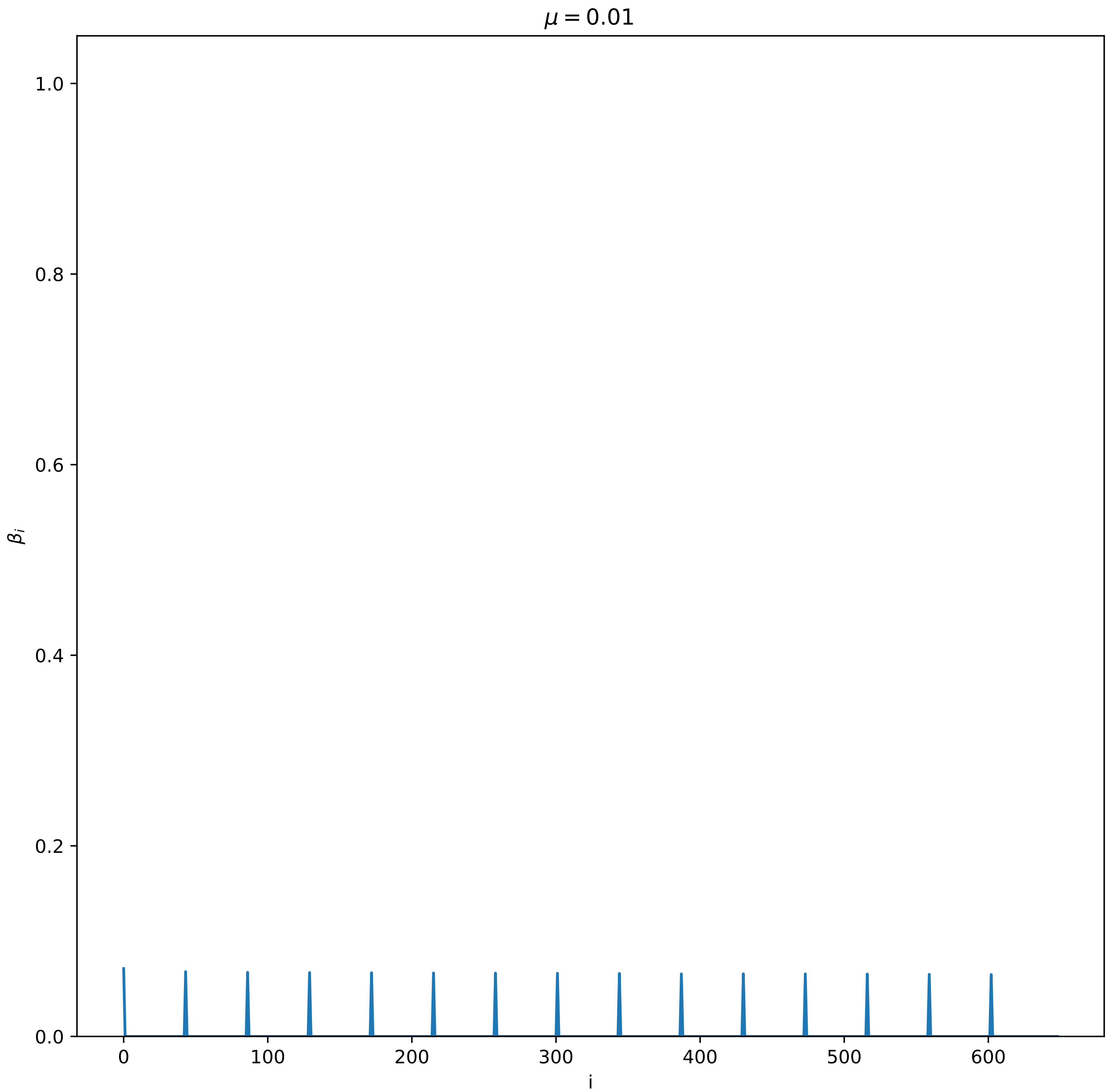
(a)

0	0	$\cdots$	0
0	0	$\cdots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	$\cdots$	0

(б)



$\mu = 0.00$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.01$
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0.6
0.4
0.2
0.0
β_i
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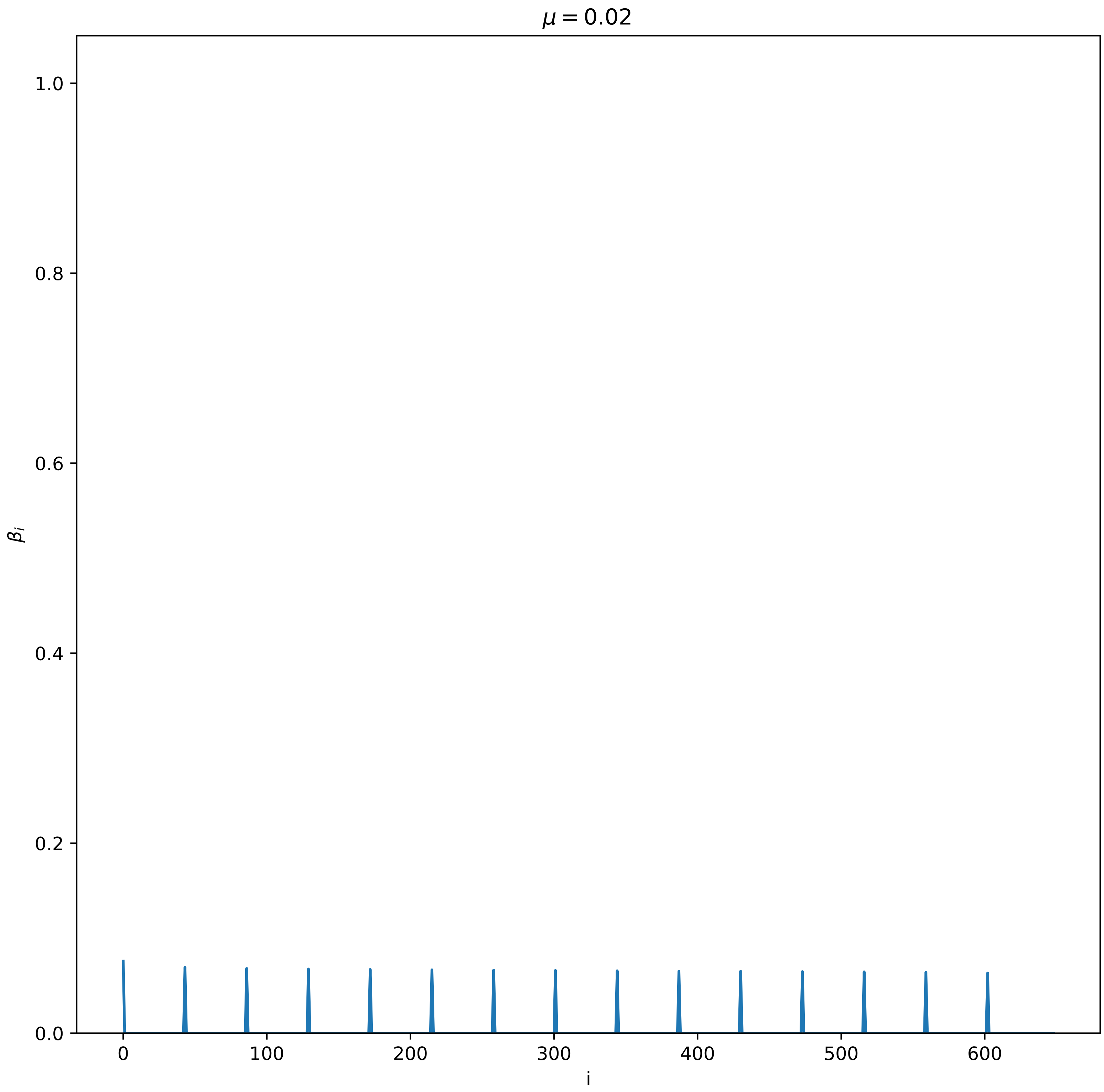
100
200
300
400
500
600
i



$\mu = 0.02$
β_i
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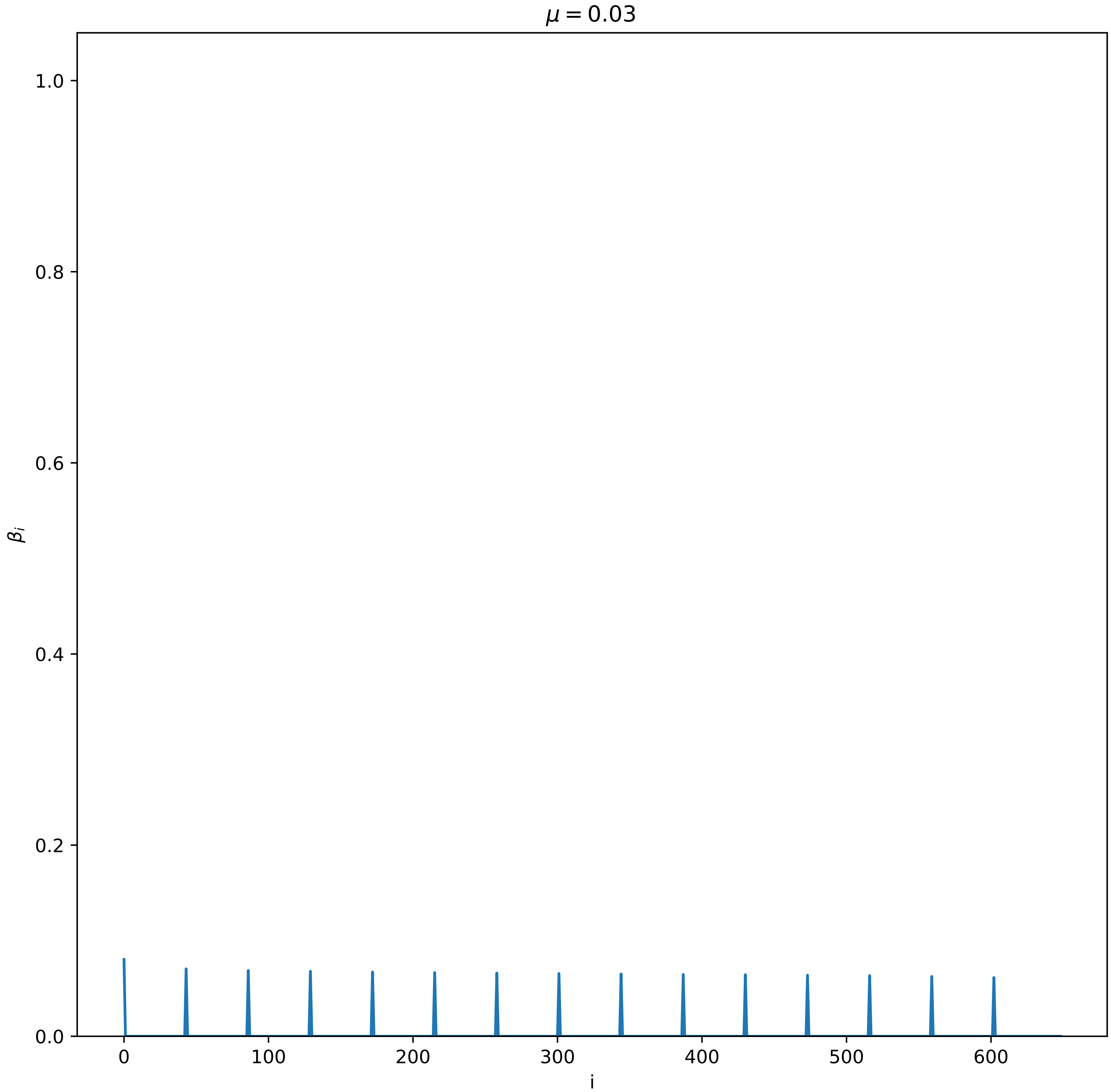
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



$\mu = 0.03$
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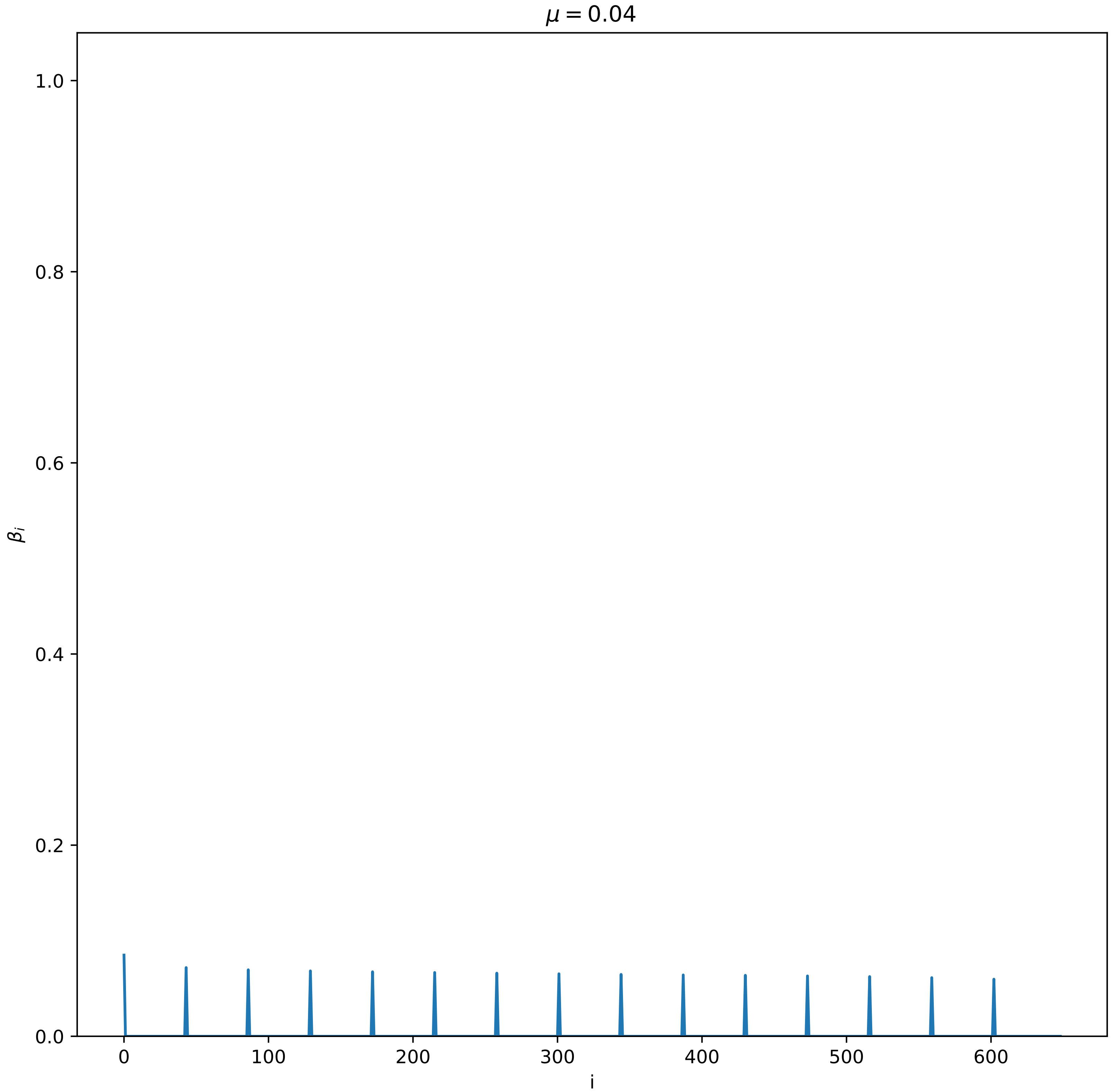
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



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0.4
0.2
0.0
β_i
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100
200
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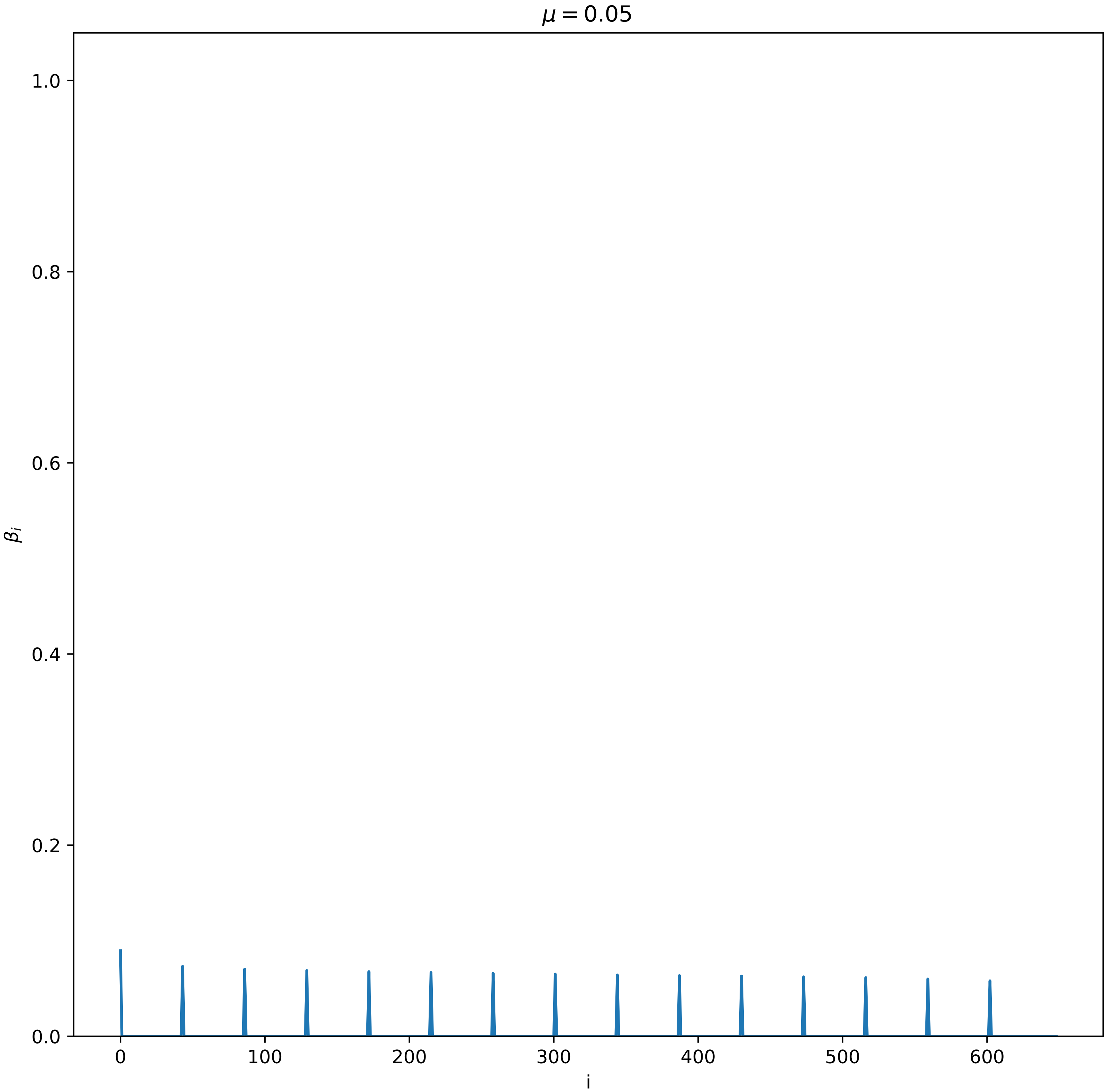
400
500
600
i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0.0
β_i
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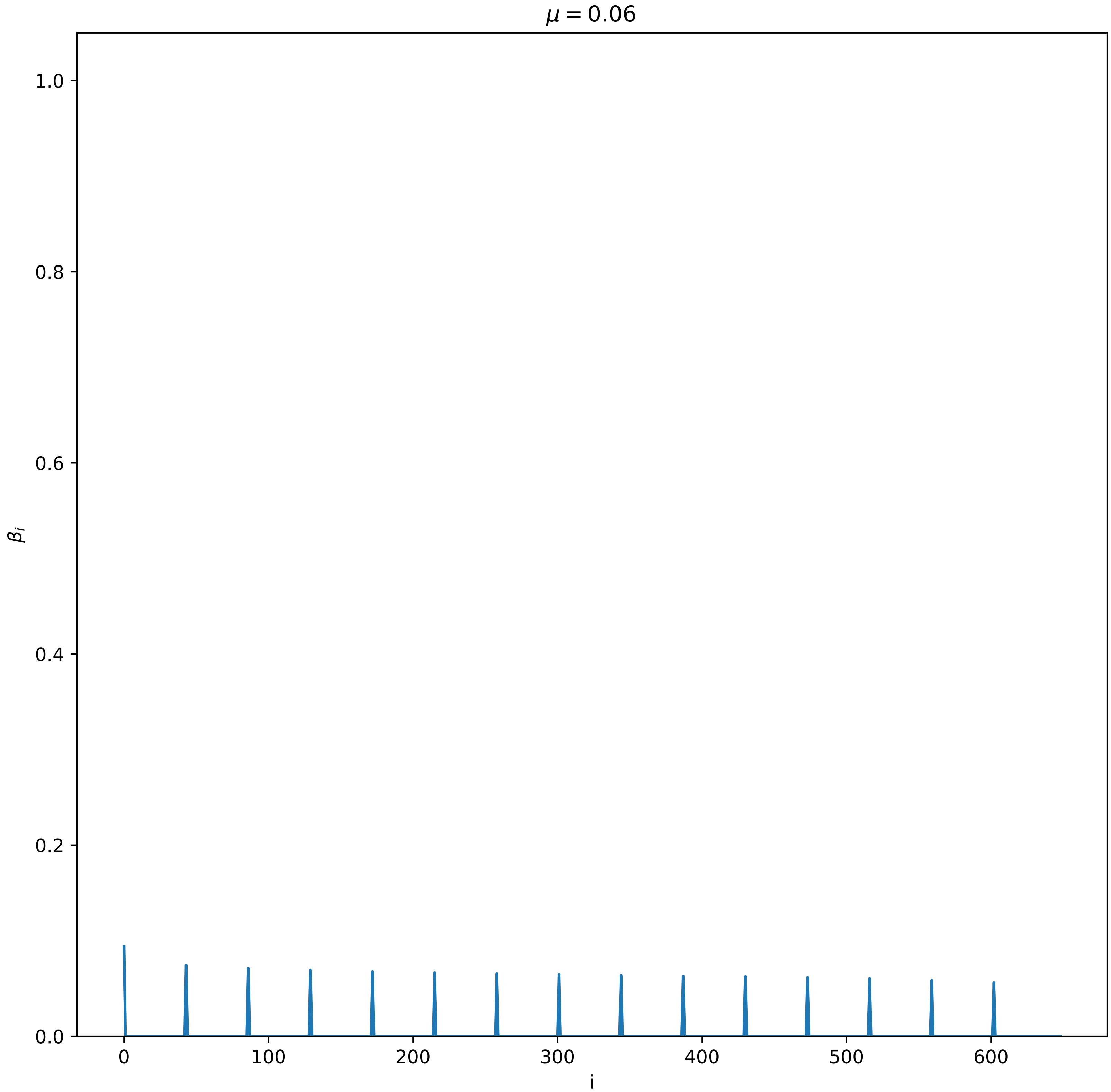
200
300
400
500
600
i



$\mu = 0.06$
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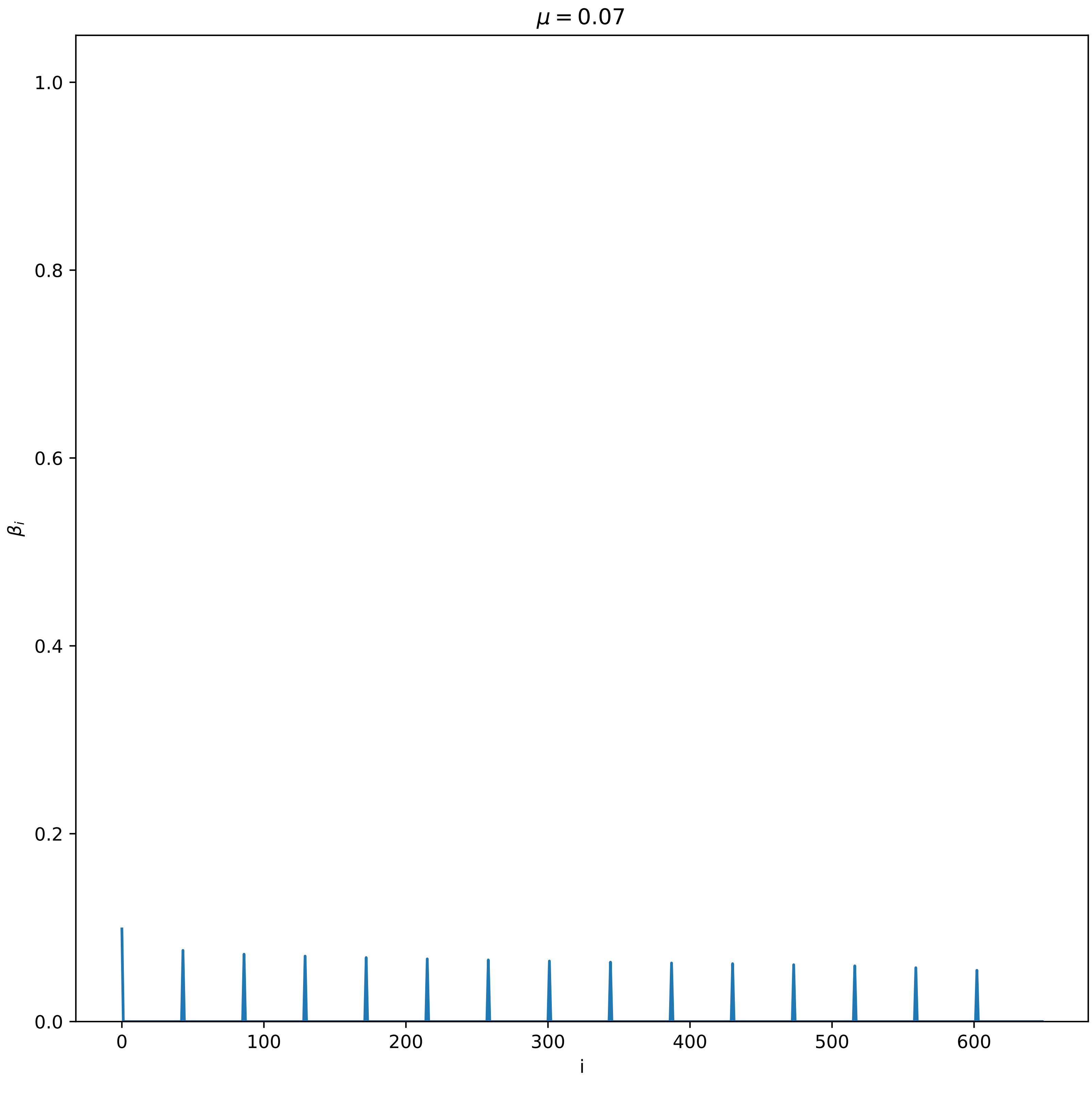
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.07$
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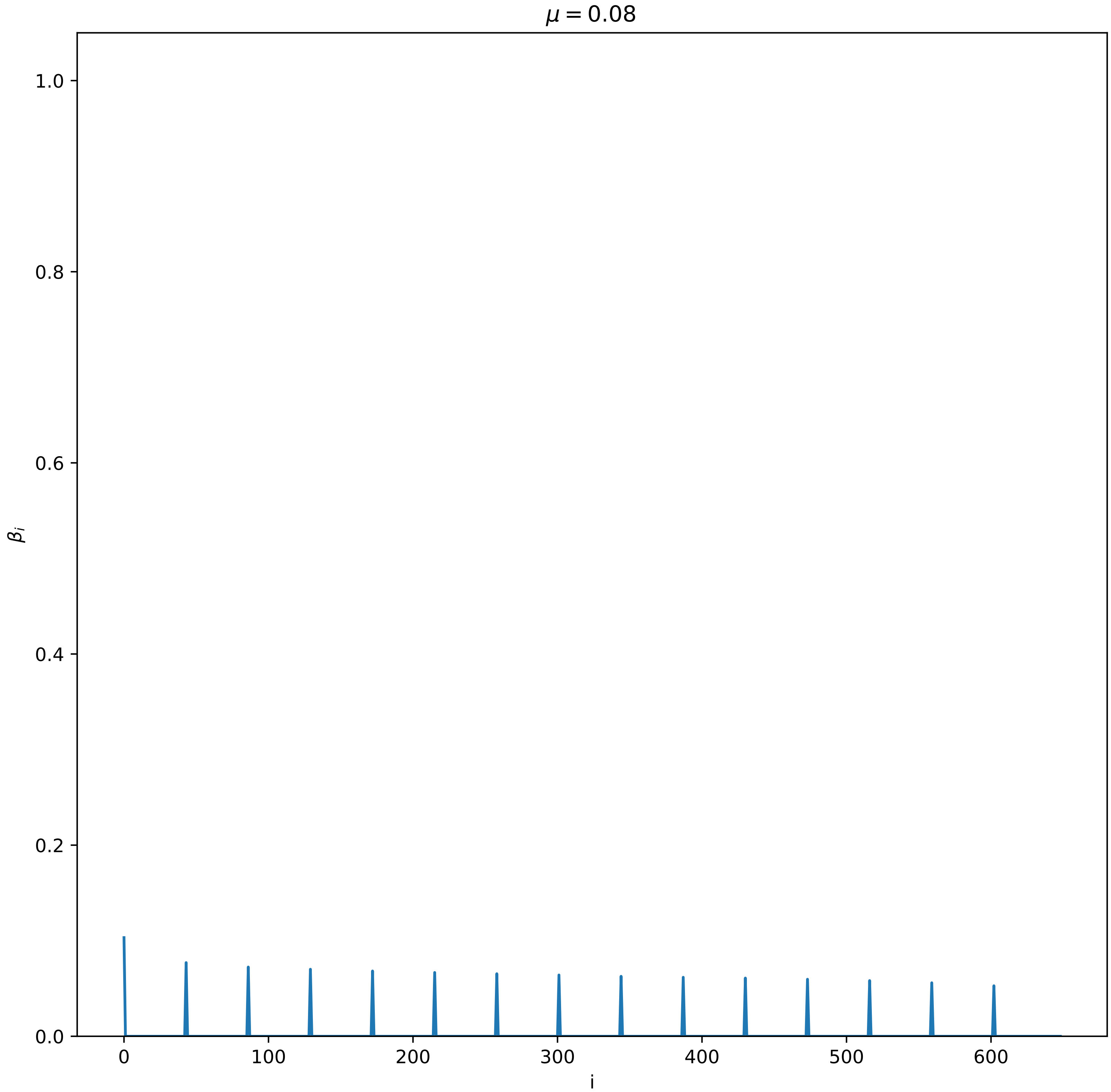
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.08$
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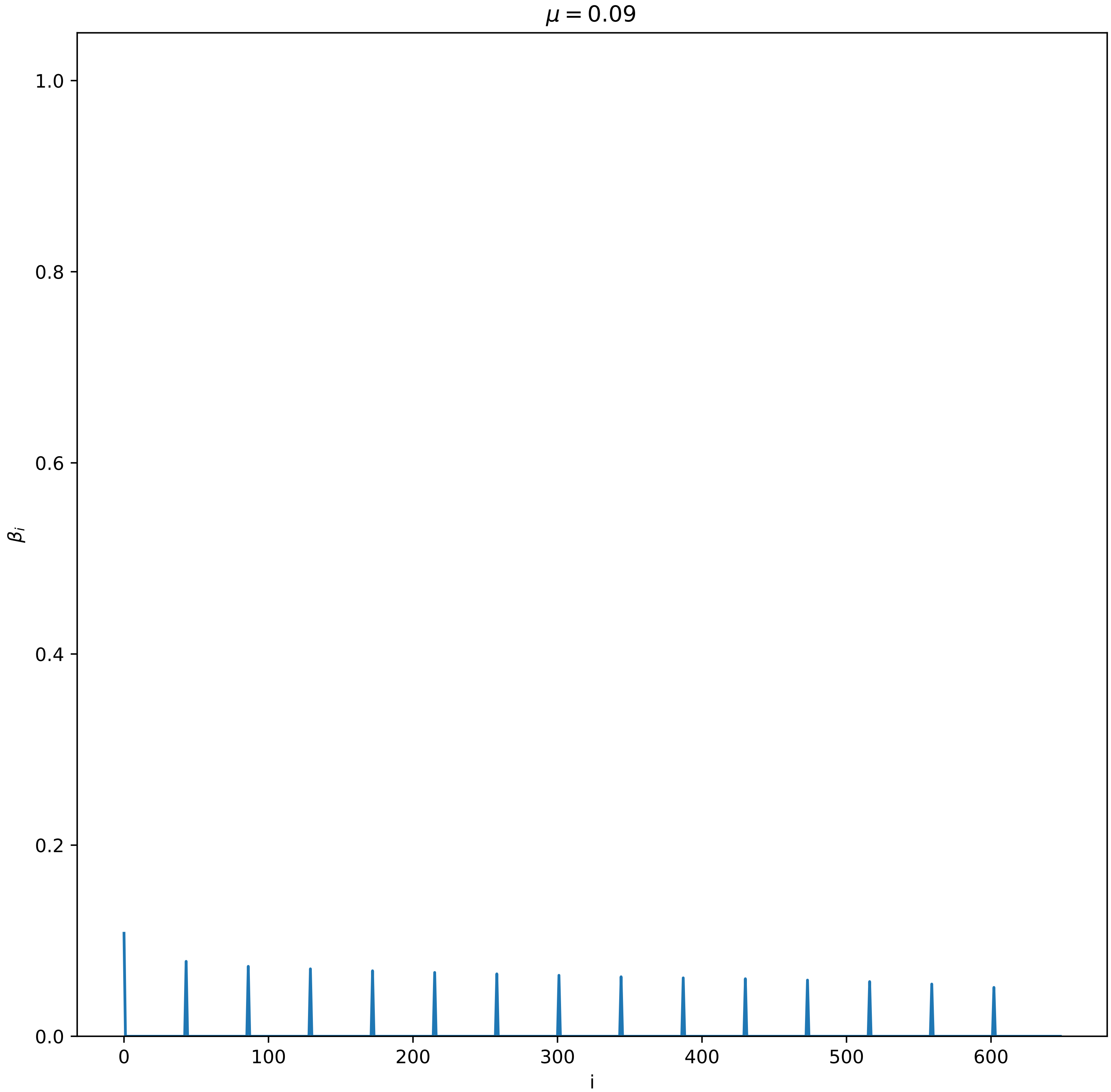
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.09$
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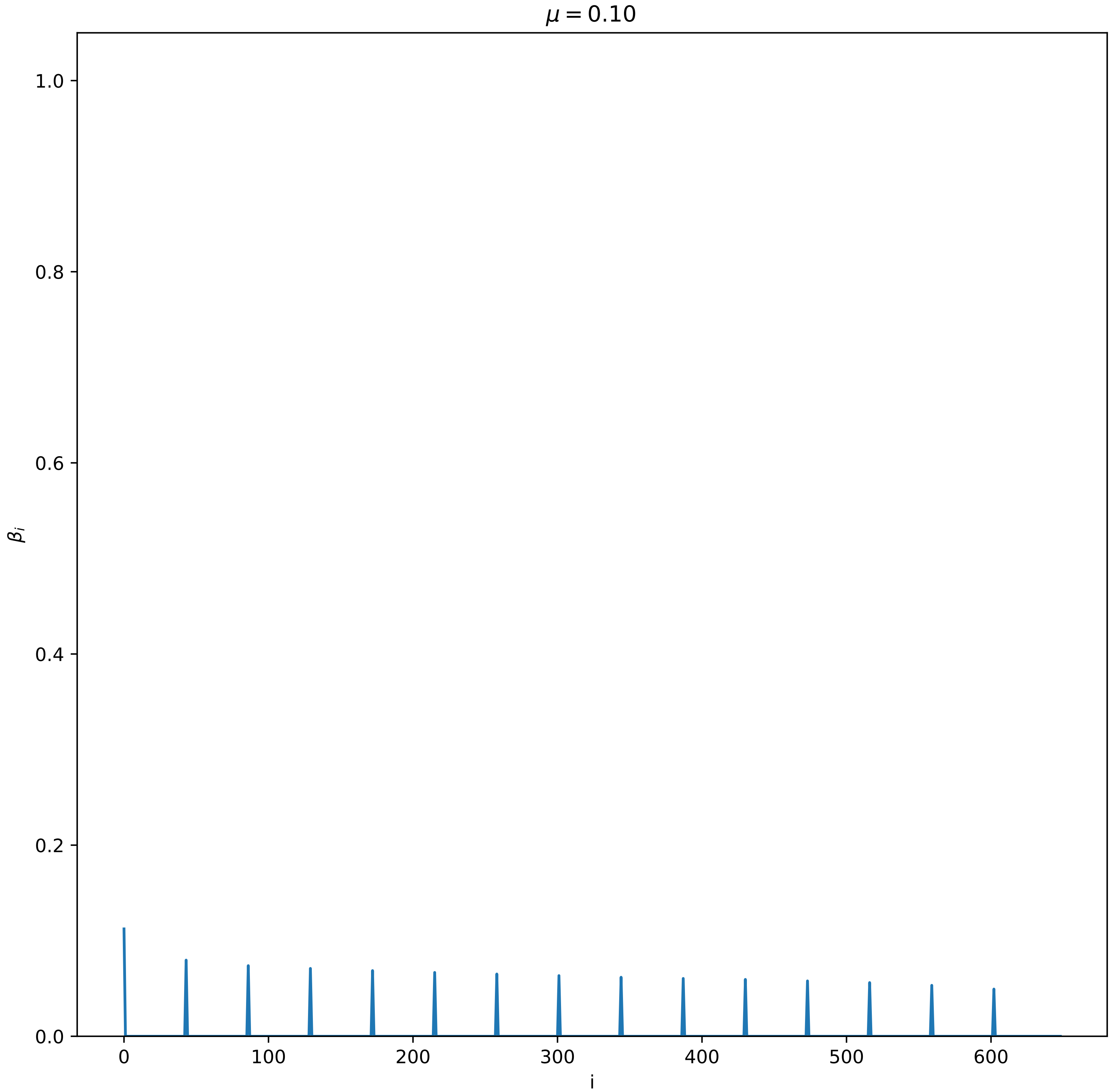
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.10$
β_i
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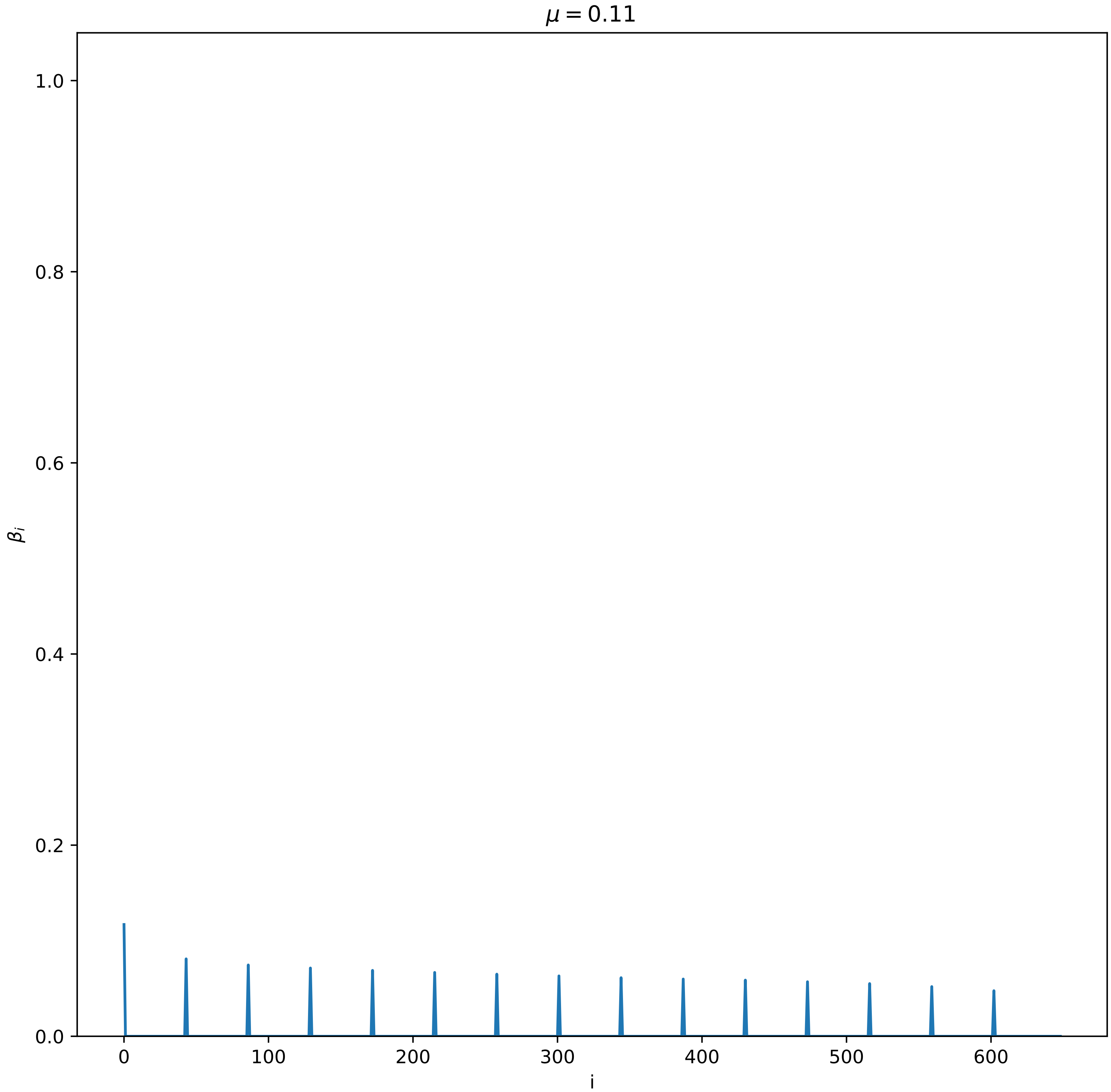
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



$\mu = 0.11$
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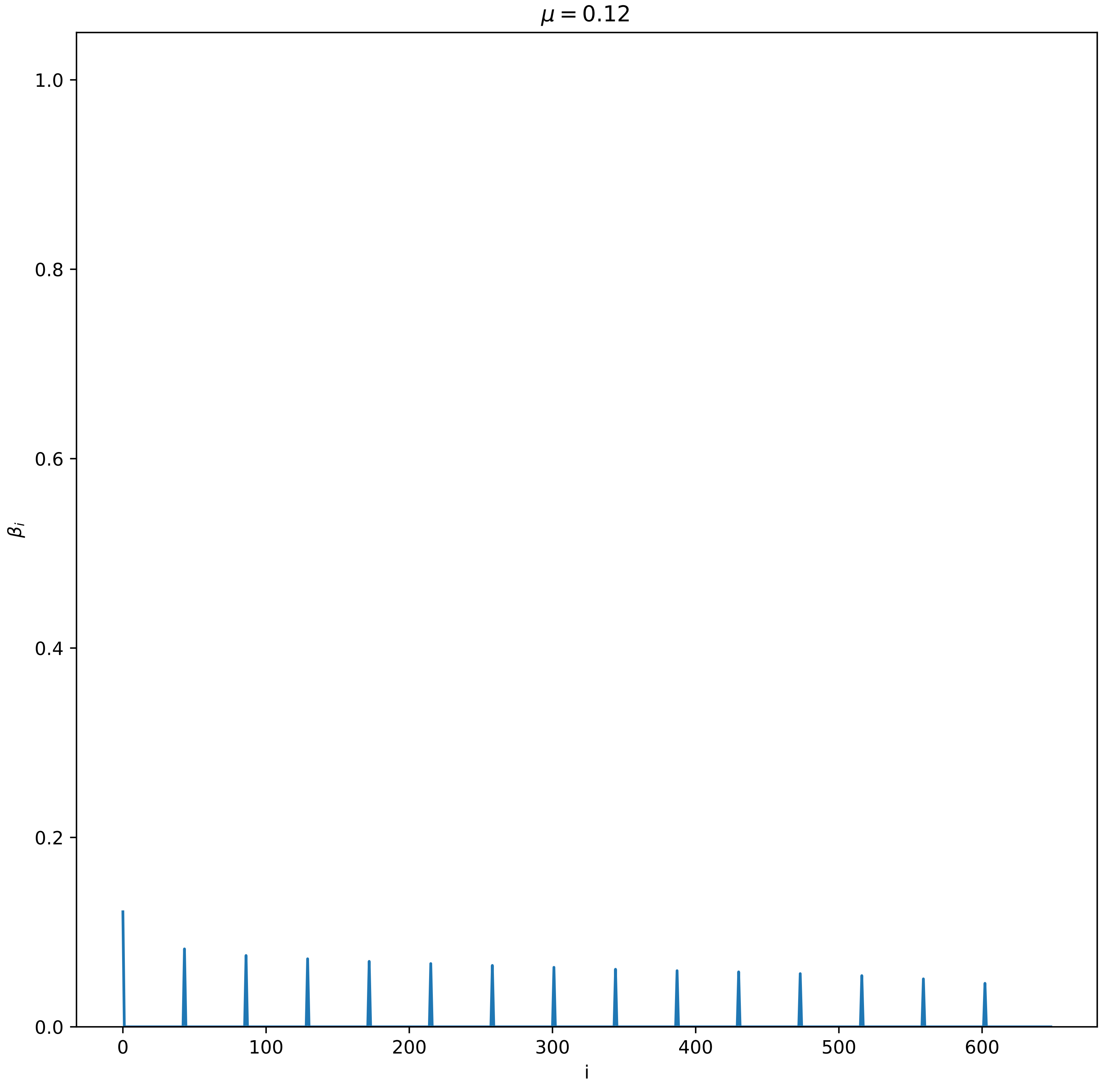
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.12$
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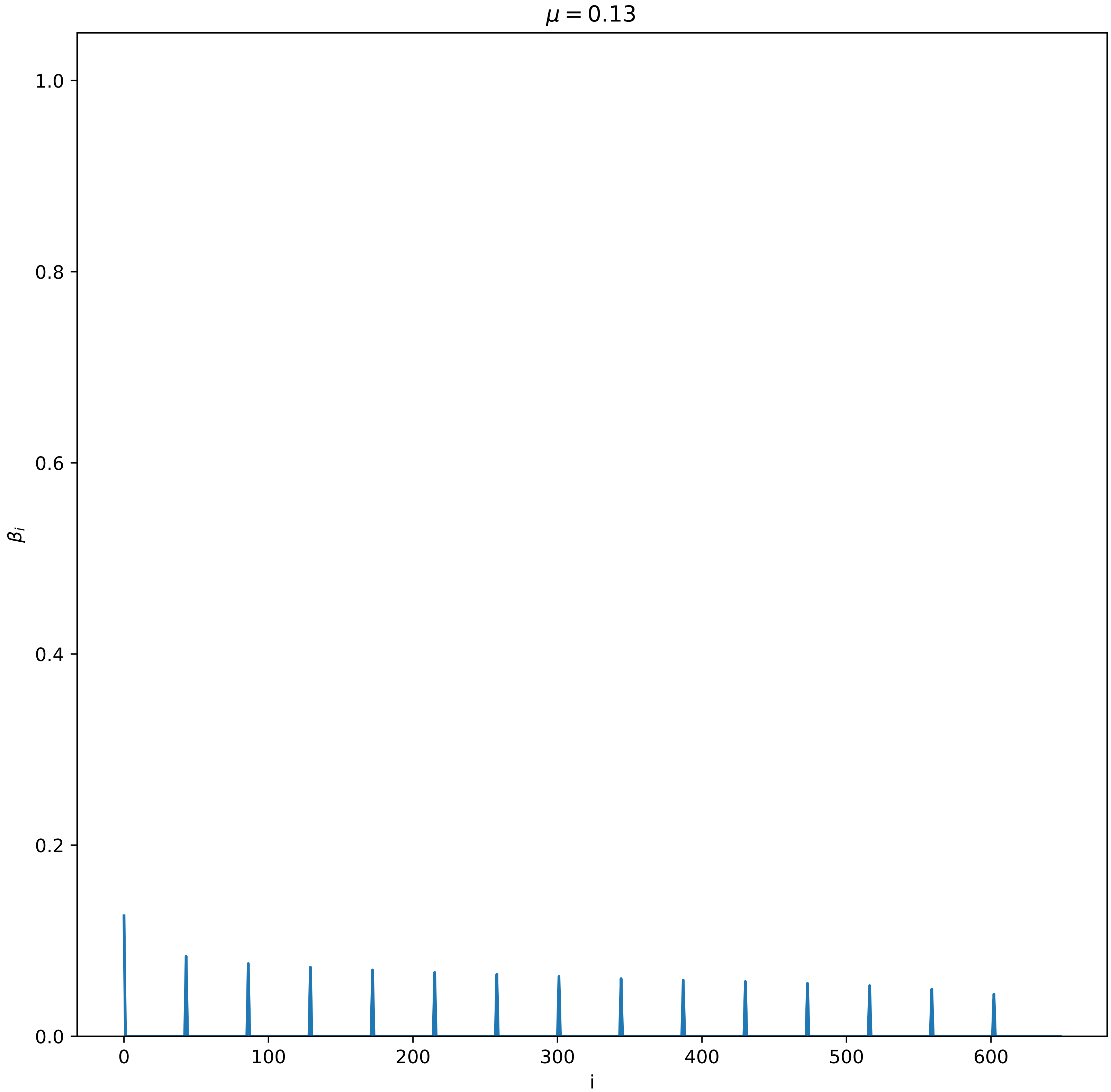
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 0.13$
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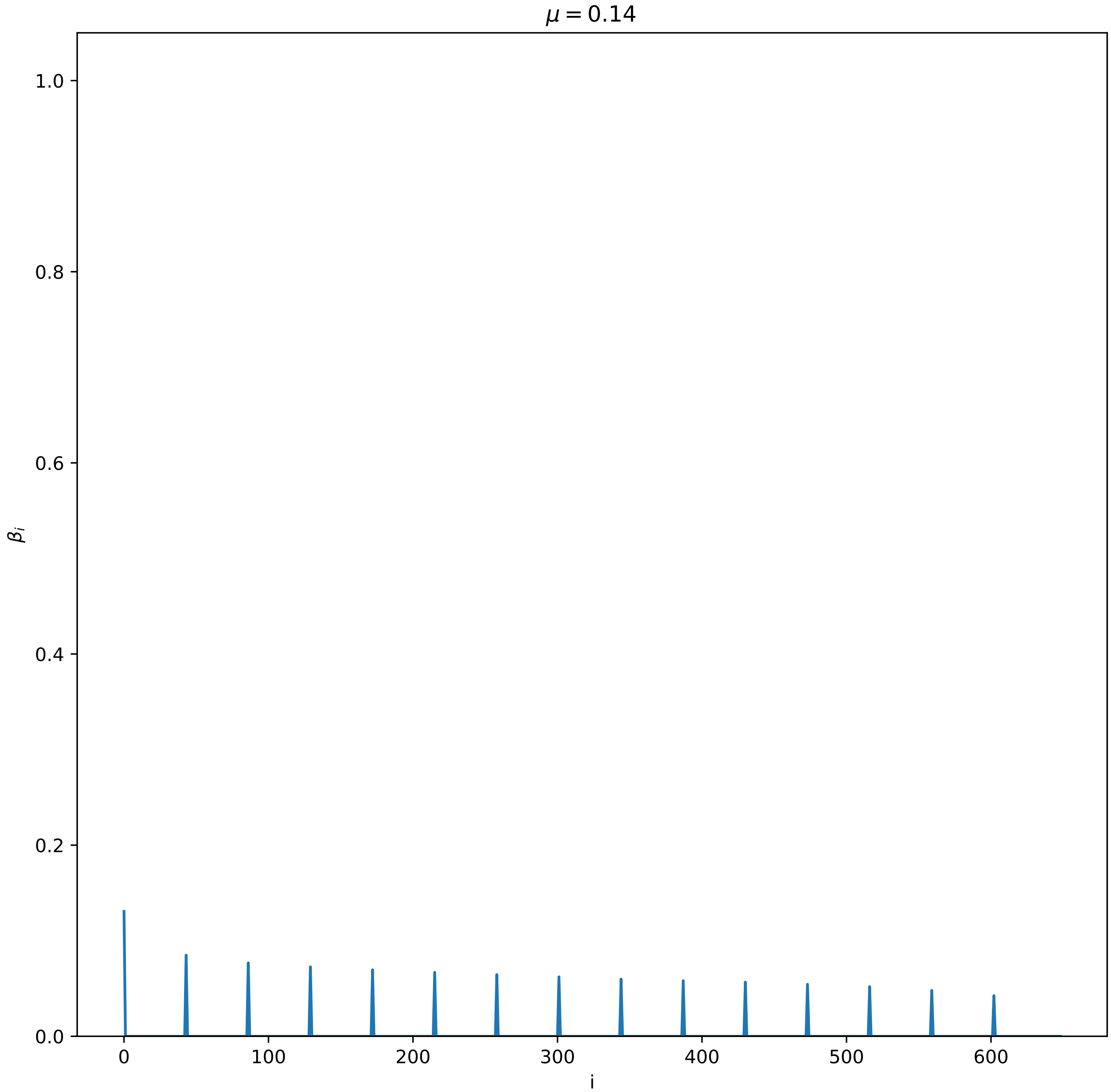
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 0.14$
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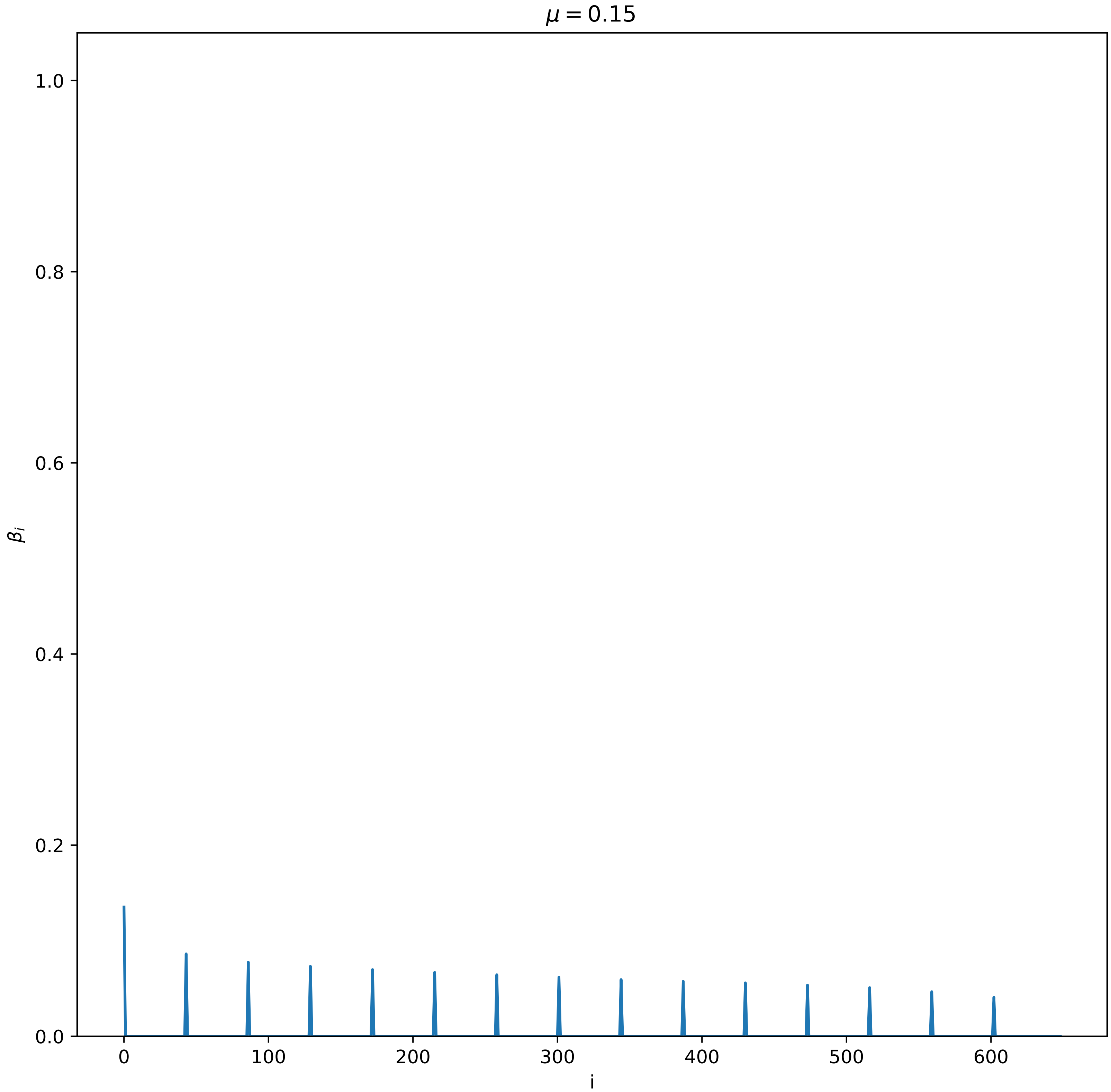
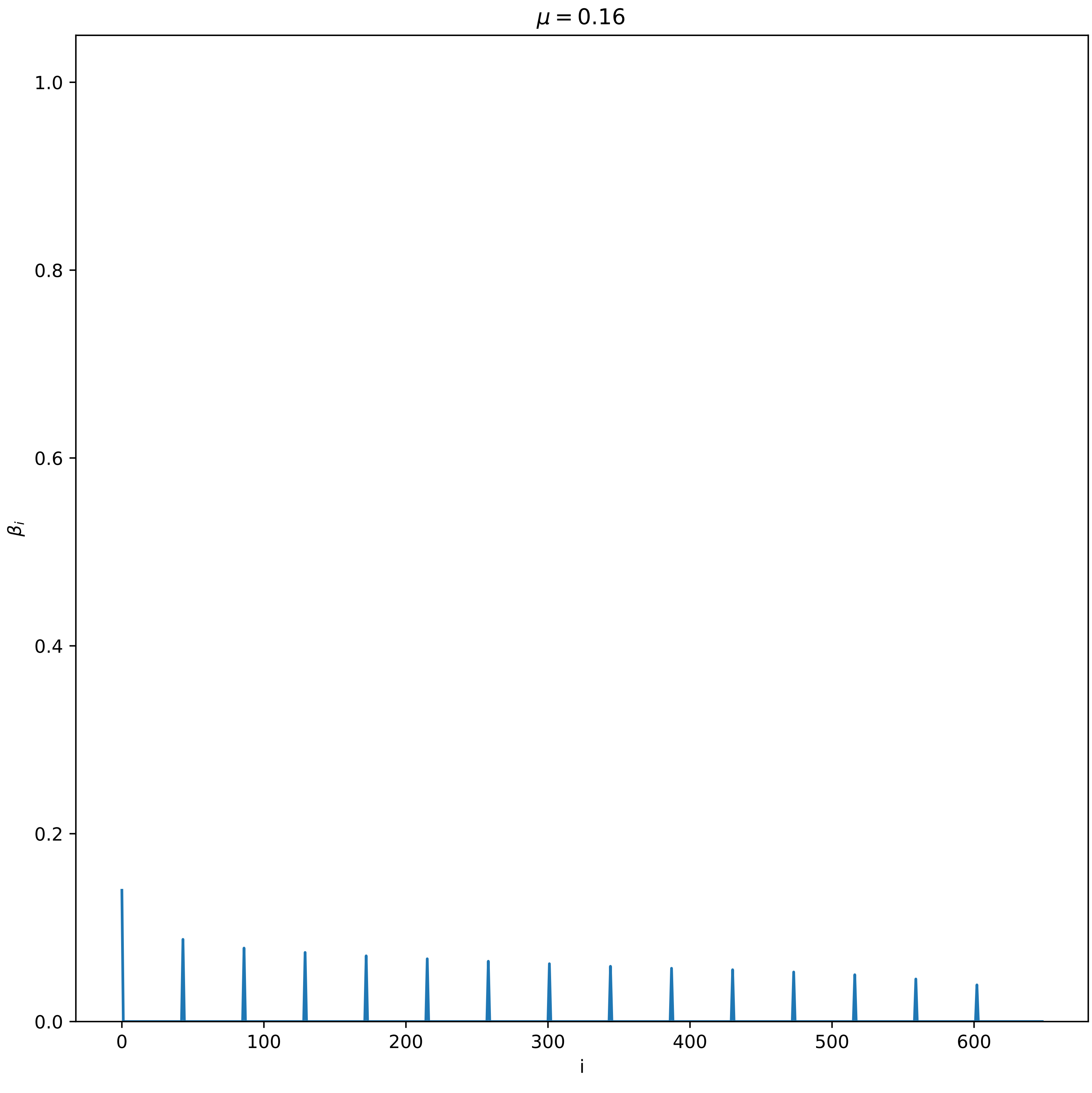
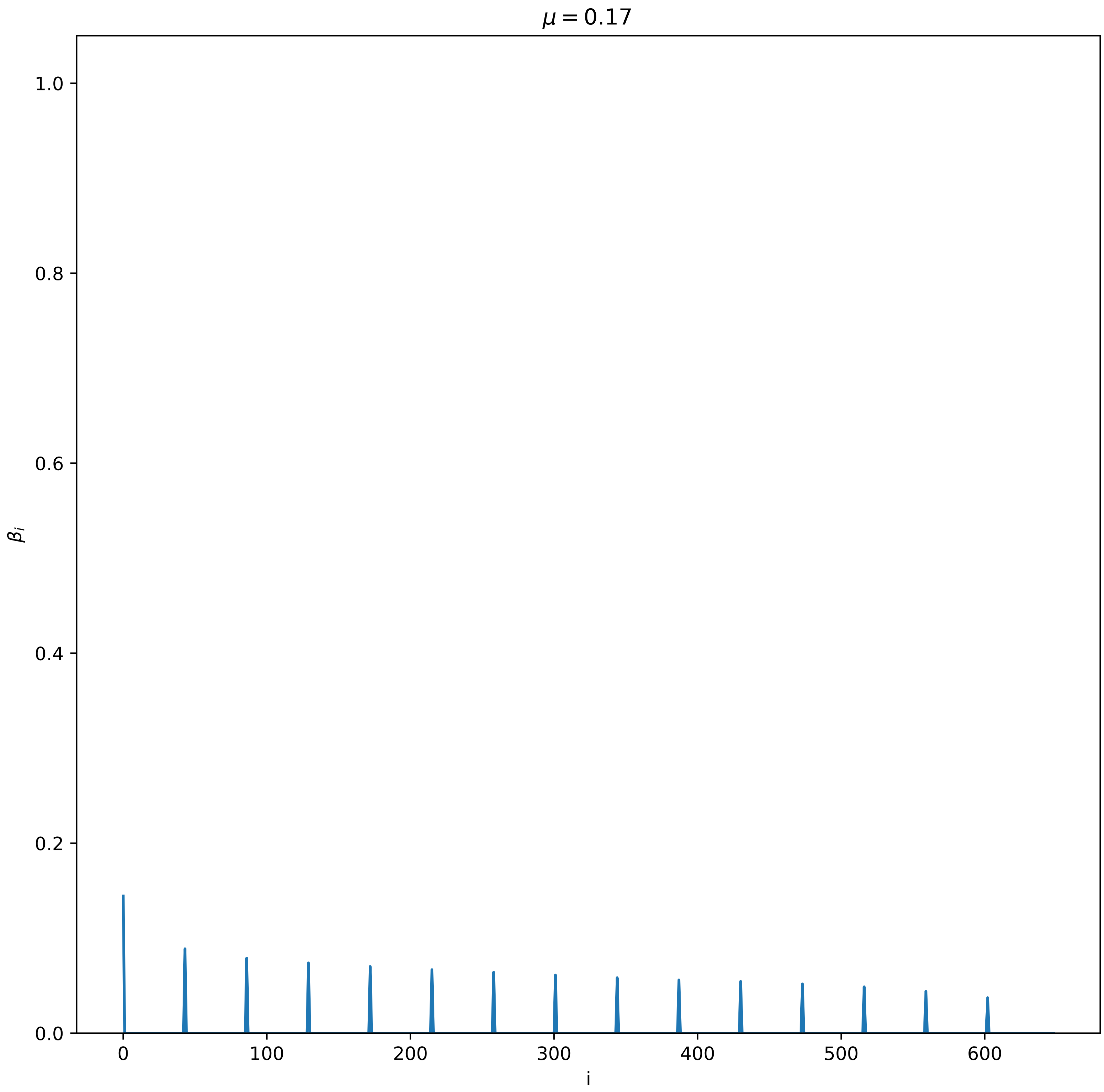
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



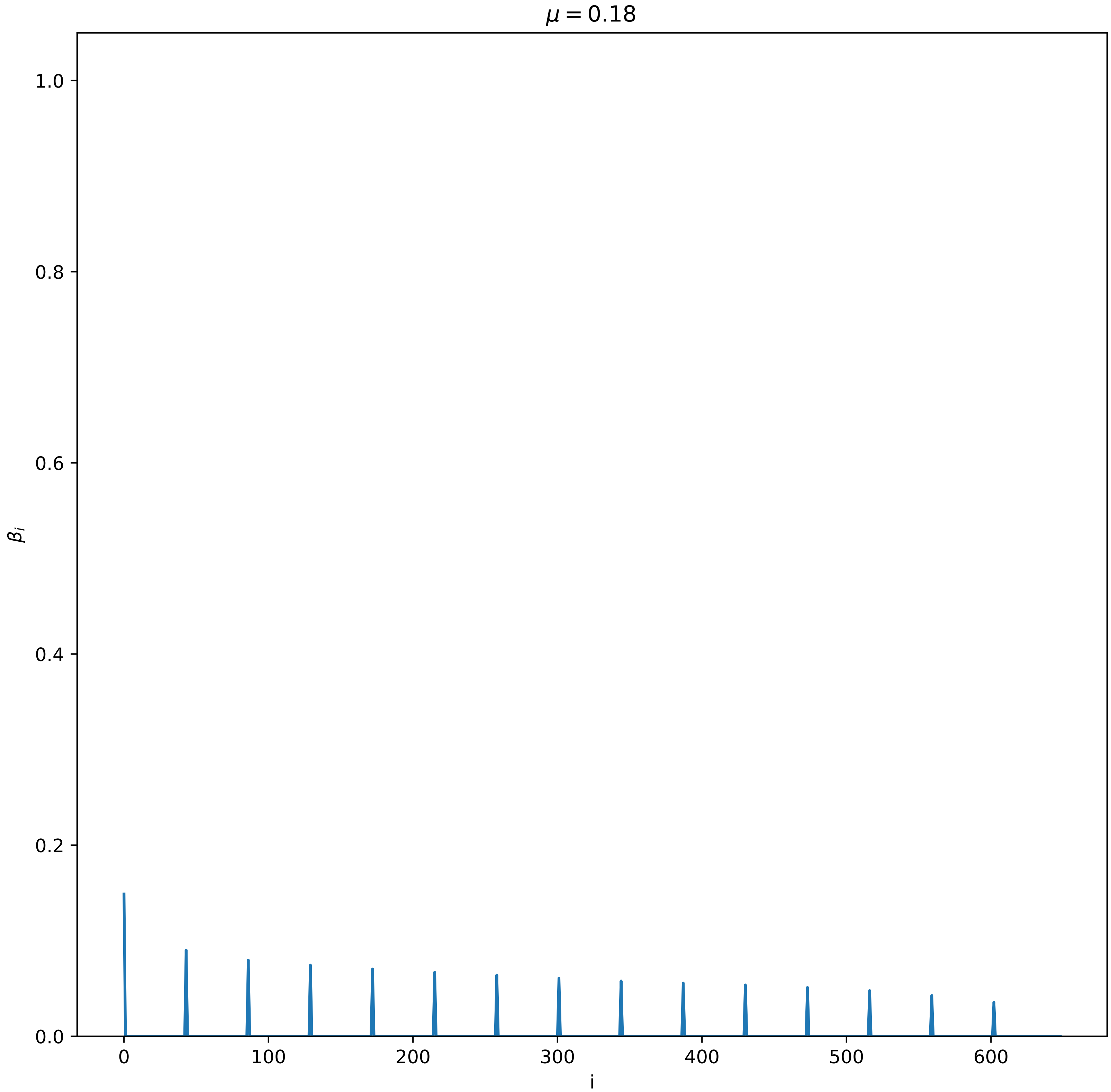
$\mu = 0.15$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.16$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.17$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i


$\mu = 0.18$
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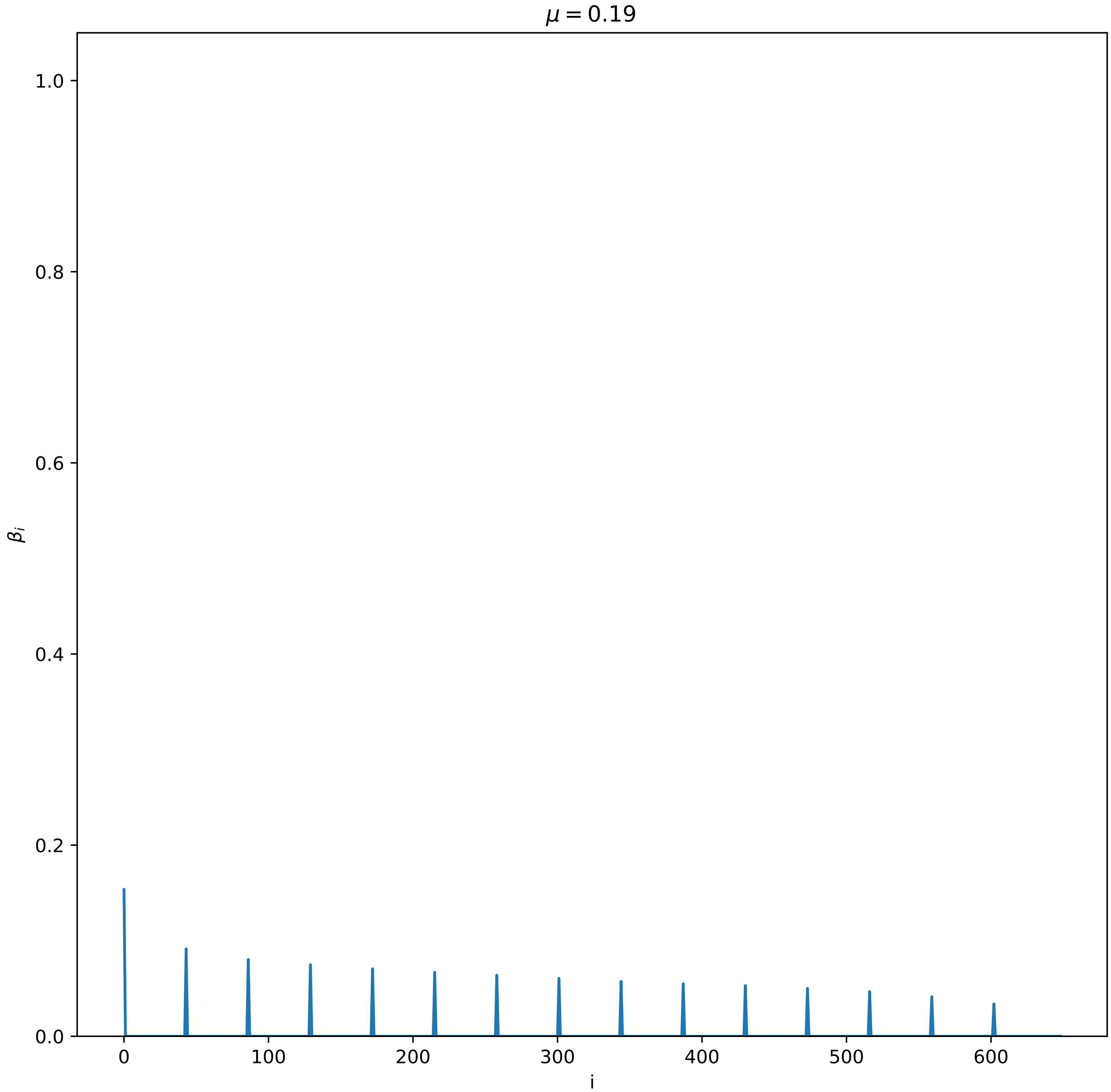
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.19$
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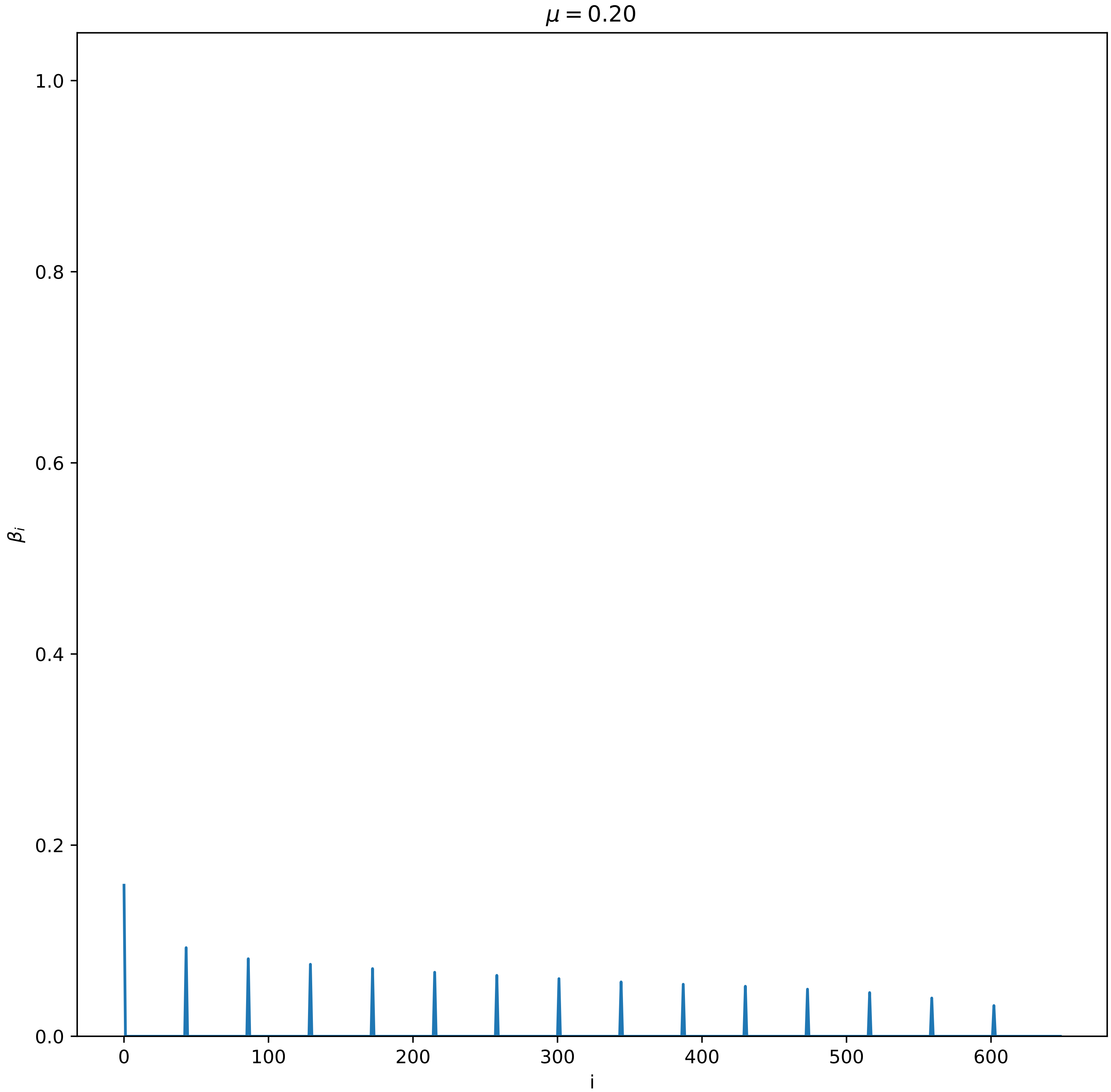
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i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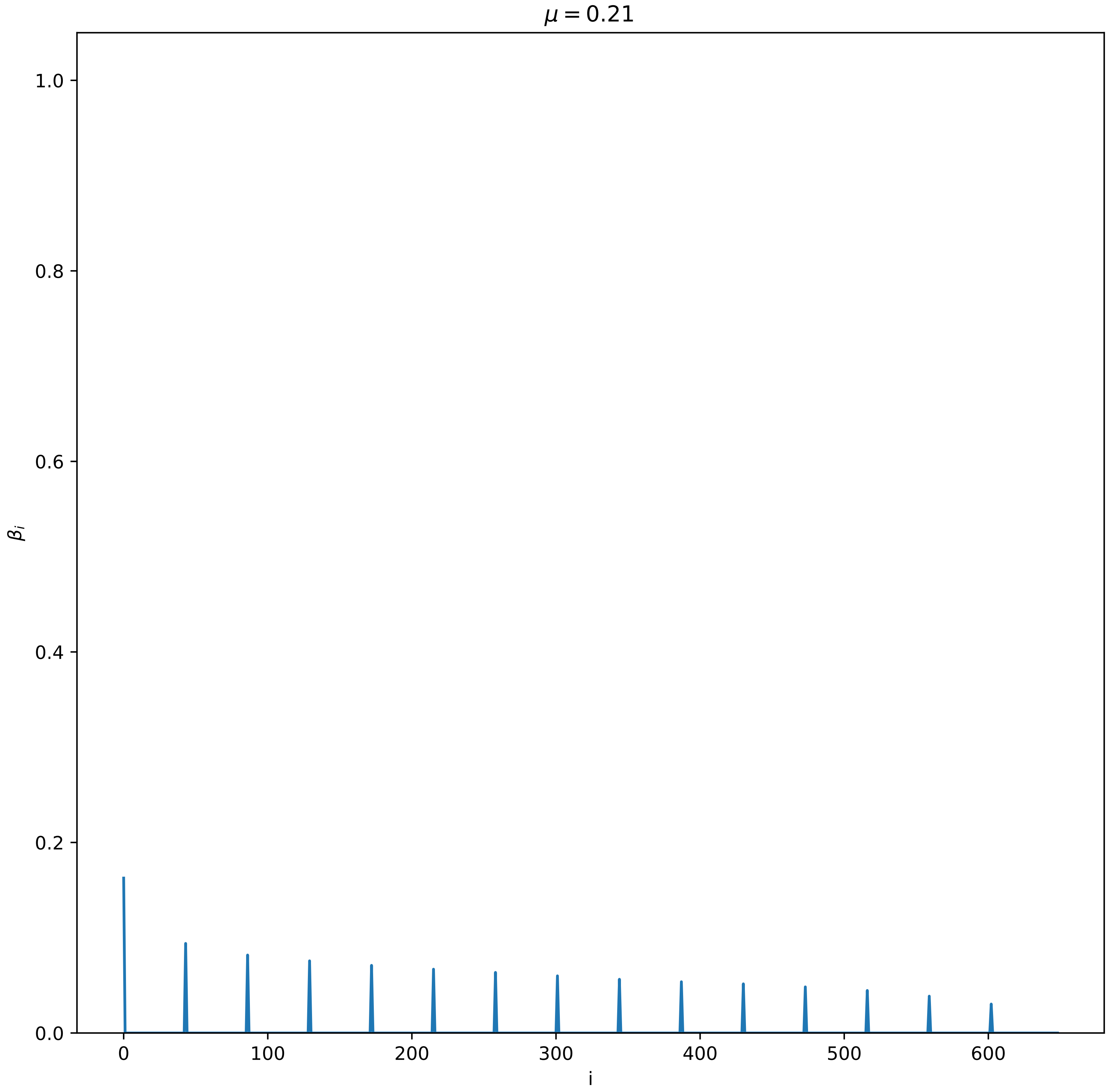
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.21$
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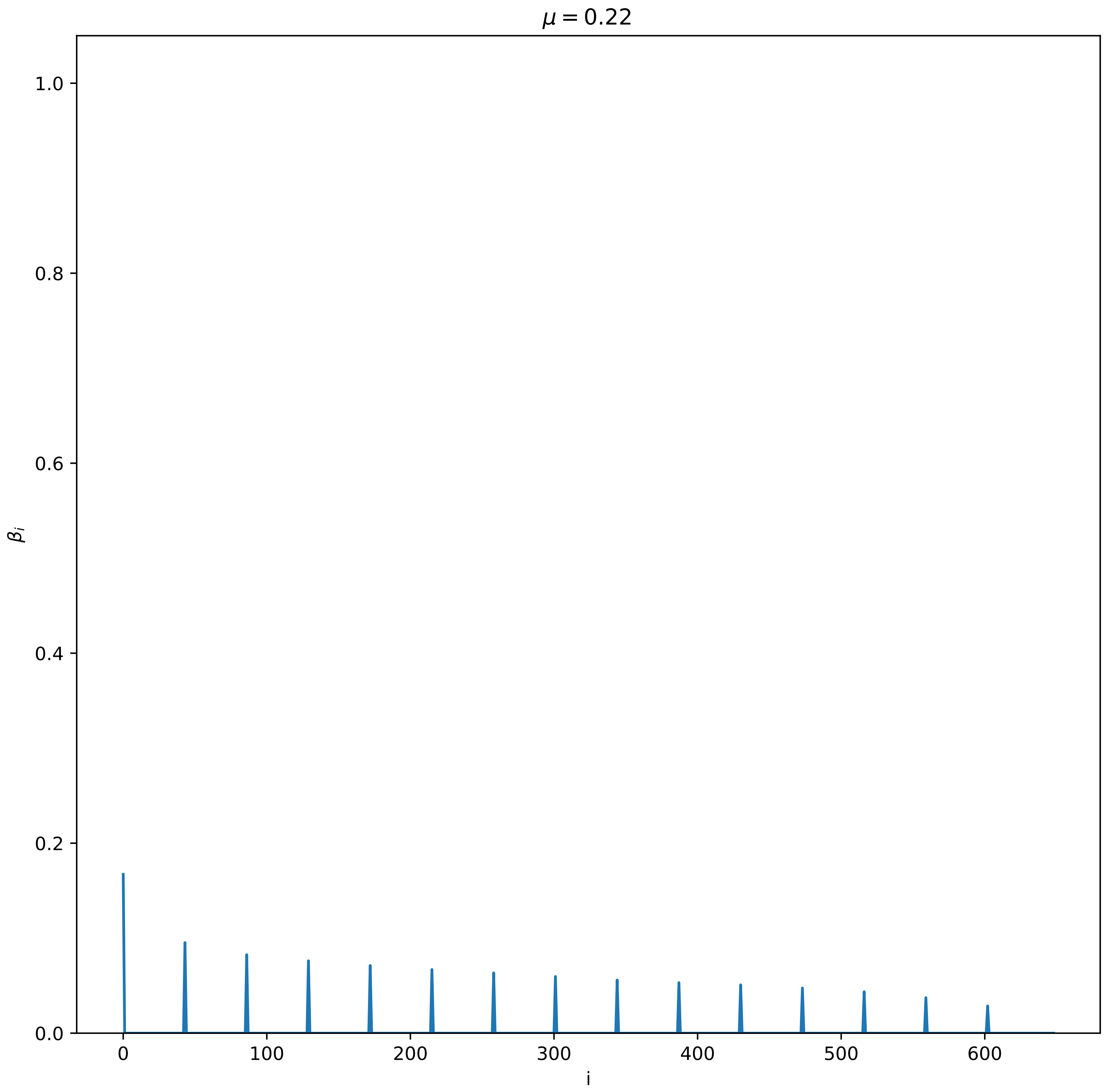
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.22$
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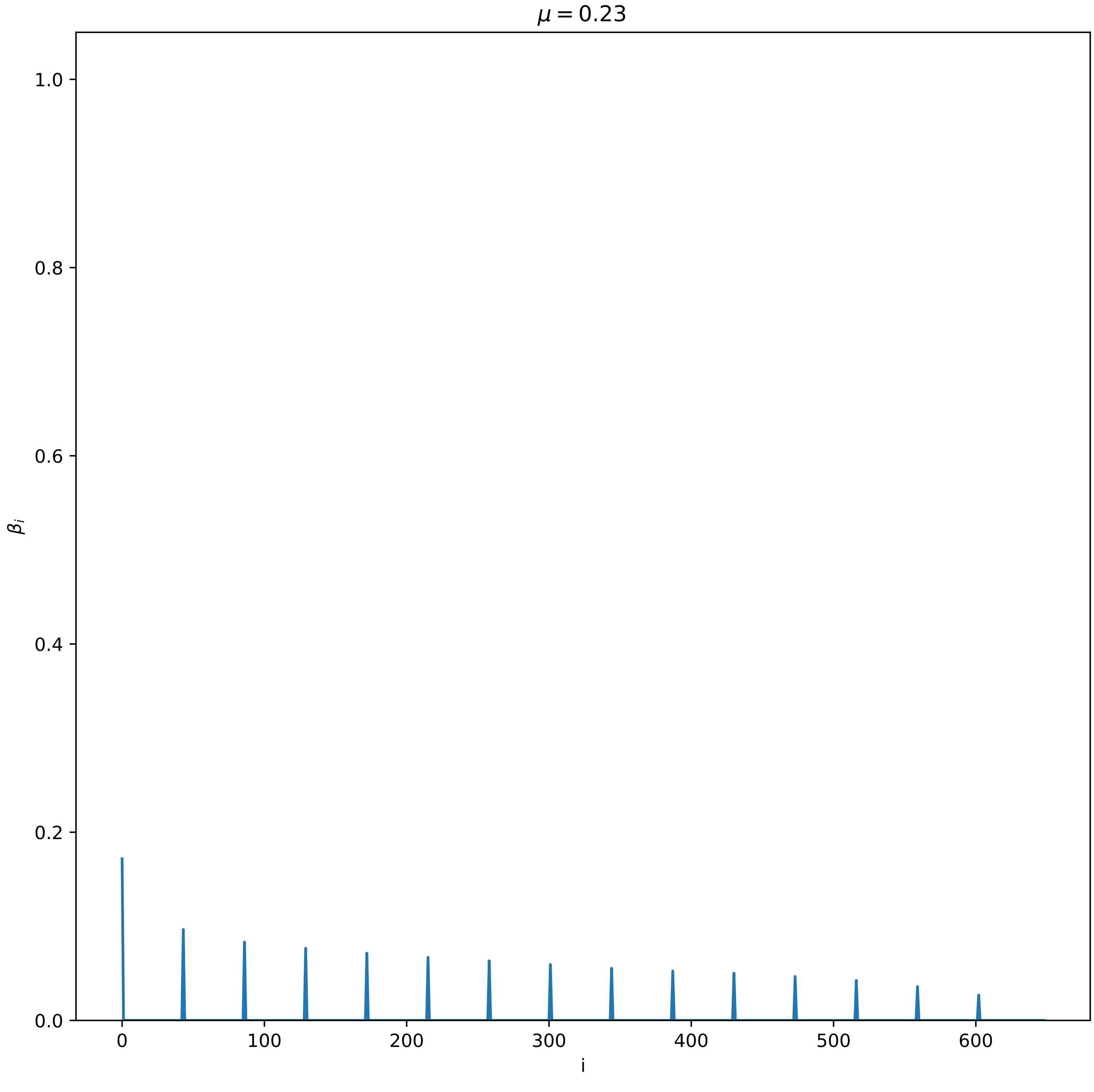
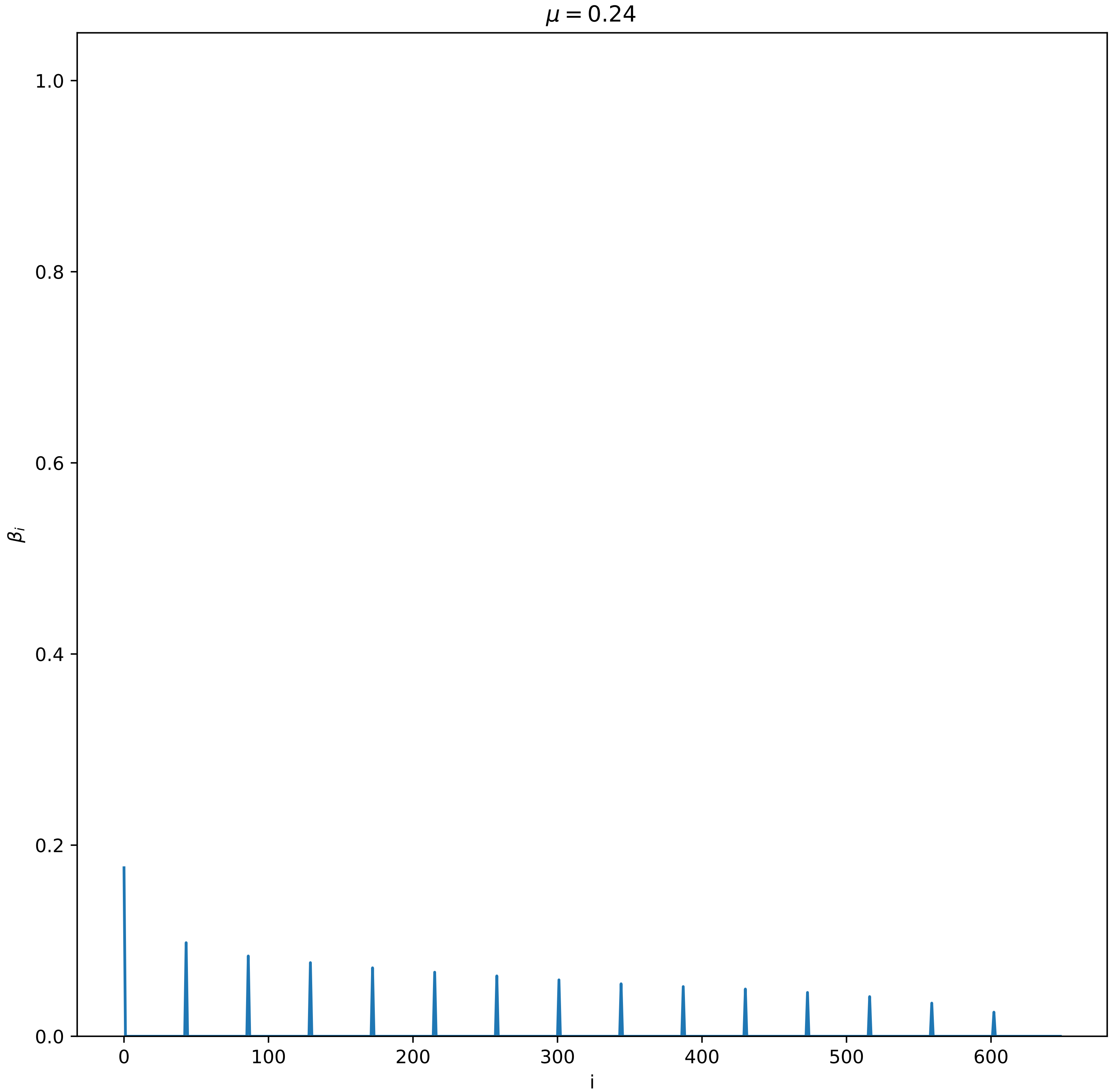
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.23$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 0.24$
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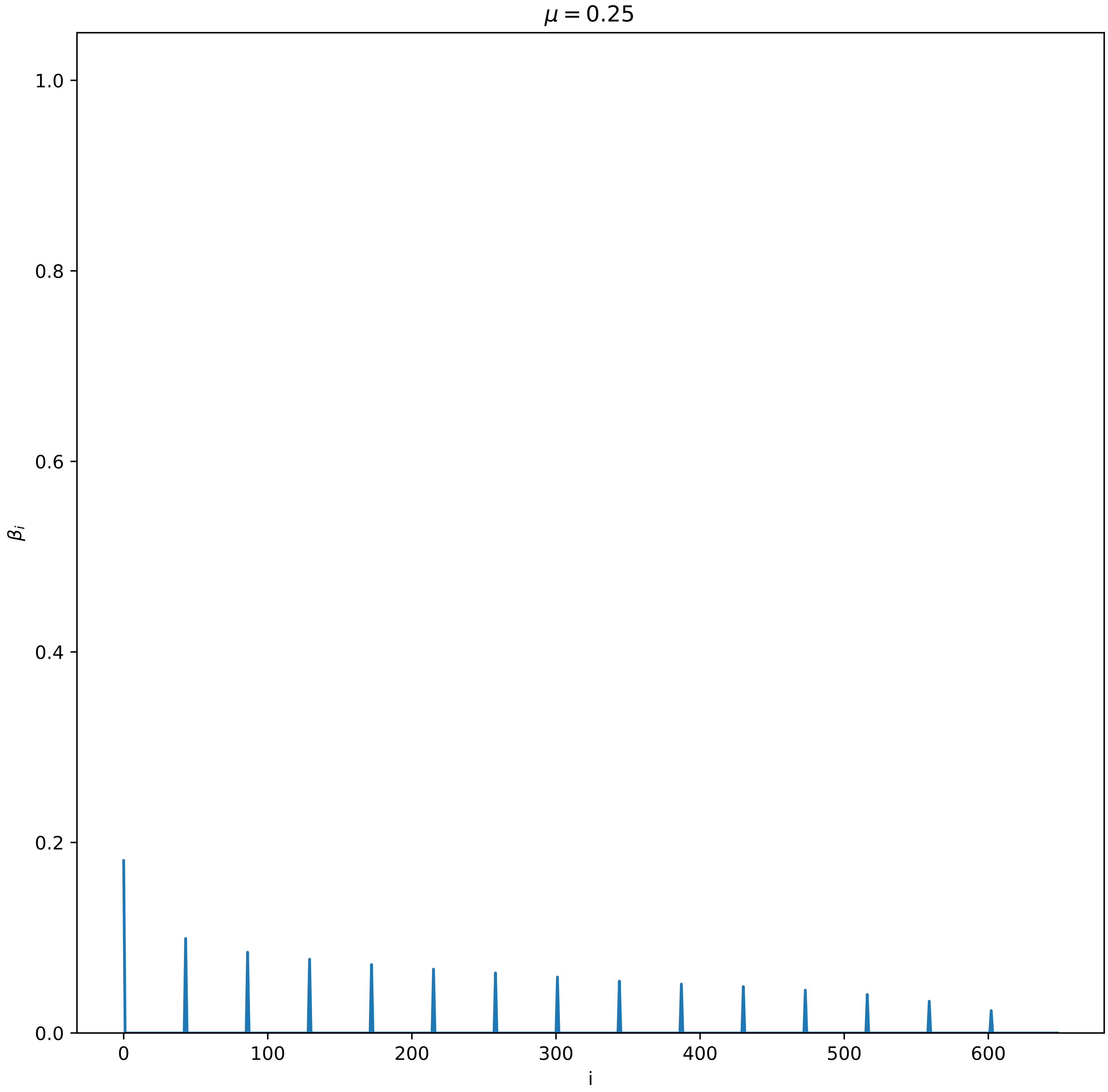
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.25$
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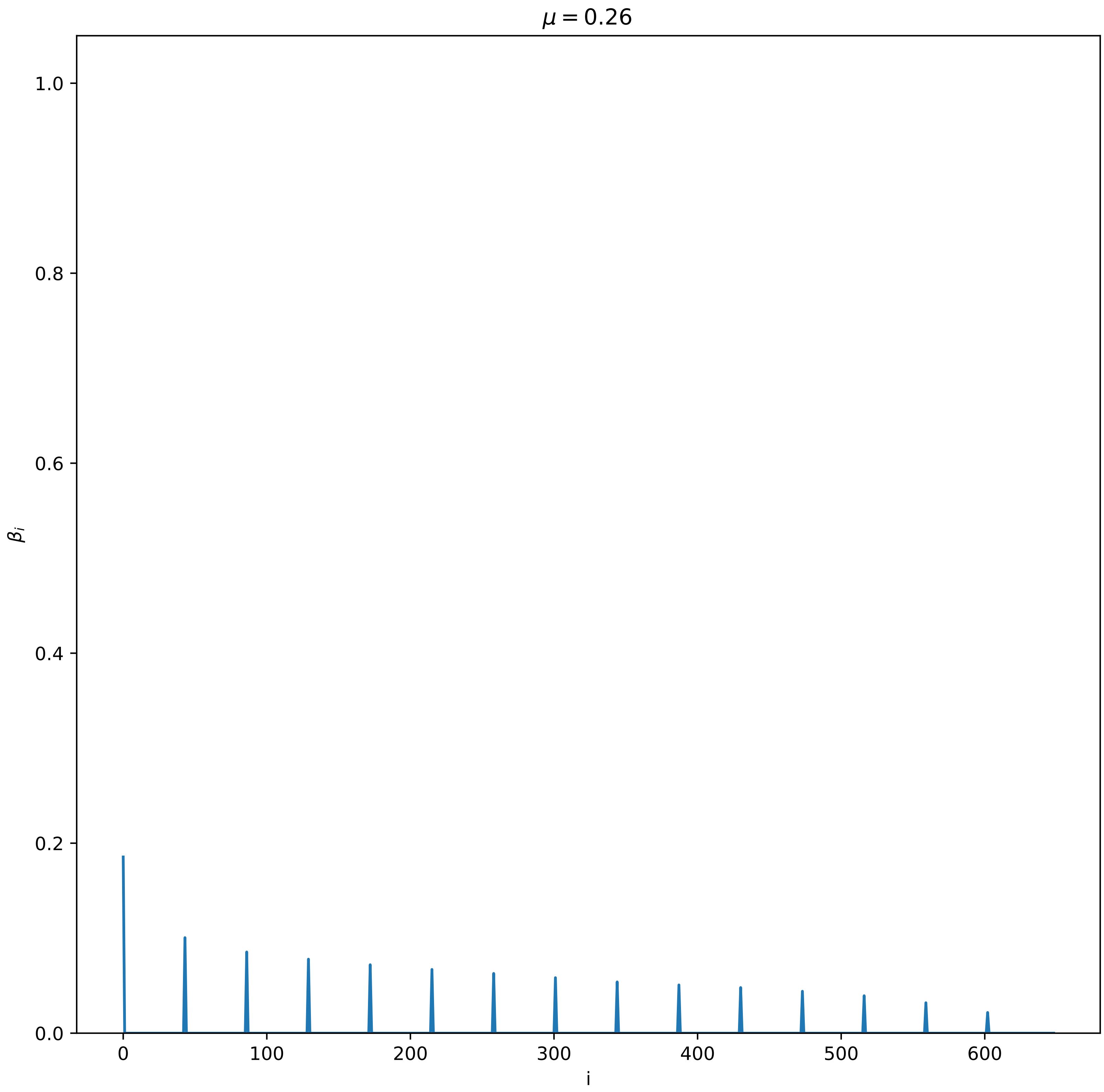
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.26$
1.0
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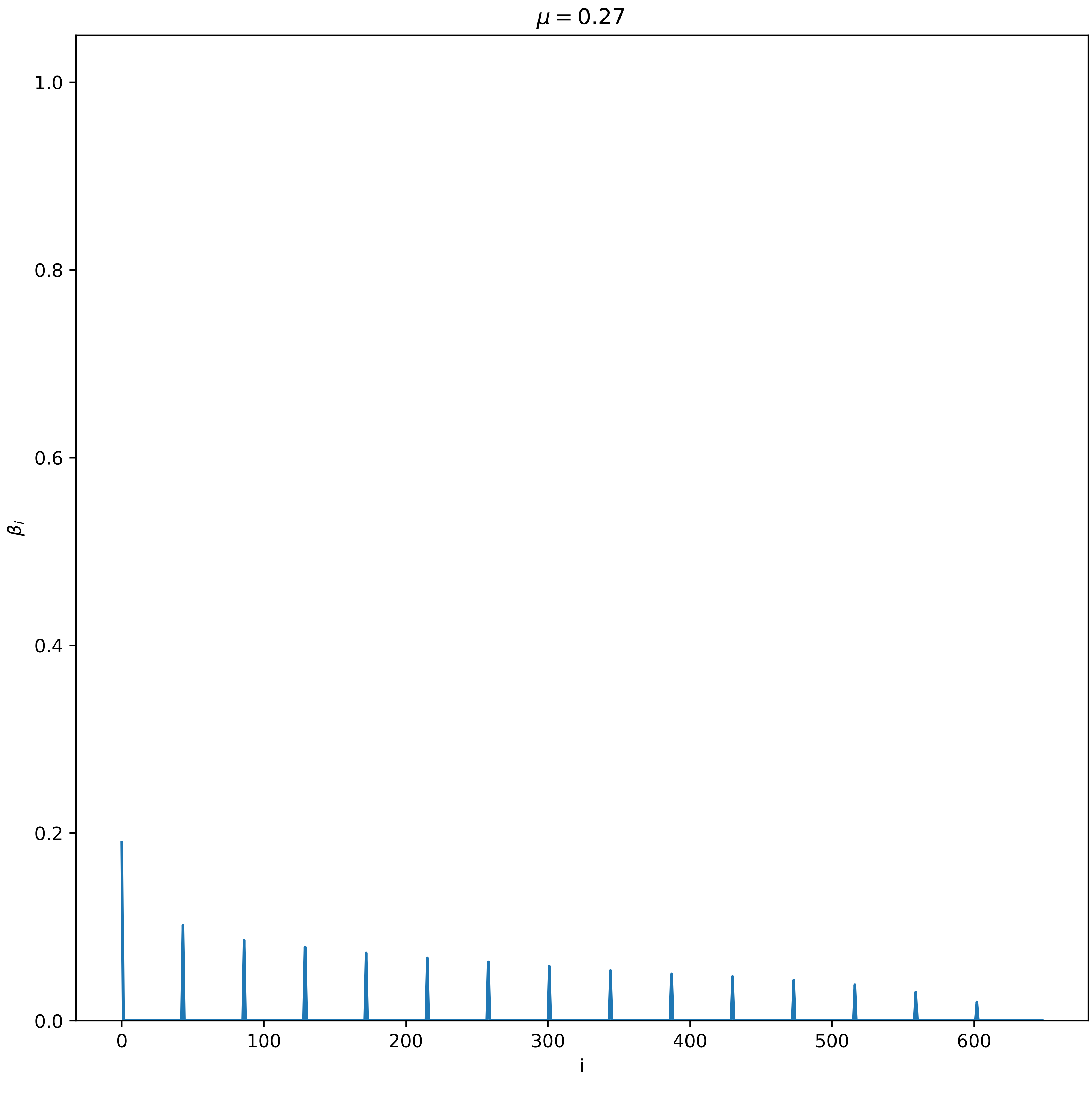
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 0.27$
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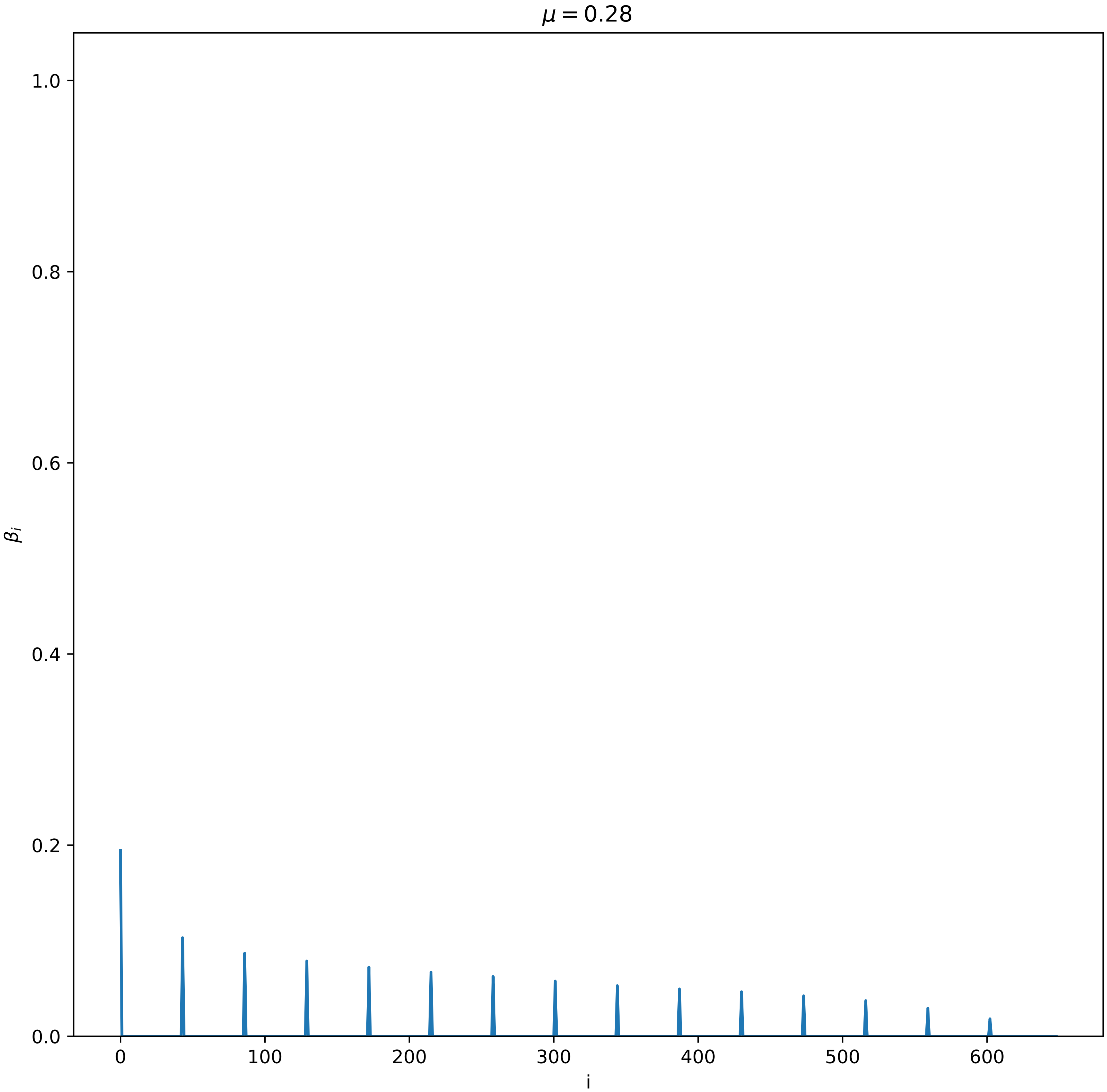
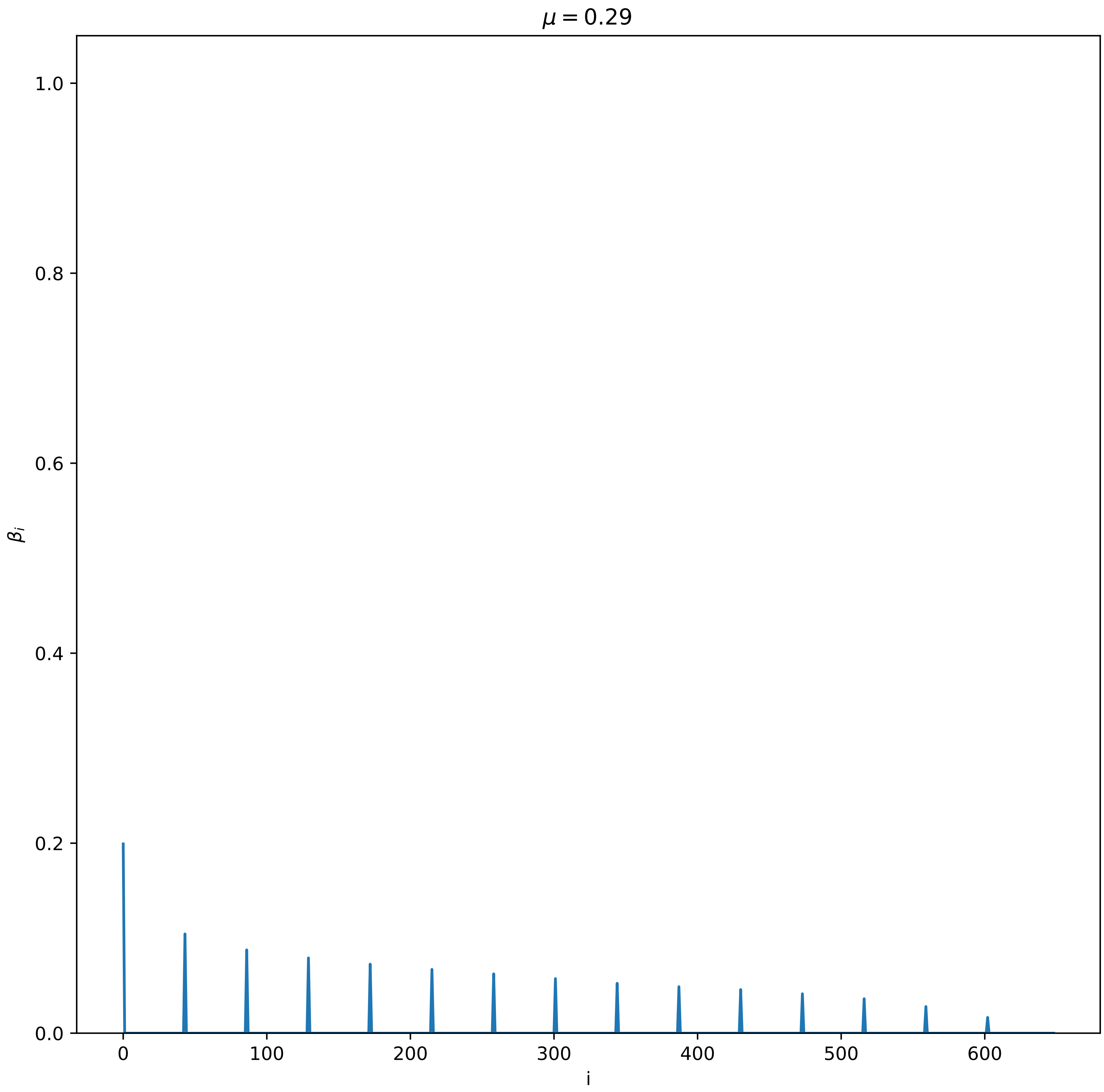
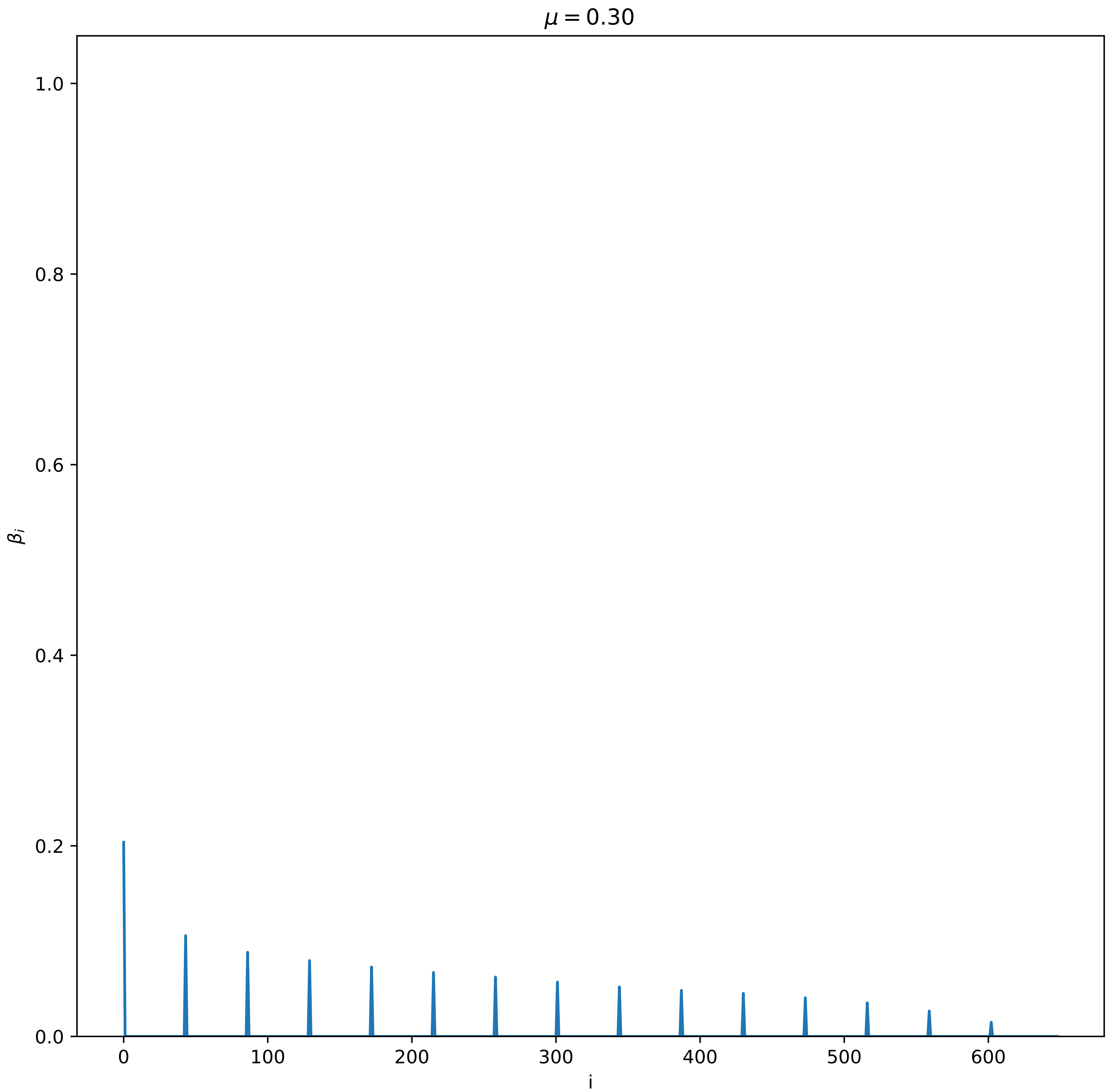
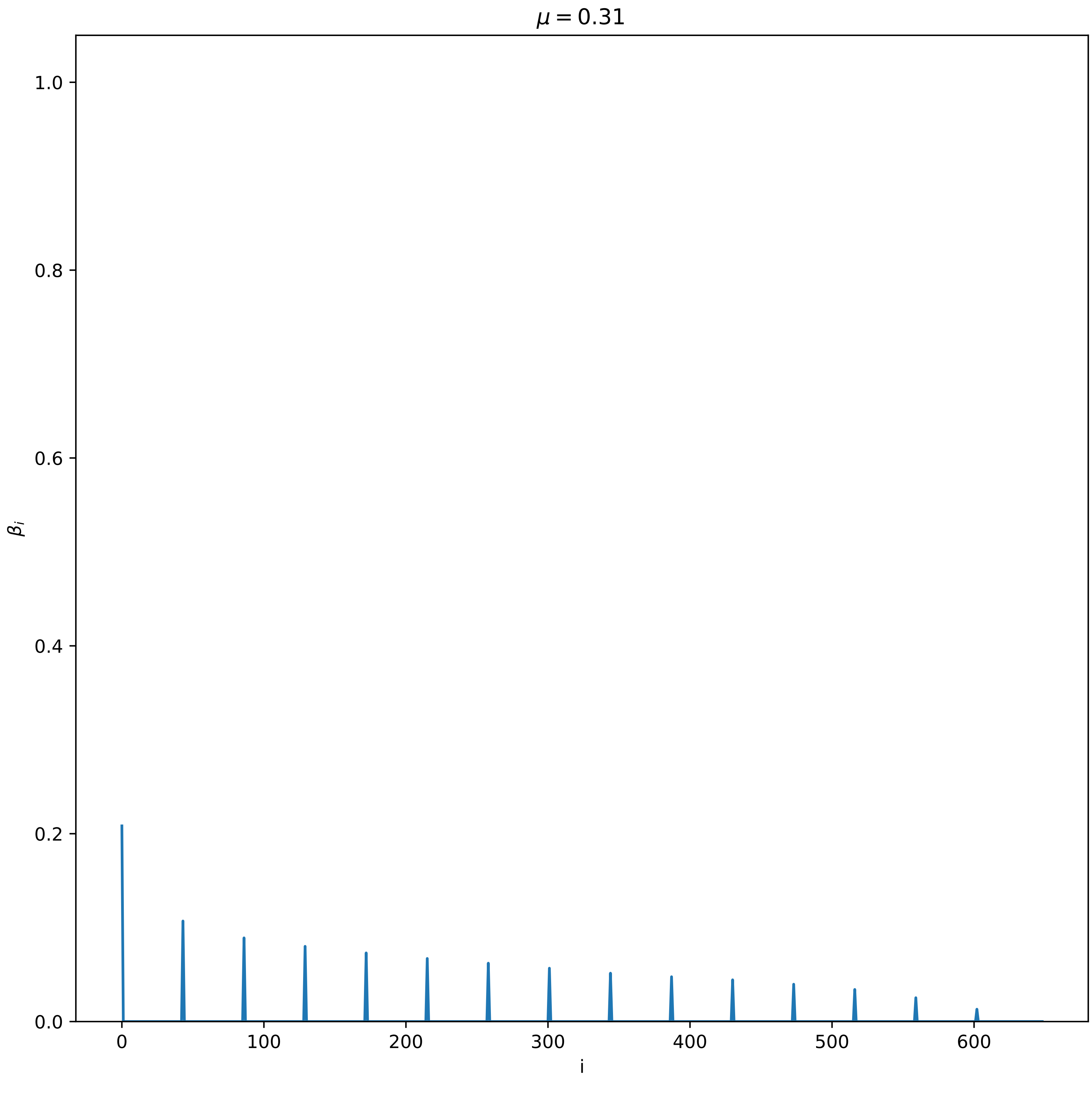
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

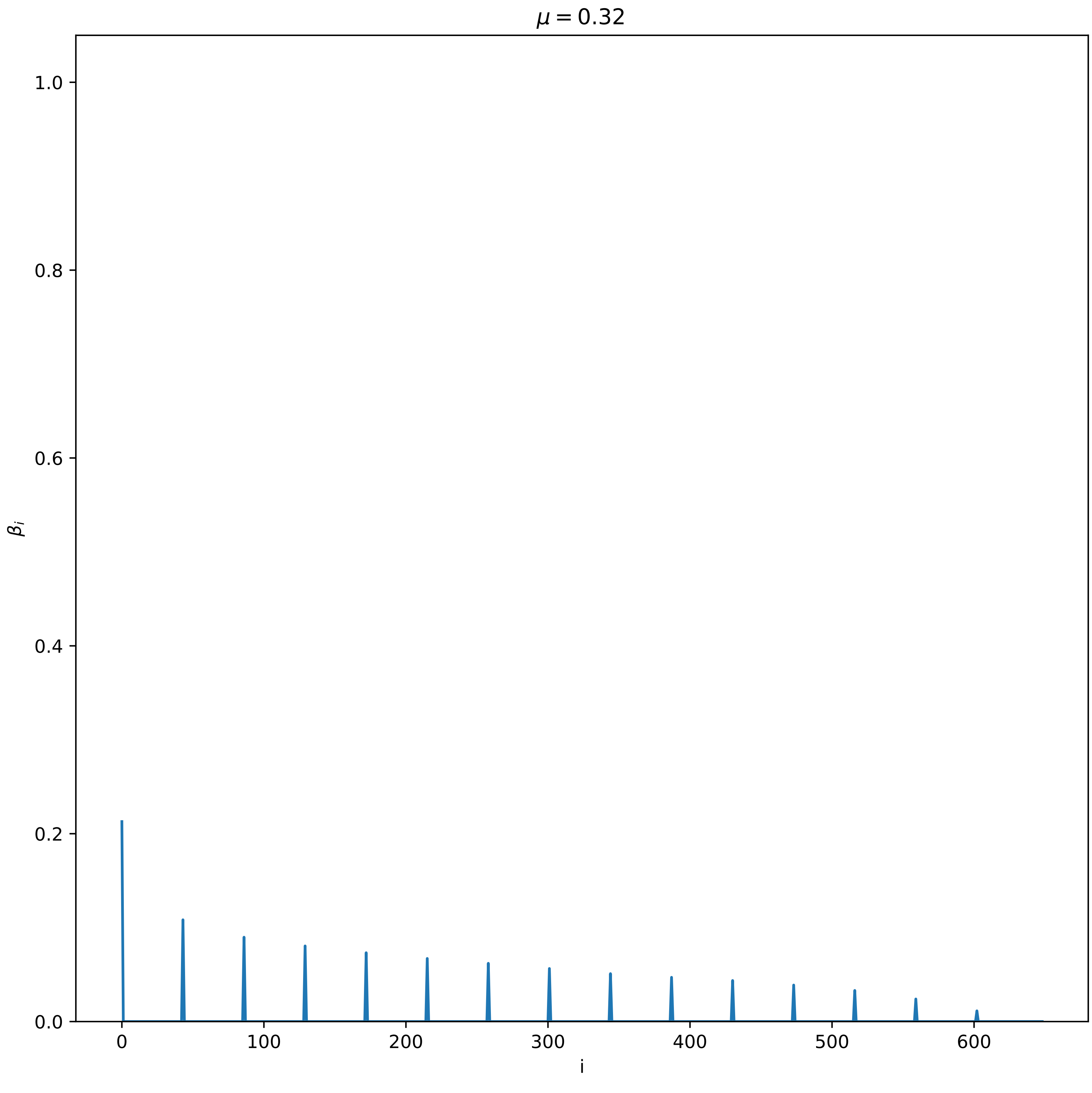

$\mu = 0.28$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 0.29$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

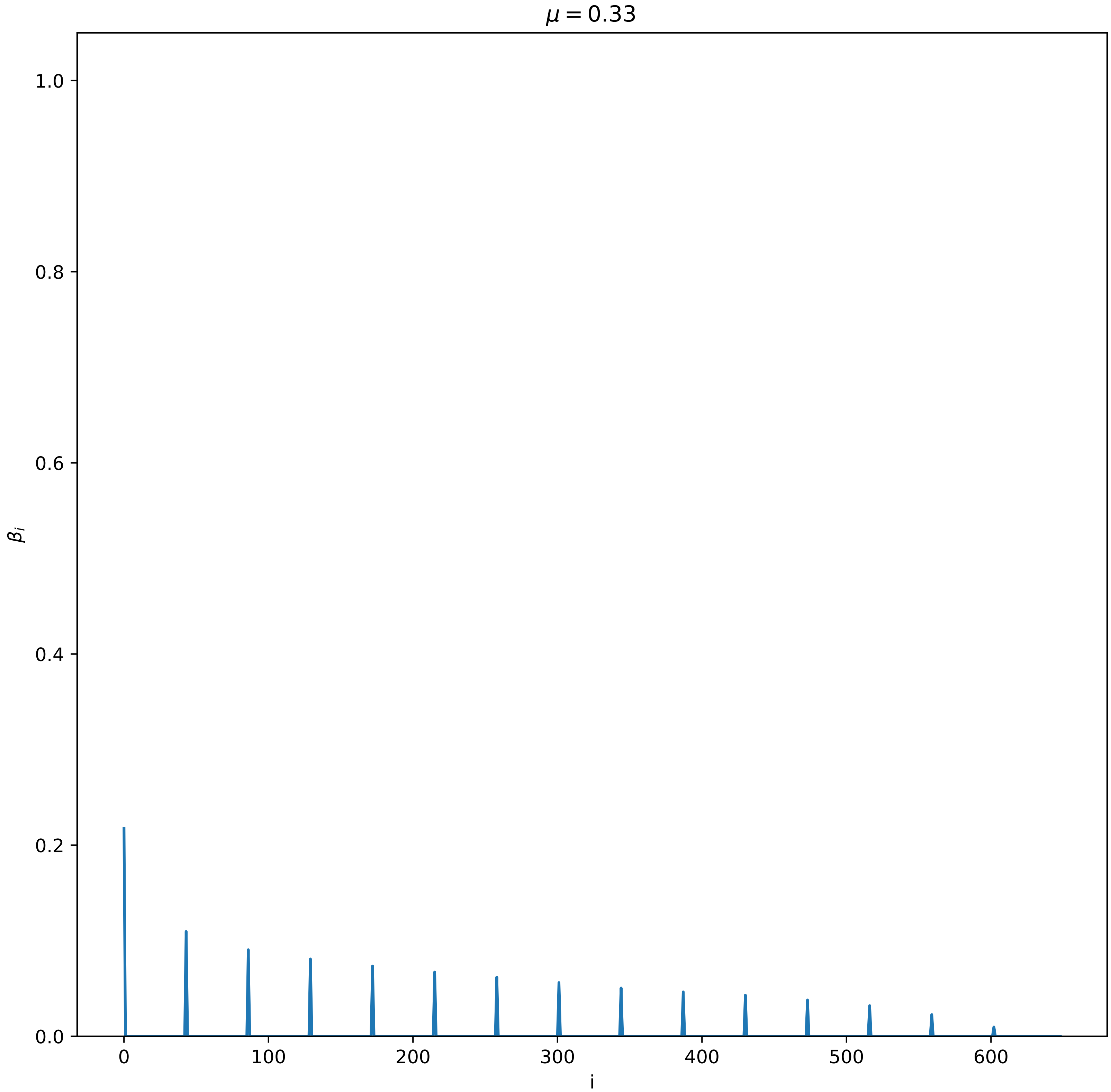


$\mu = 0.30$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

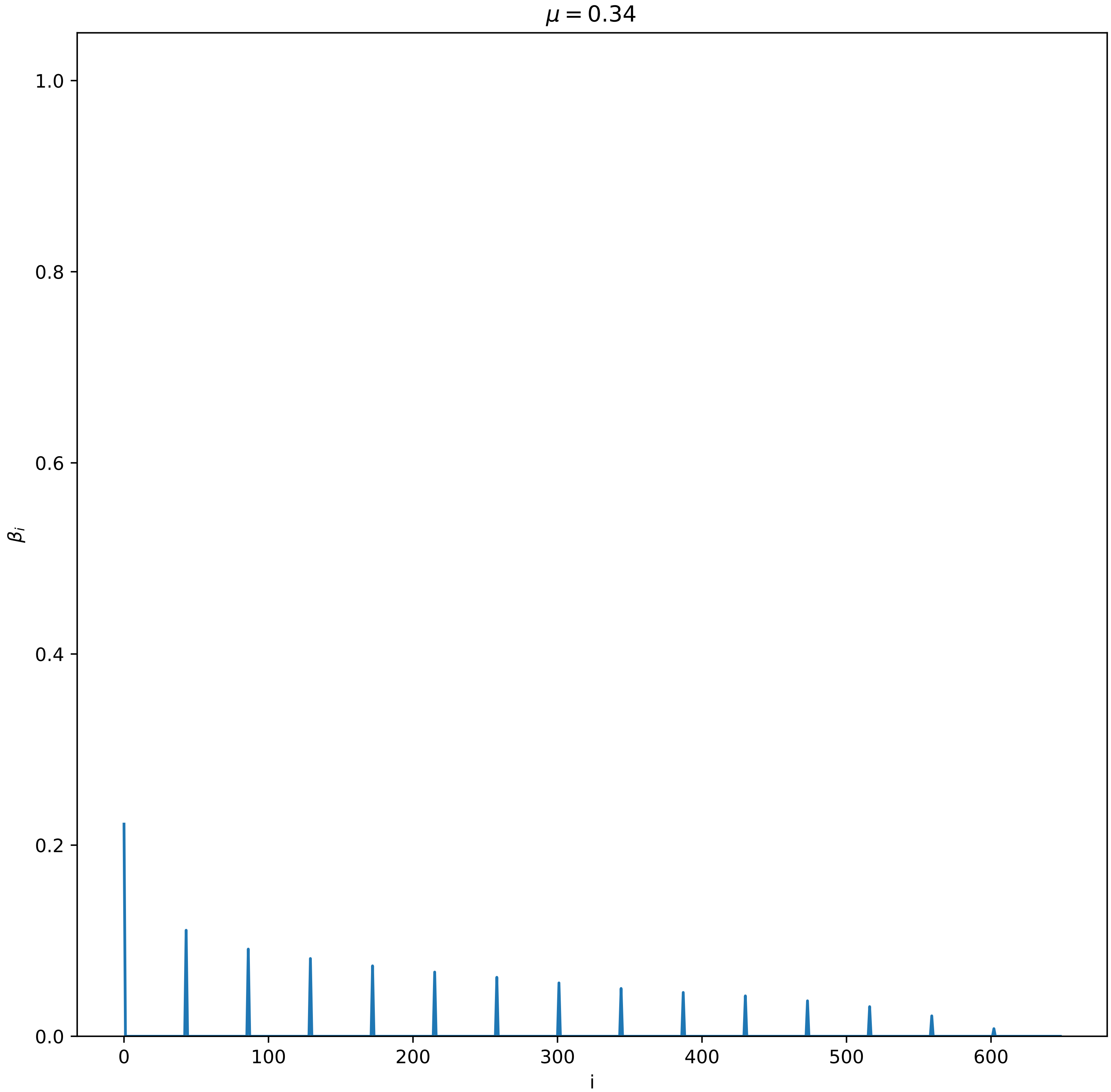

$\mu = 0.31$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



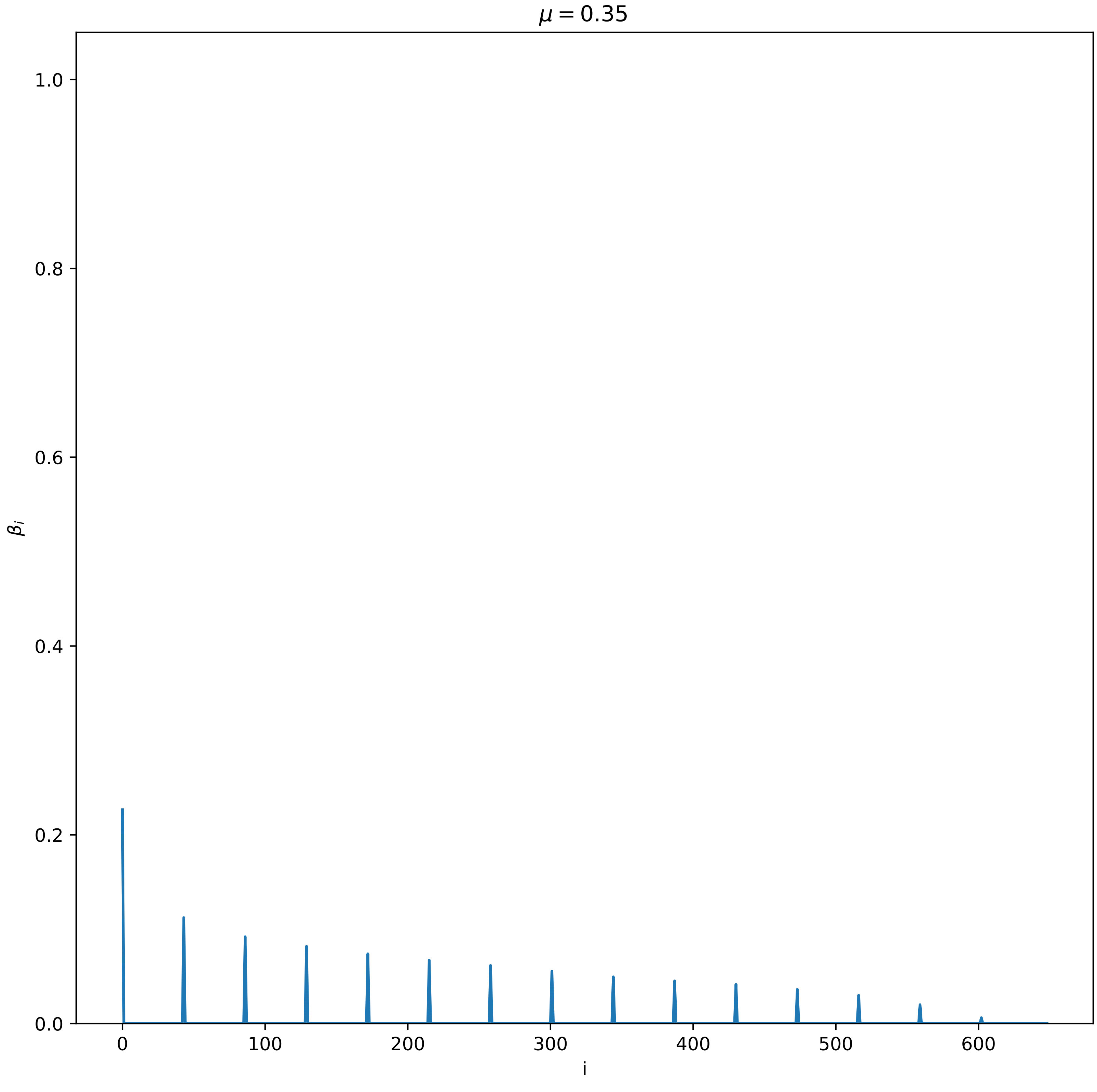
$\mu = 0.32$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



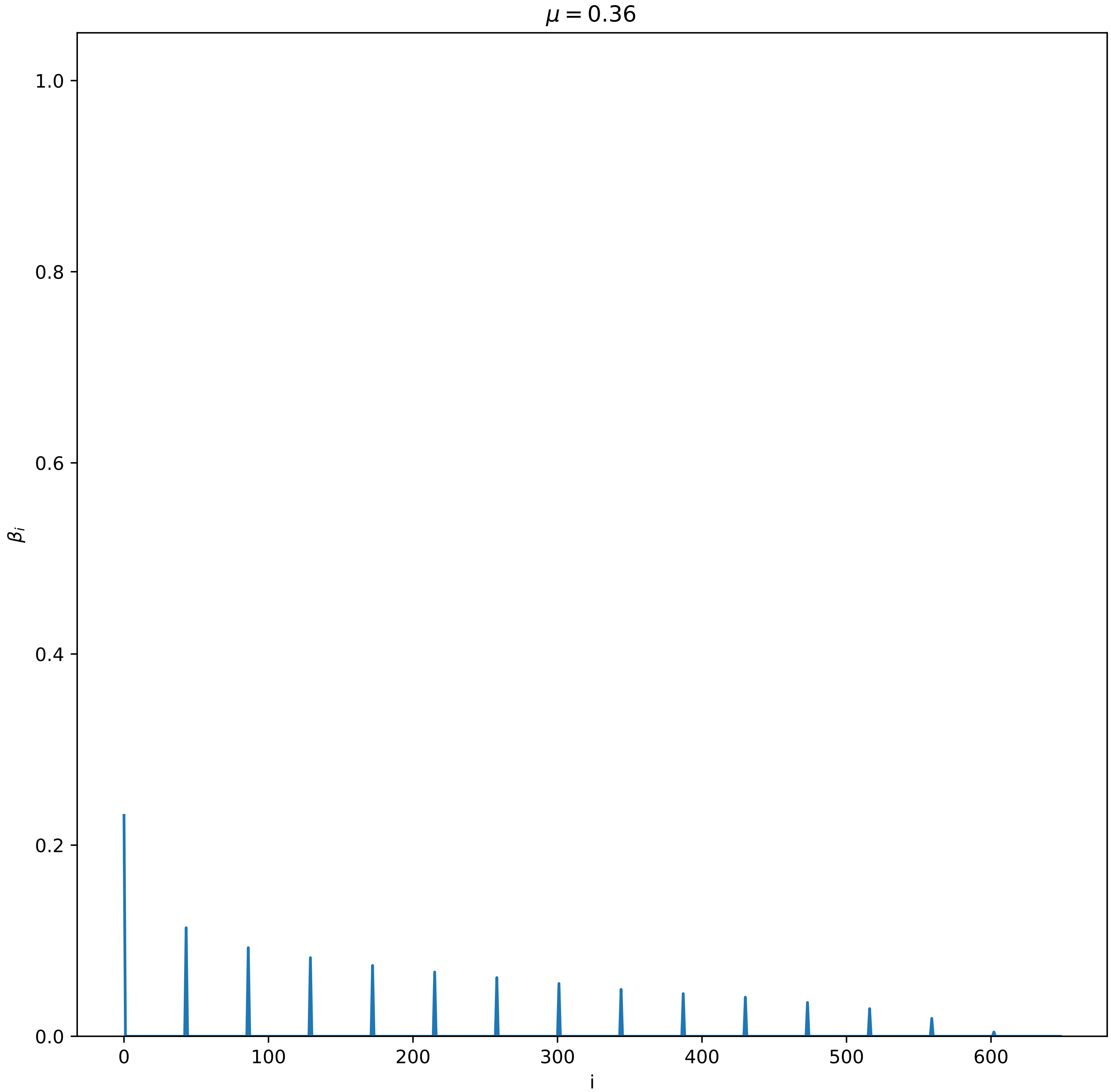
$\mu = 0.33$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

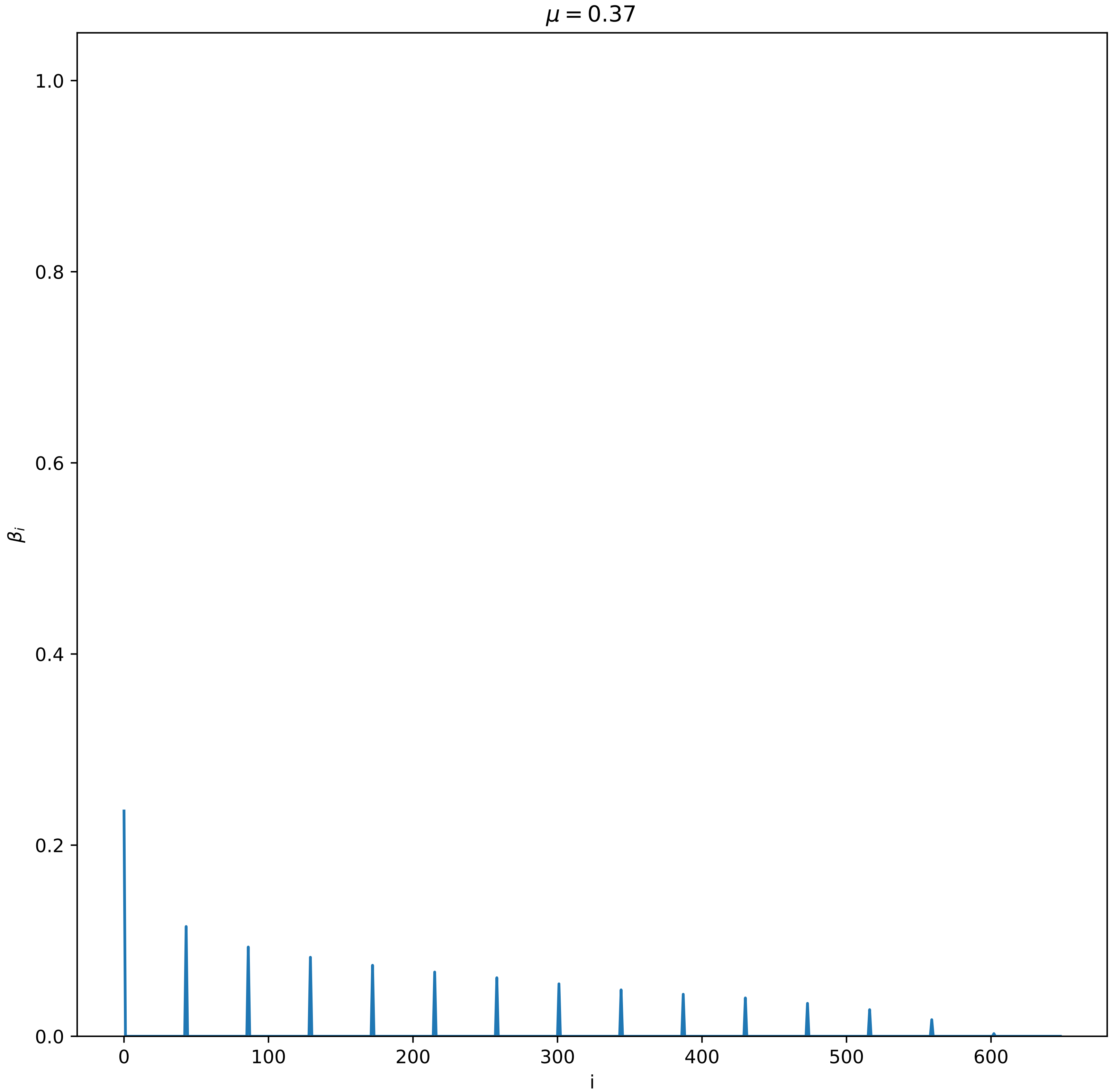


$\mu = 0.34$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

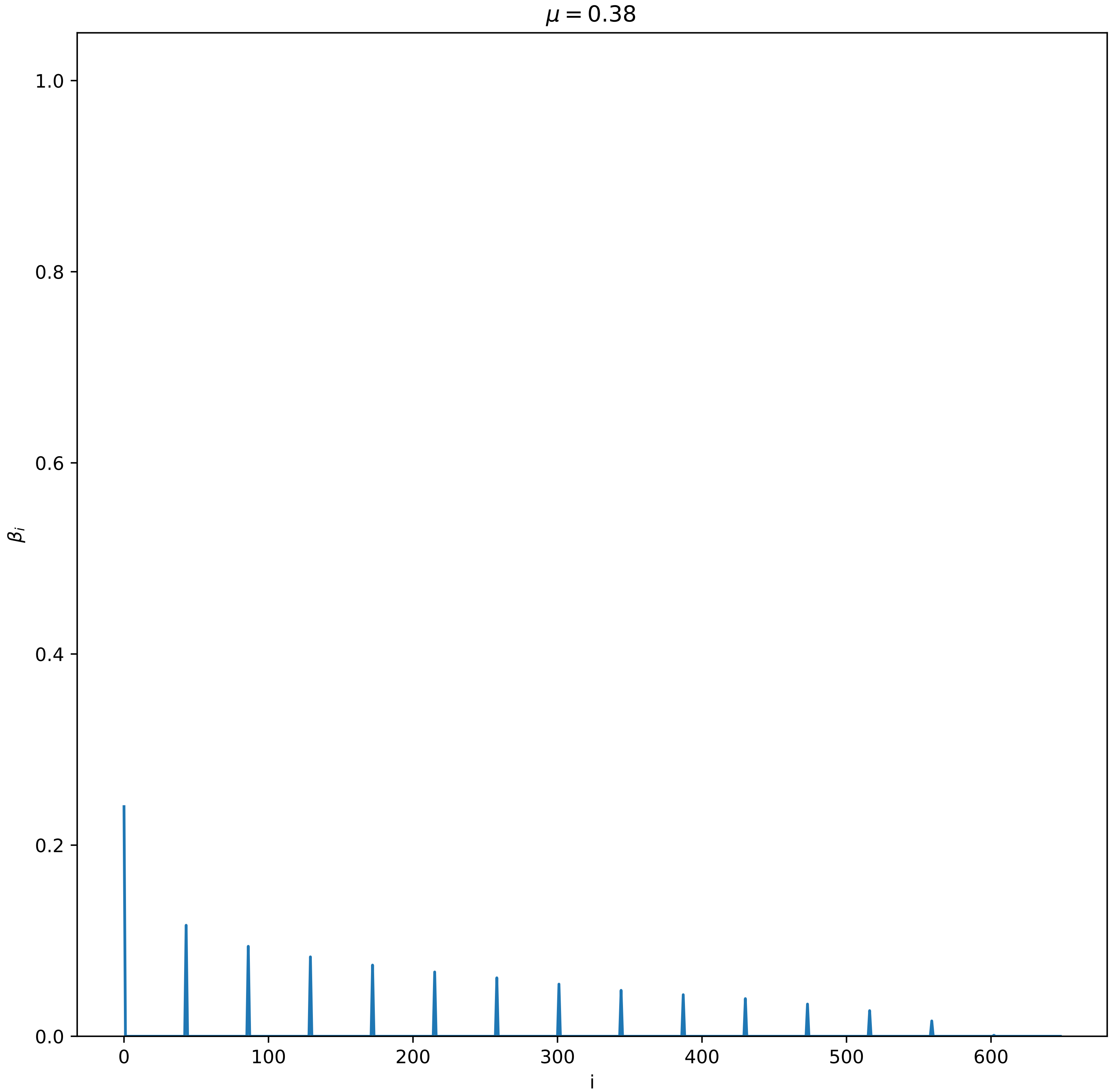


$\mu = 0.35$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

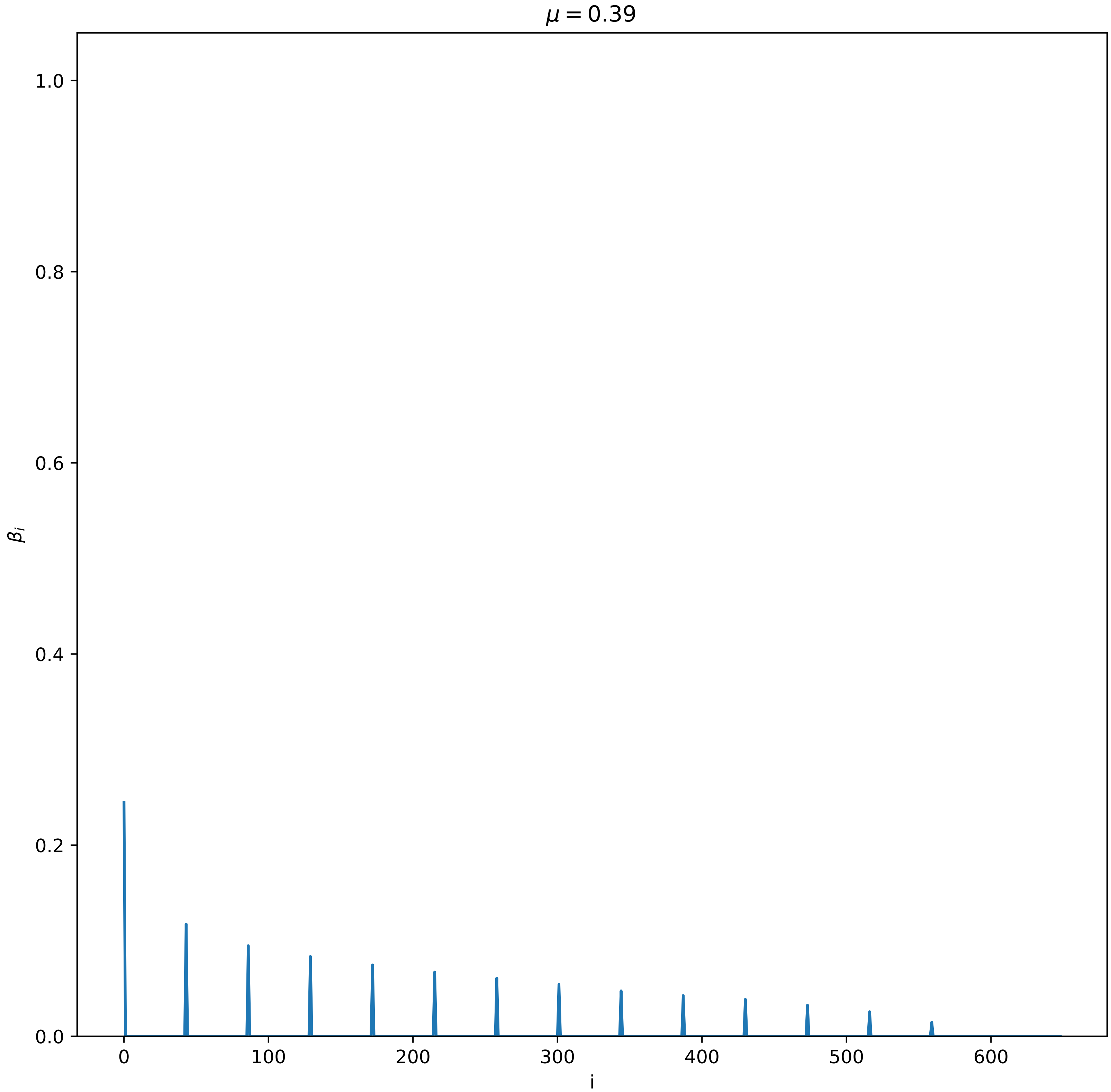

$\mu = 0.36$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



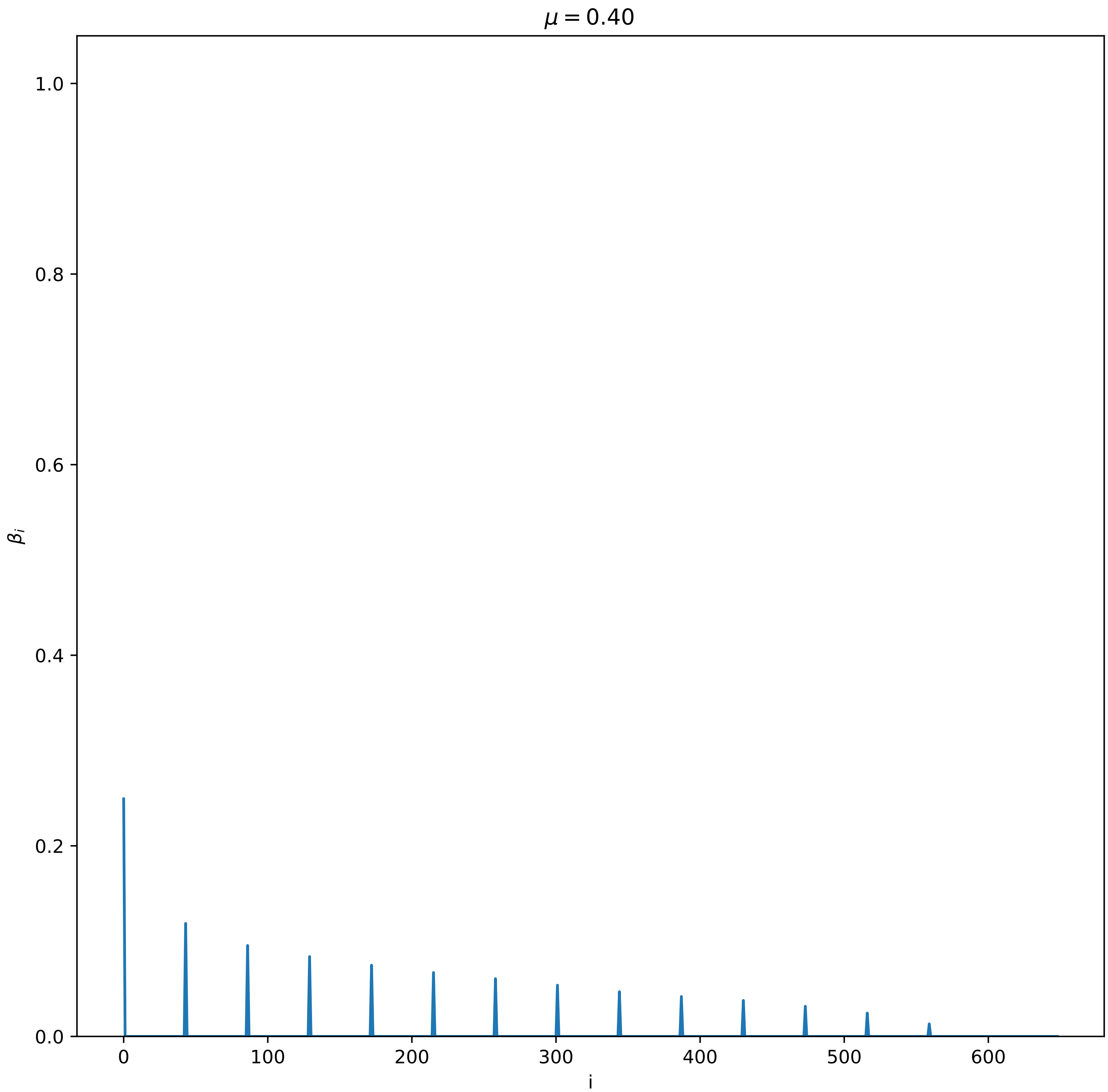
$\mu = 0.37$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



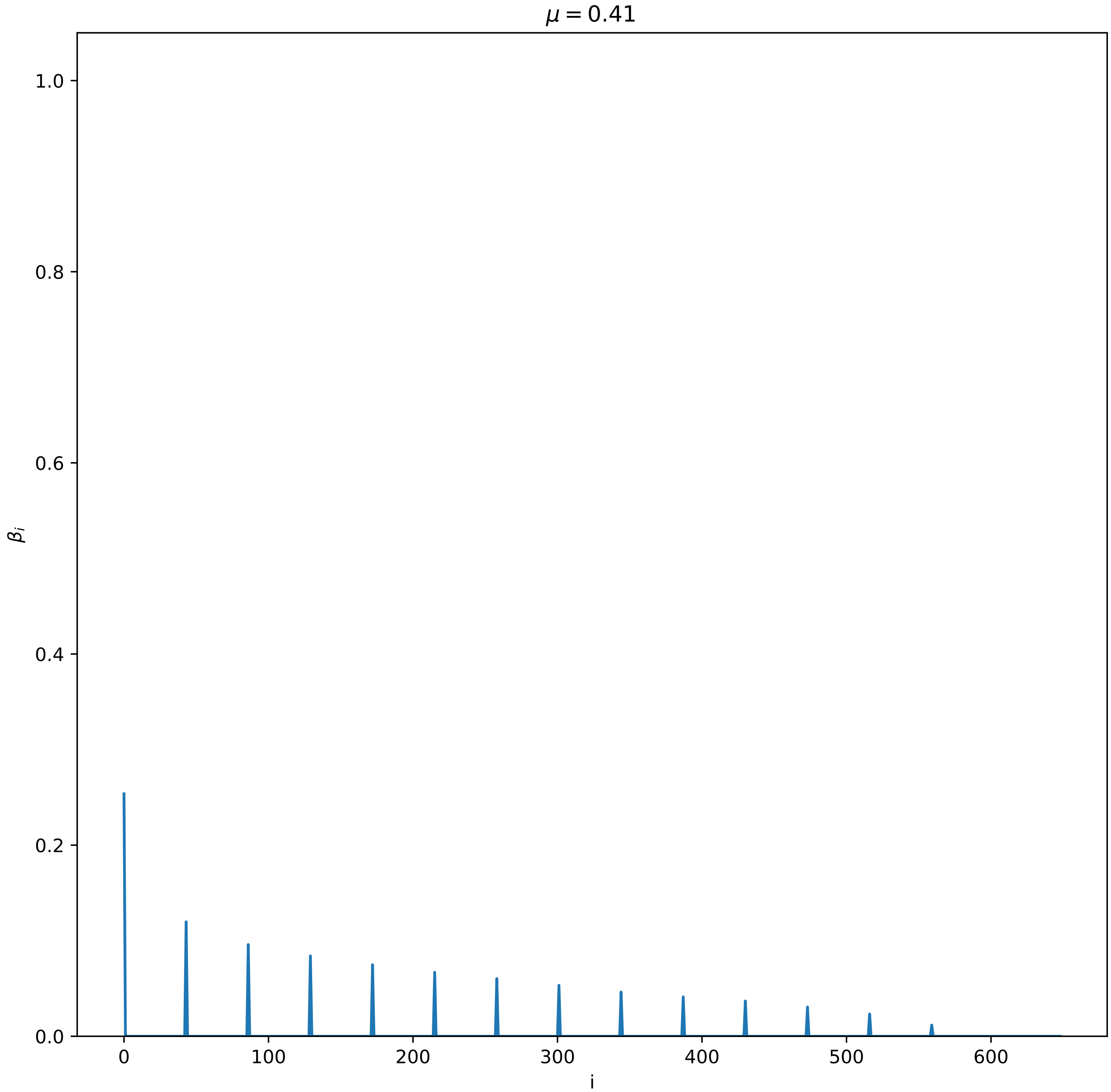
$\mu = 0.38$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



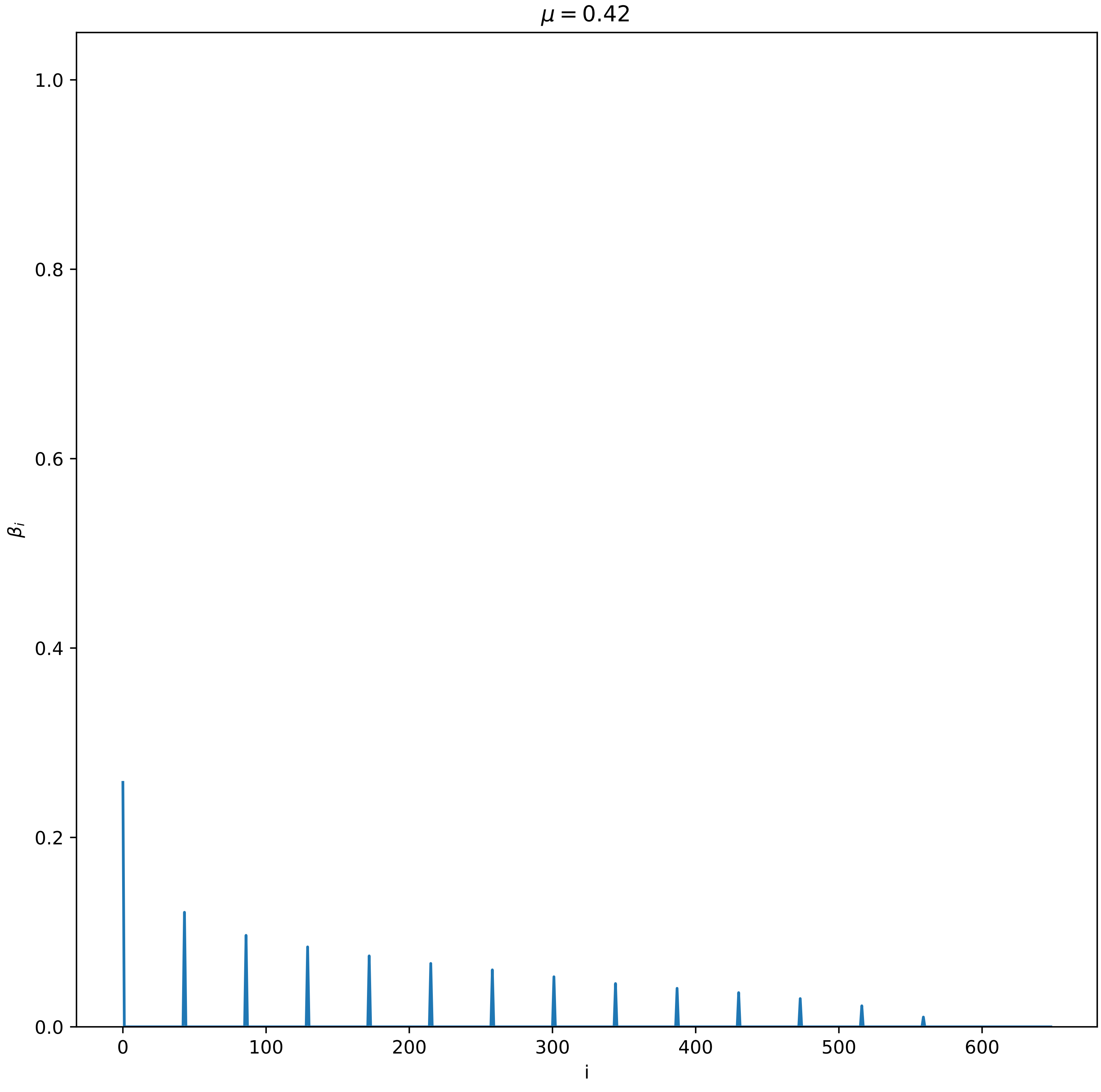
$\mu = 0.39$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

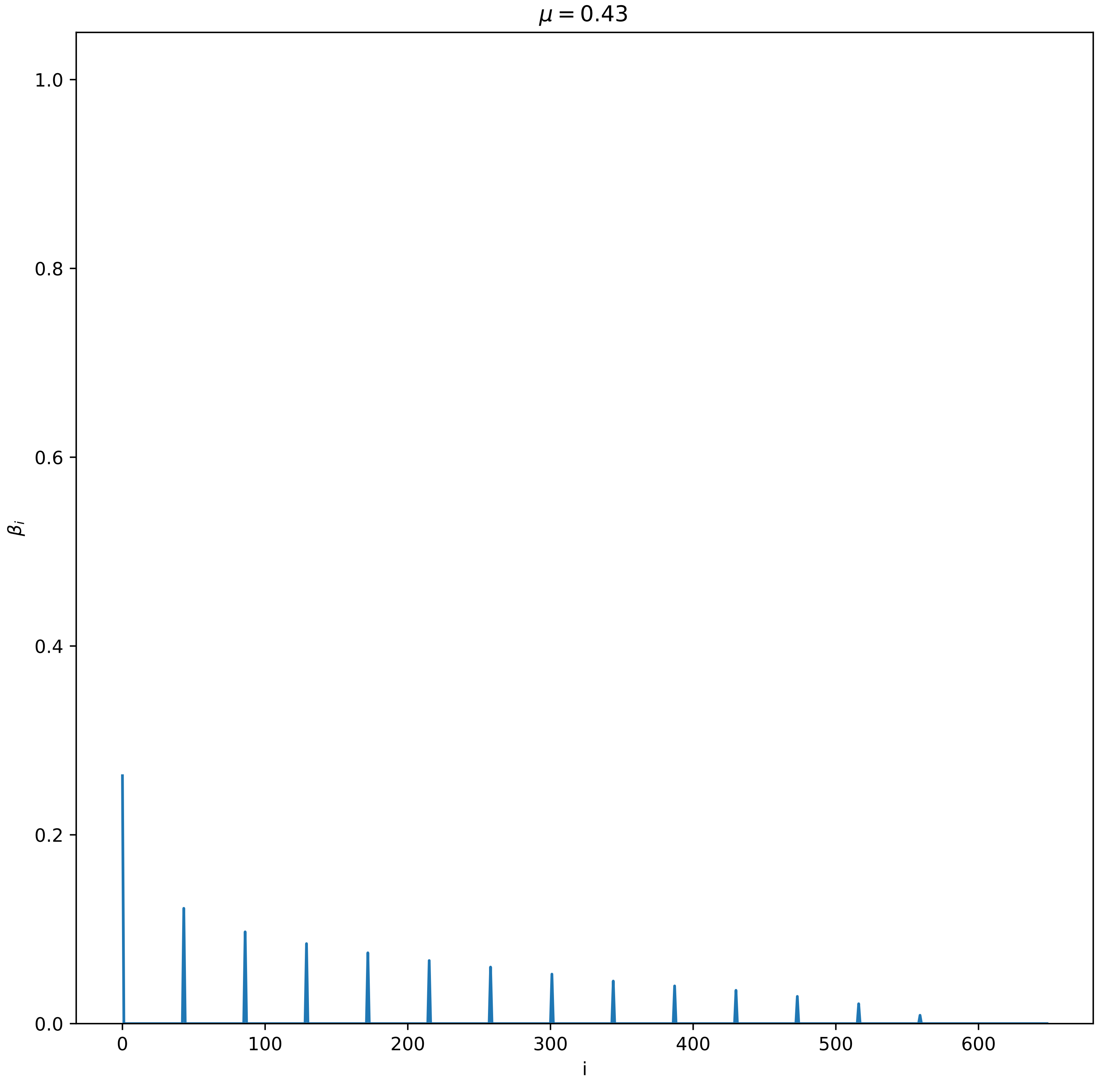


$\mu = 0.40$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

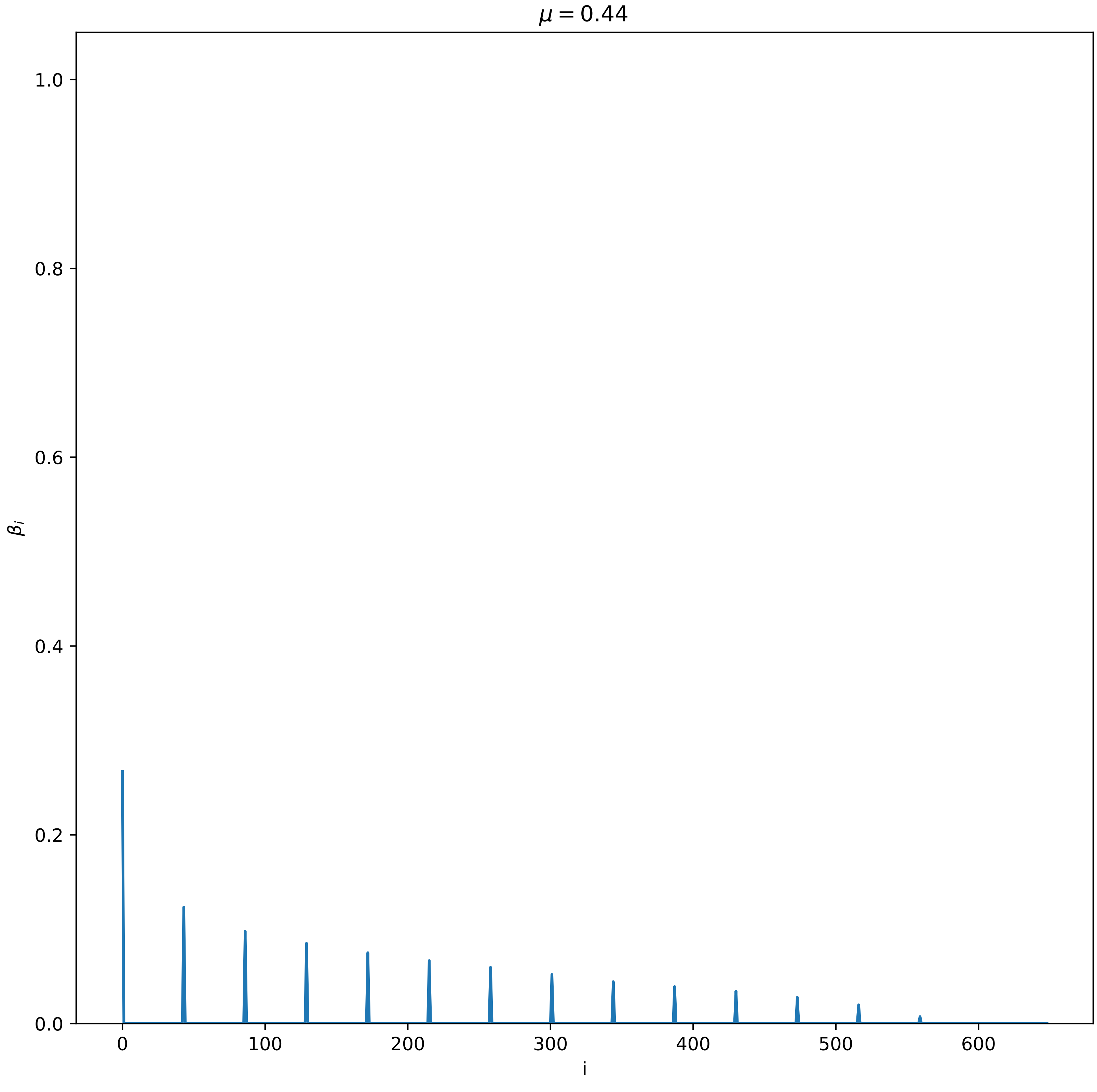


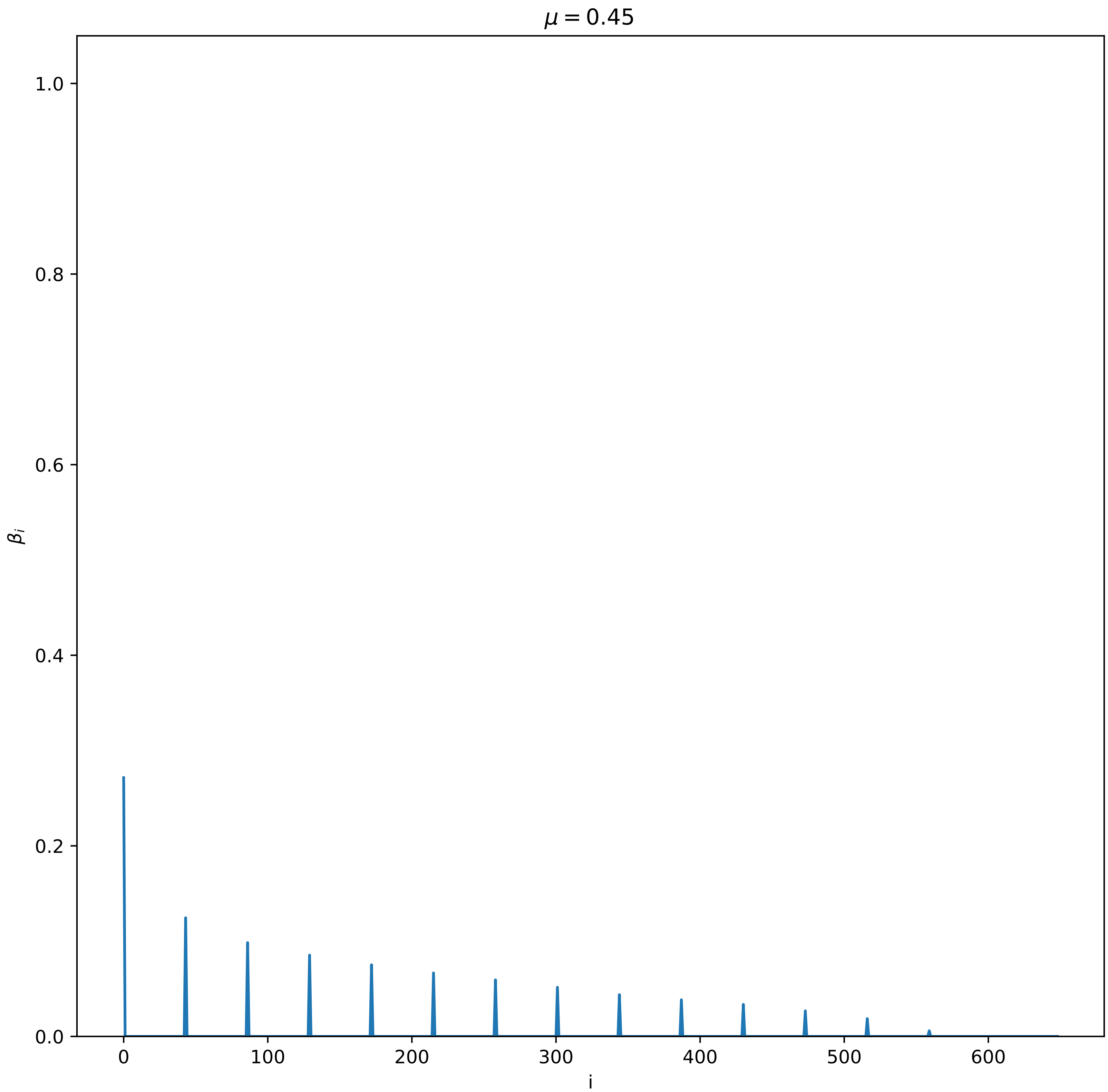
$\mu = 0.41$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 0.42$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

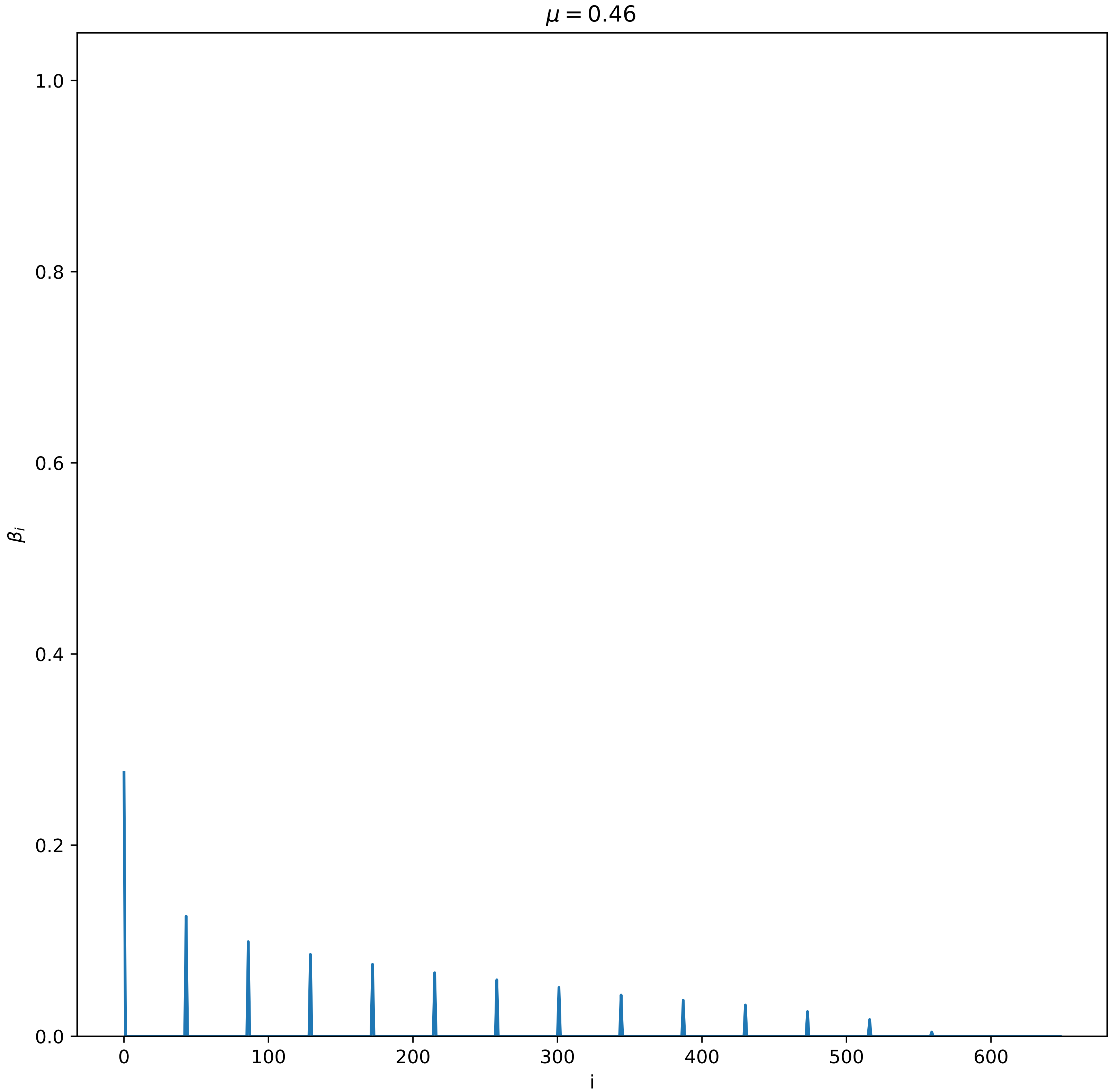


$\mu = 0.43$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

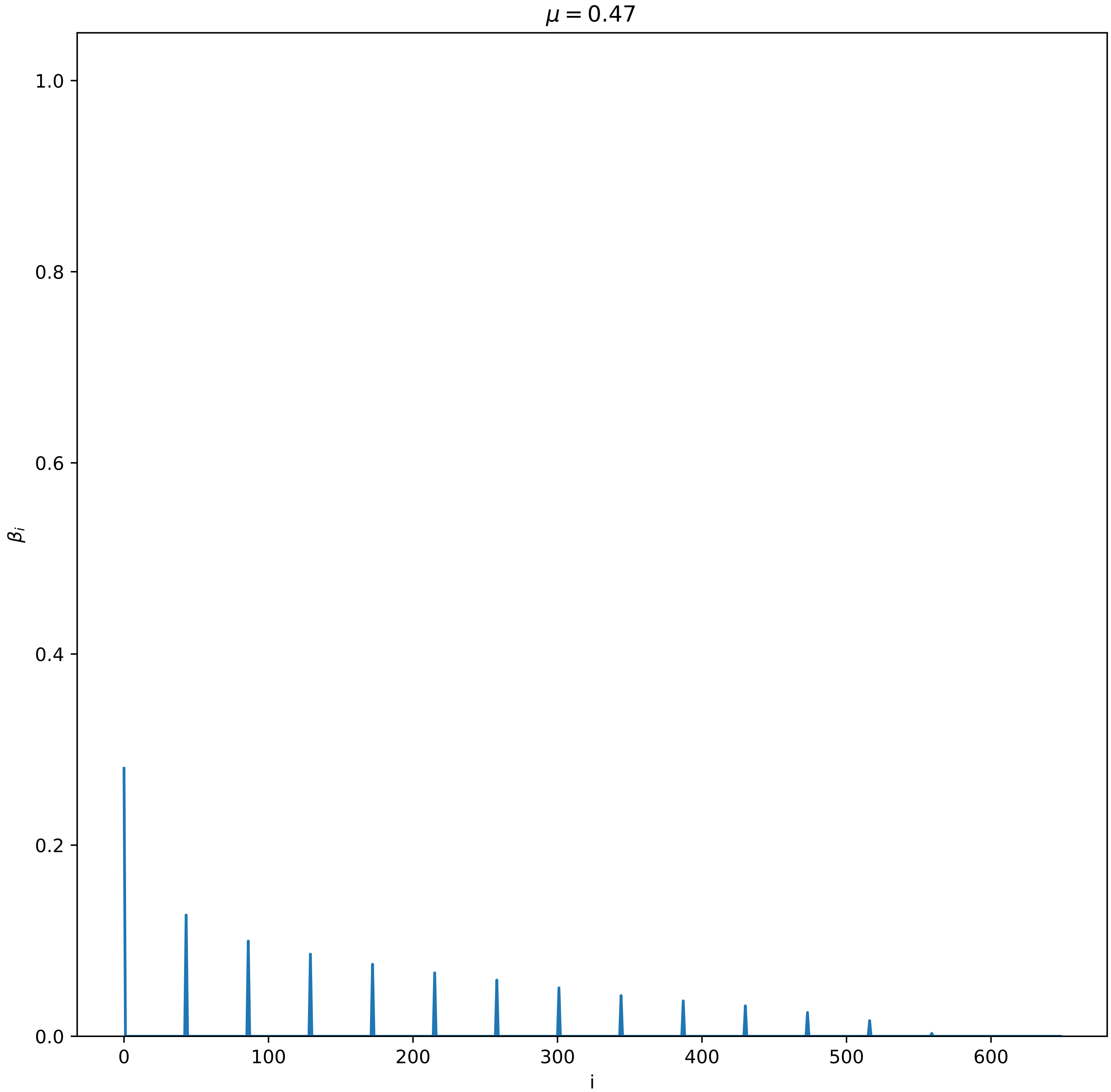

$\mu = 0.44$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



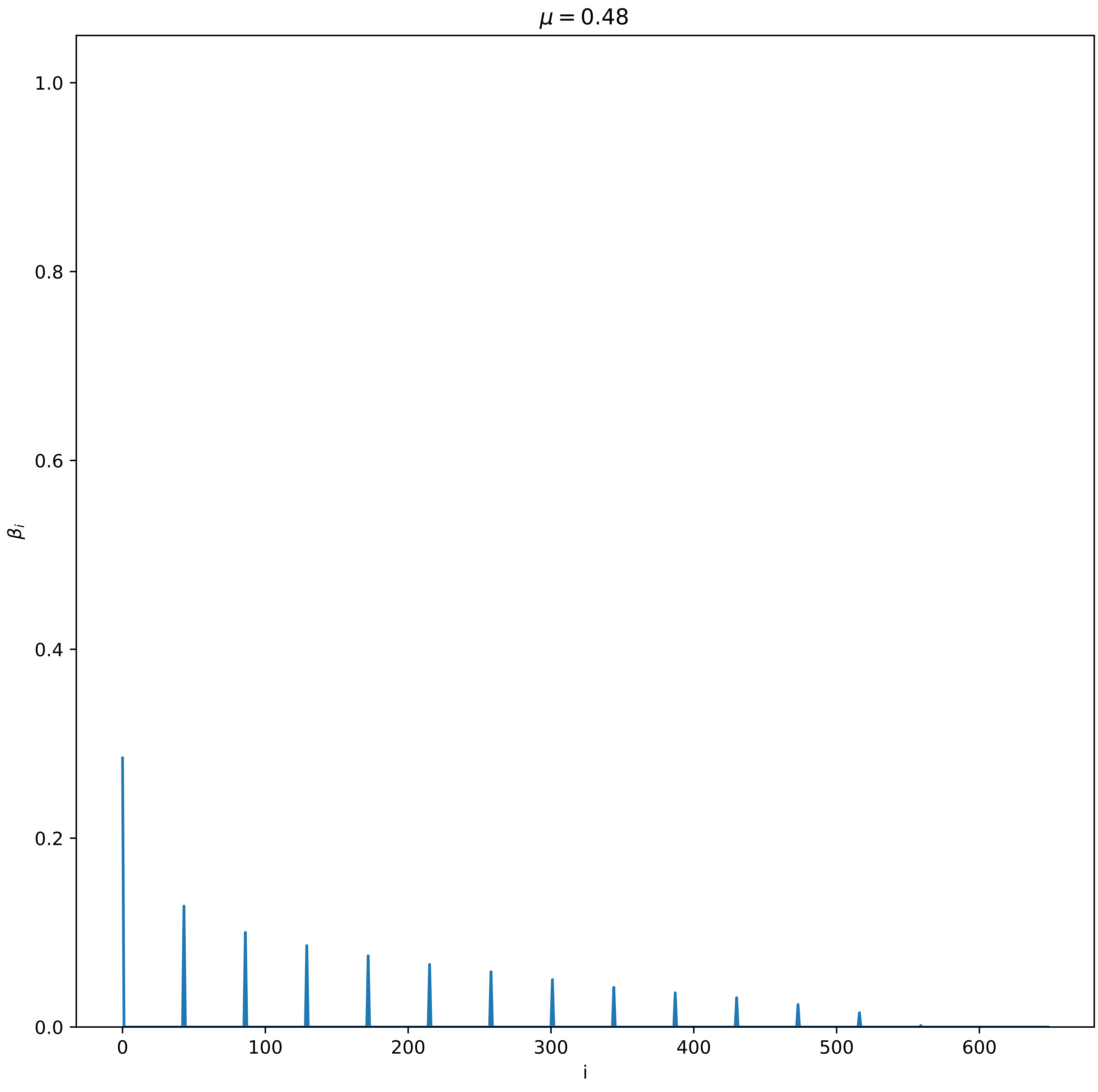
$\mu = 0.45$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



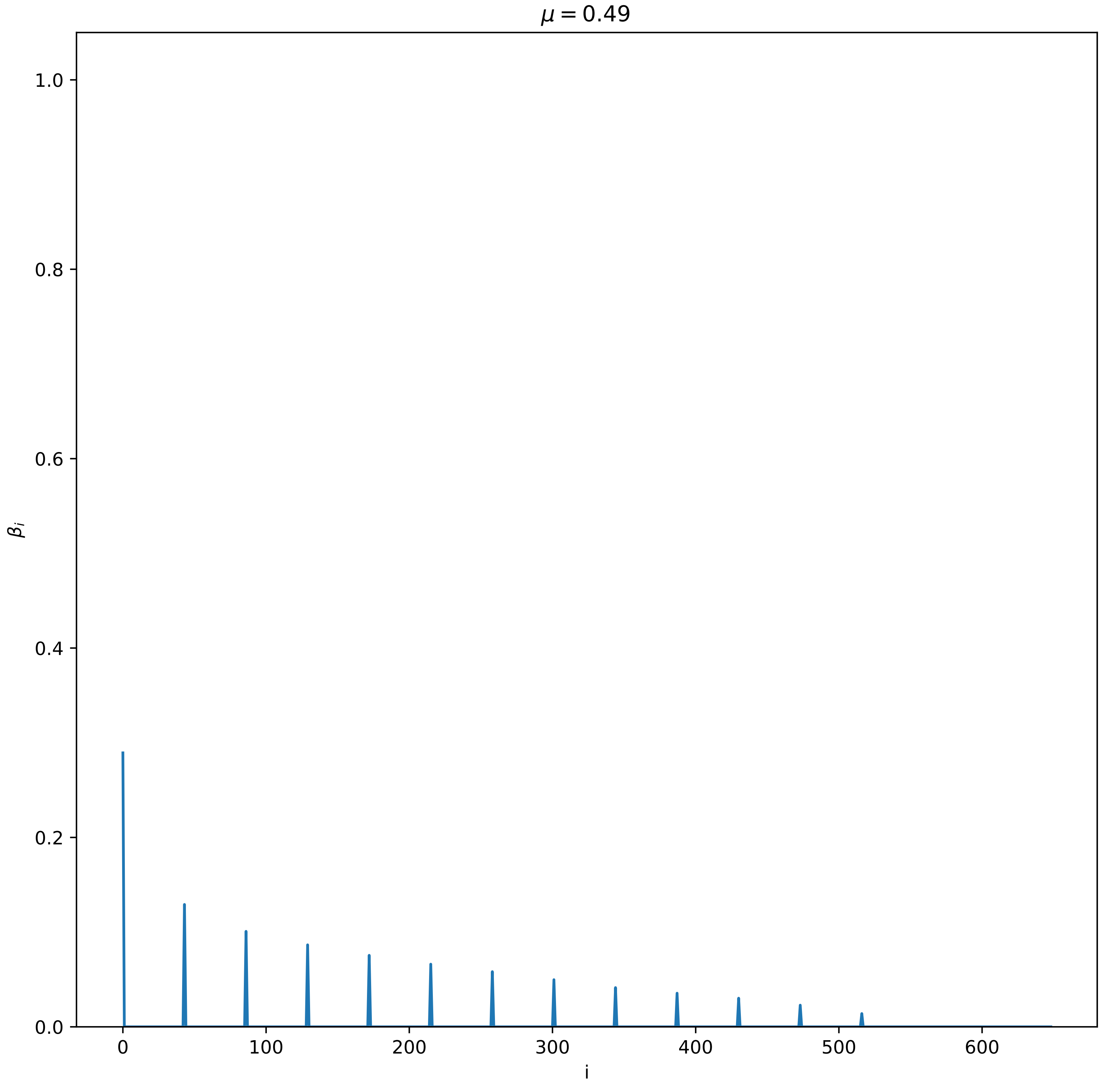
$\mu = 0.46$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



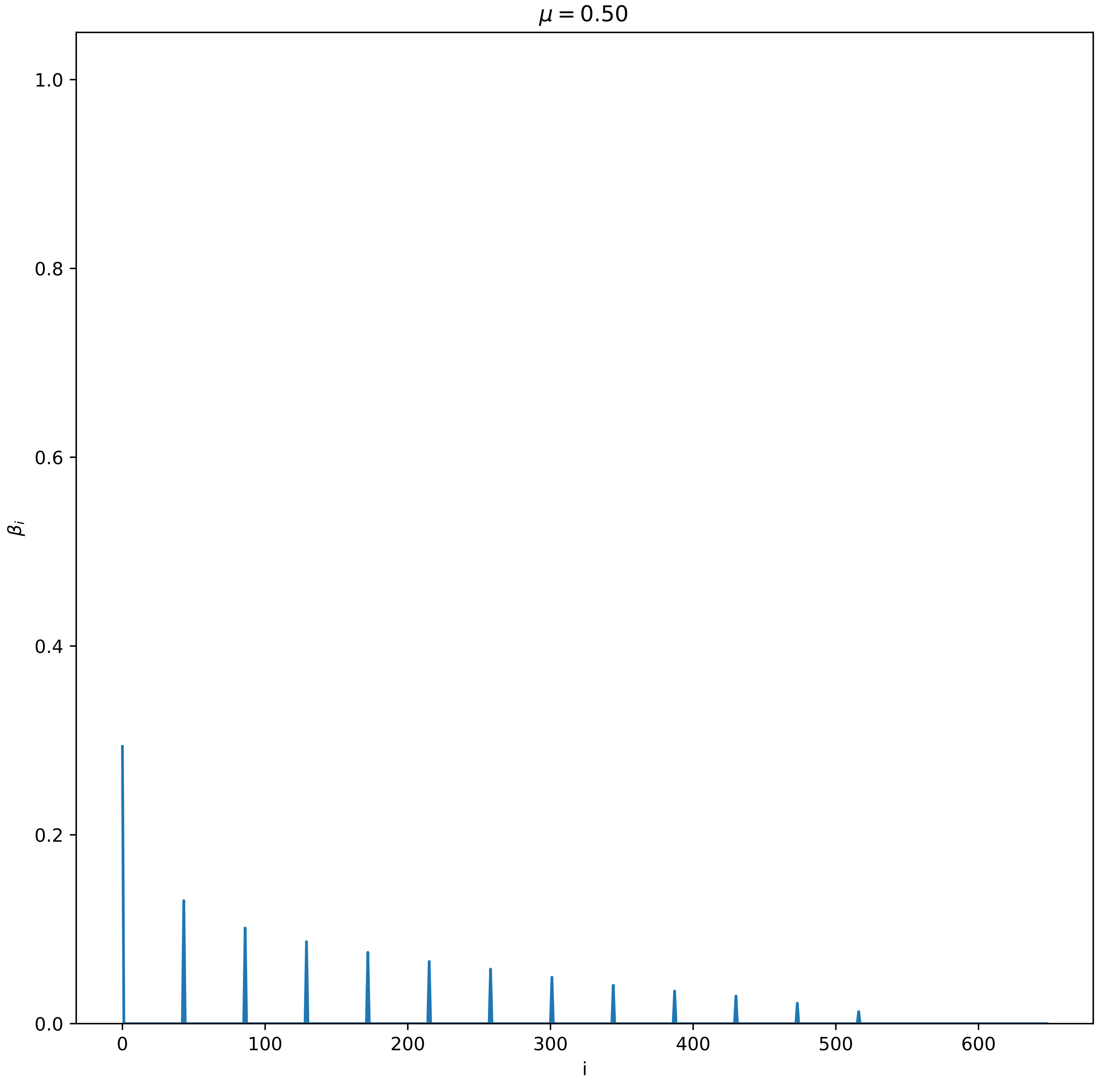
$\mu = 0.47$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



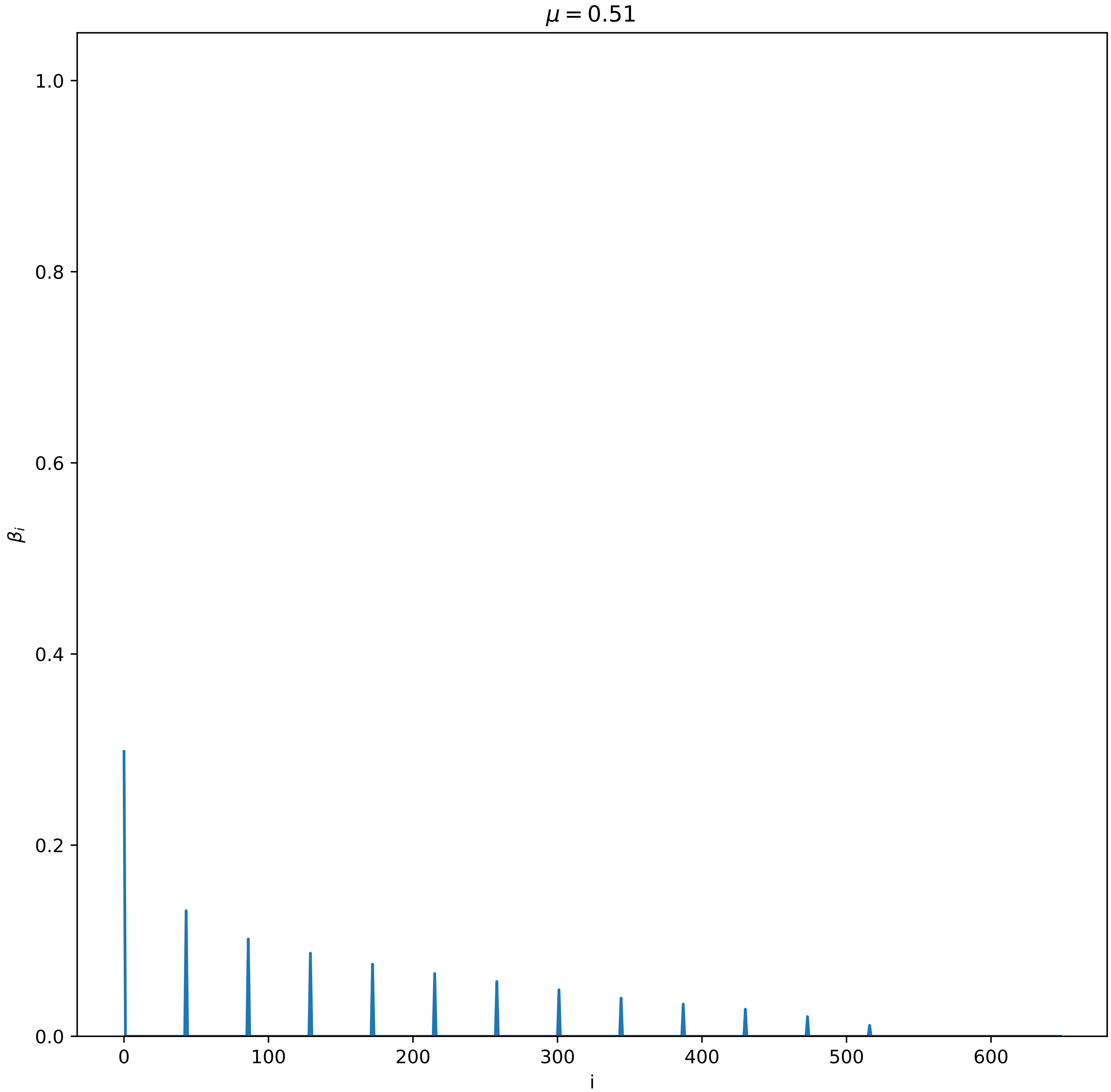
$\mu = 0.48$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

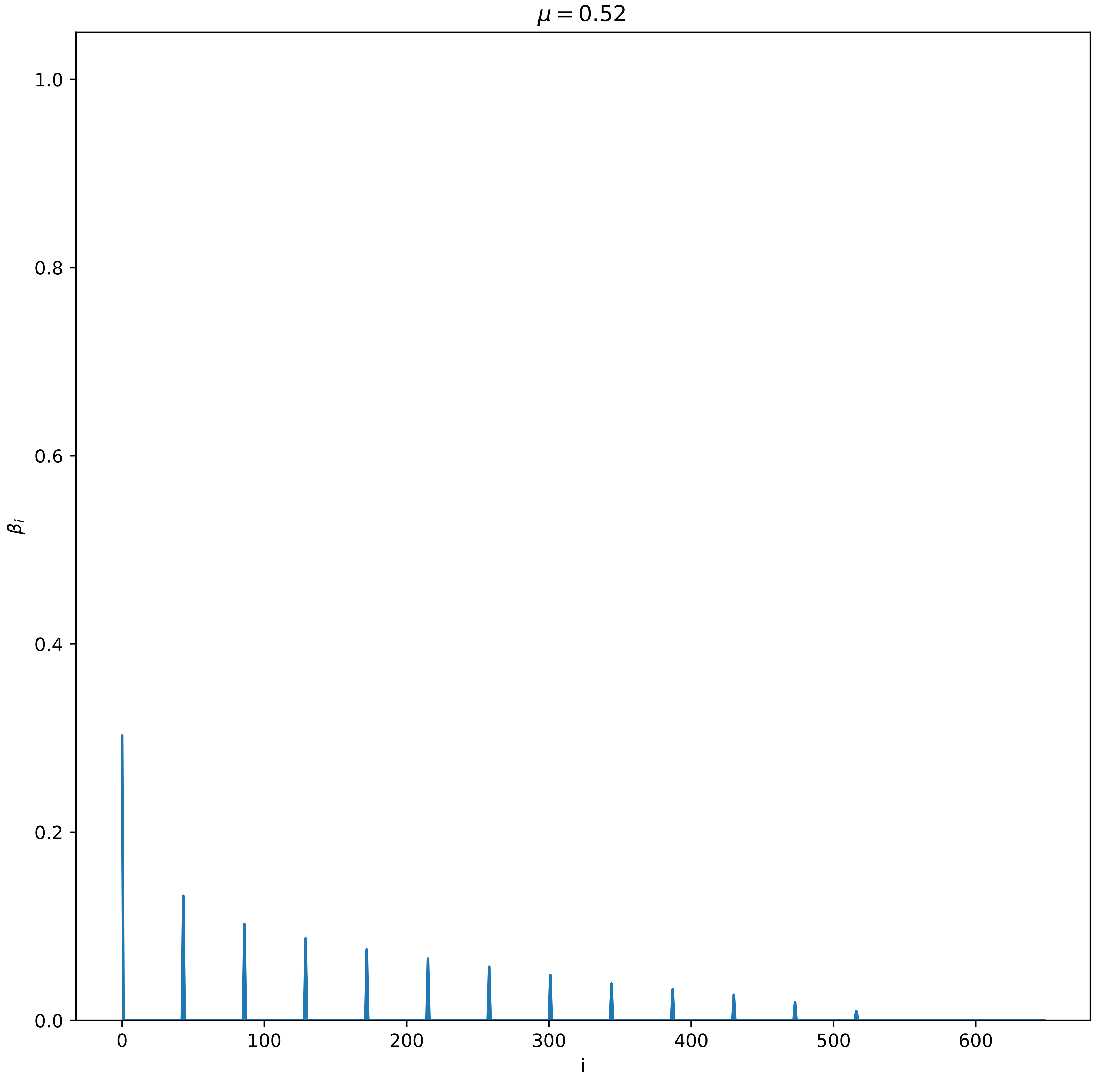


$\mu = 0.49$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

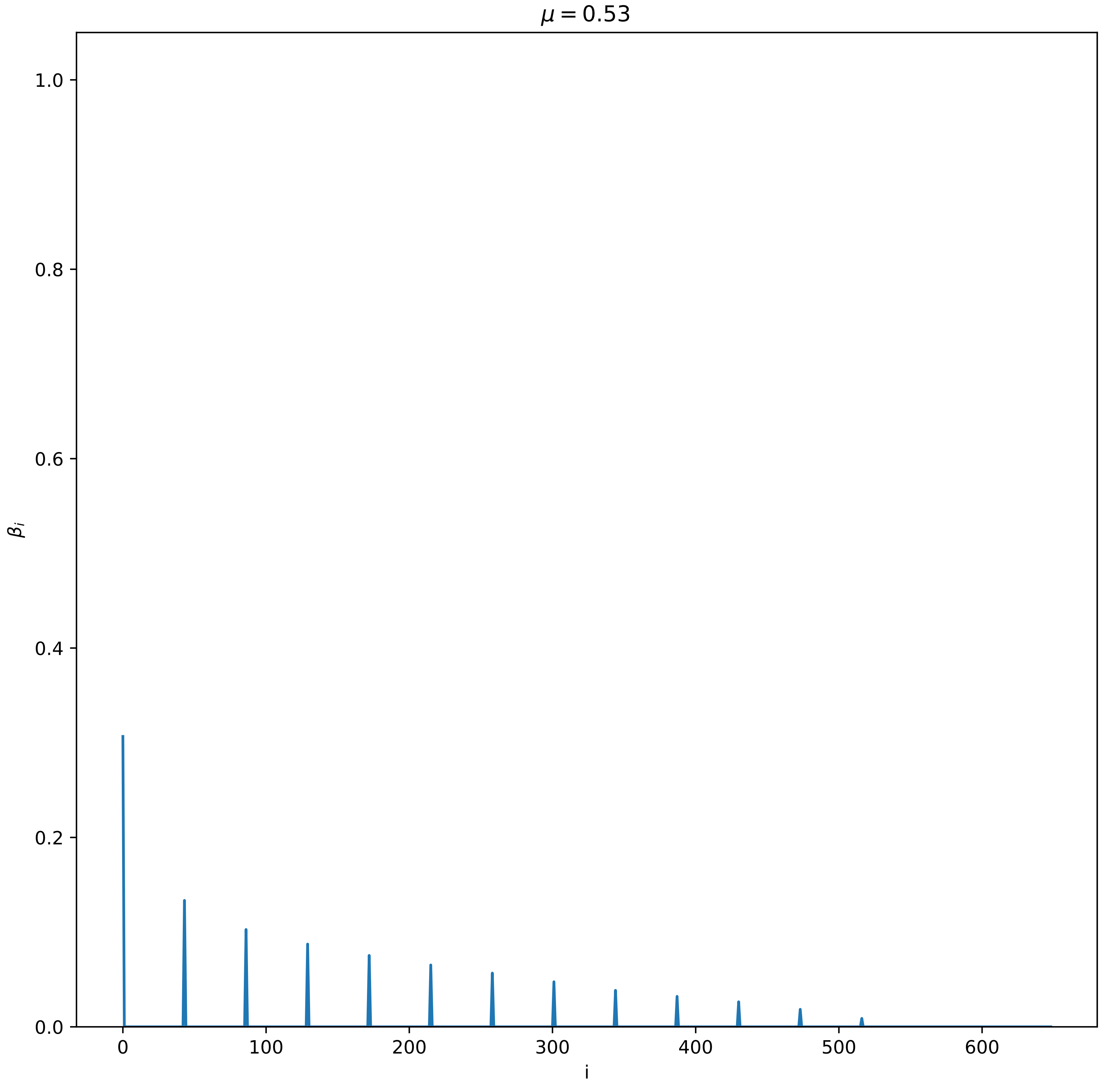


$\mu = 0.50$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

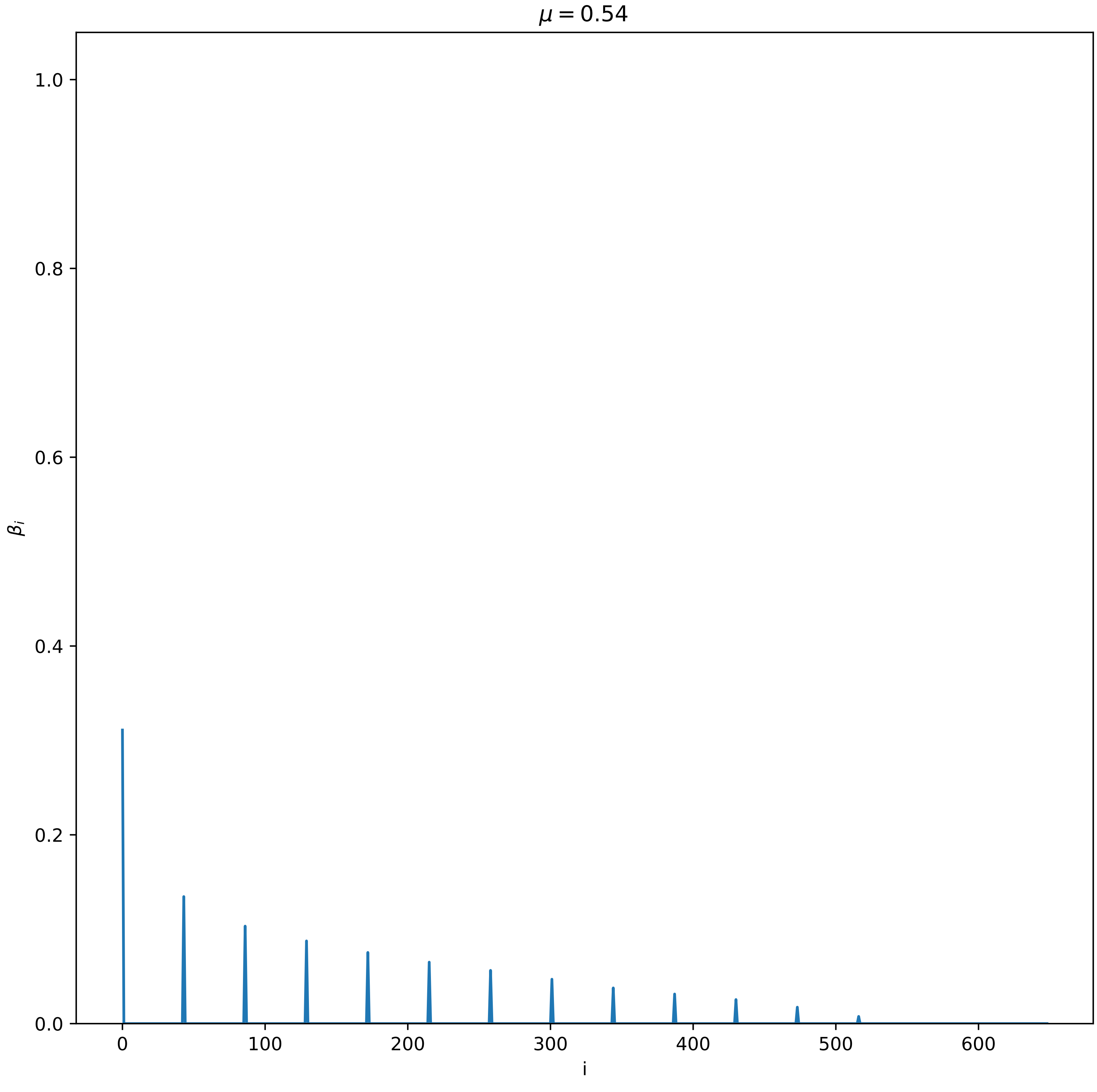

$\mu = 0.51$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



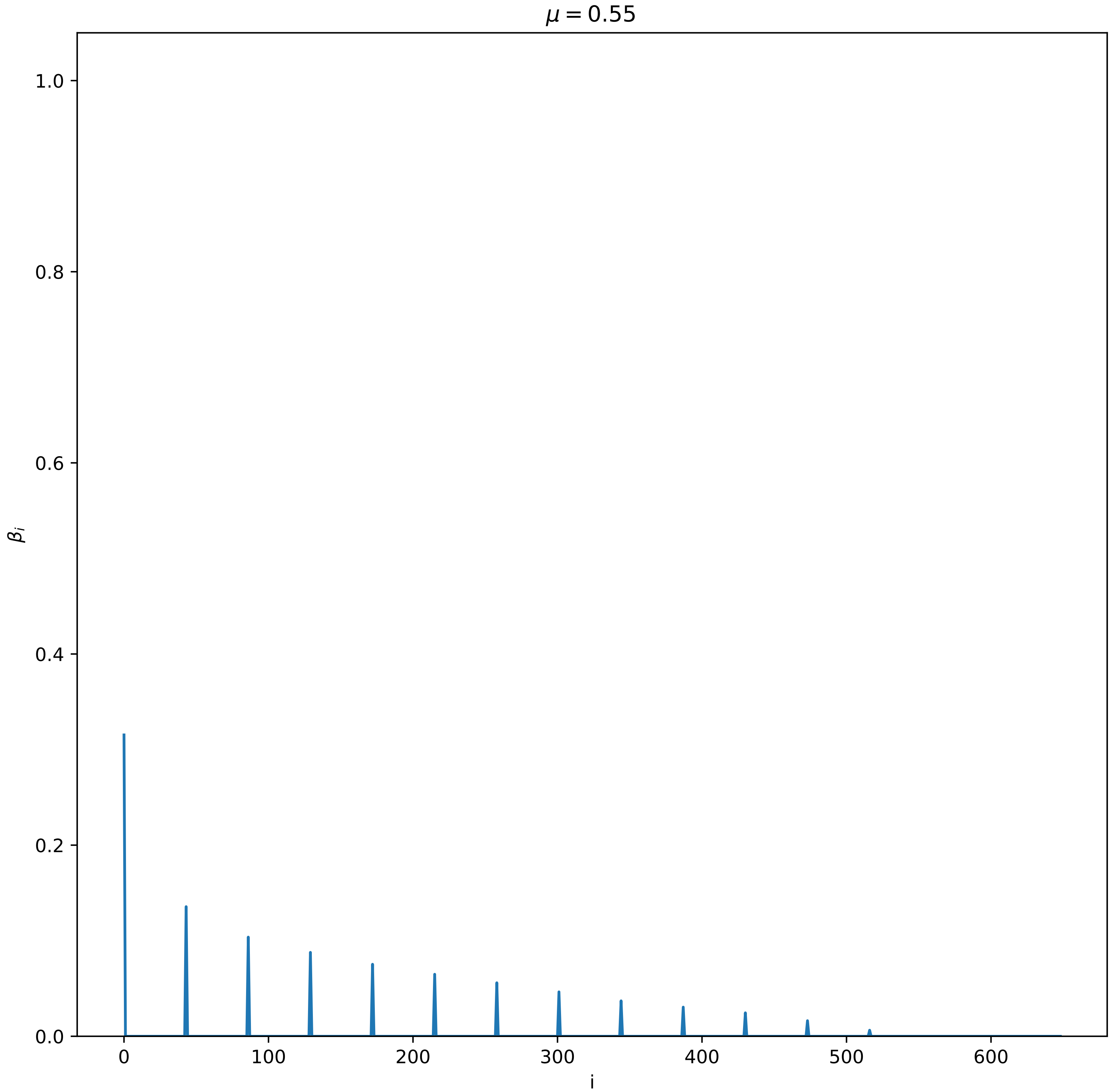
$\mu = 0.52$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

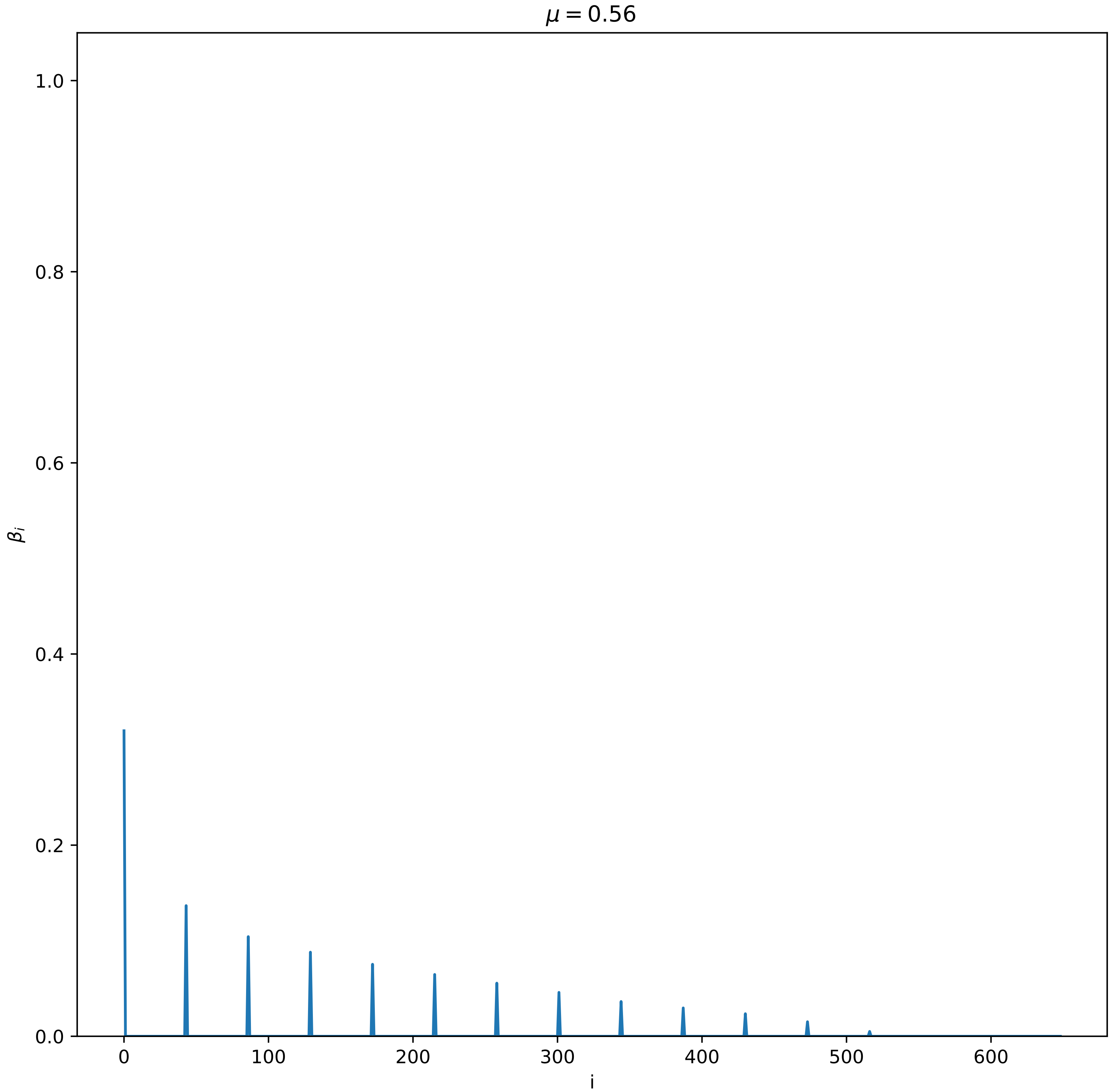


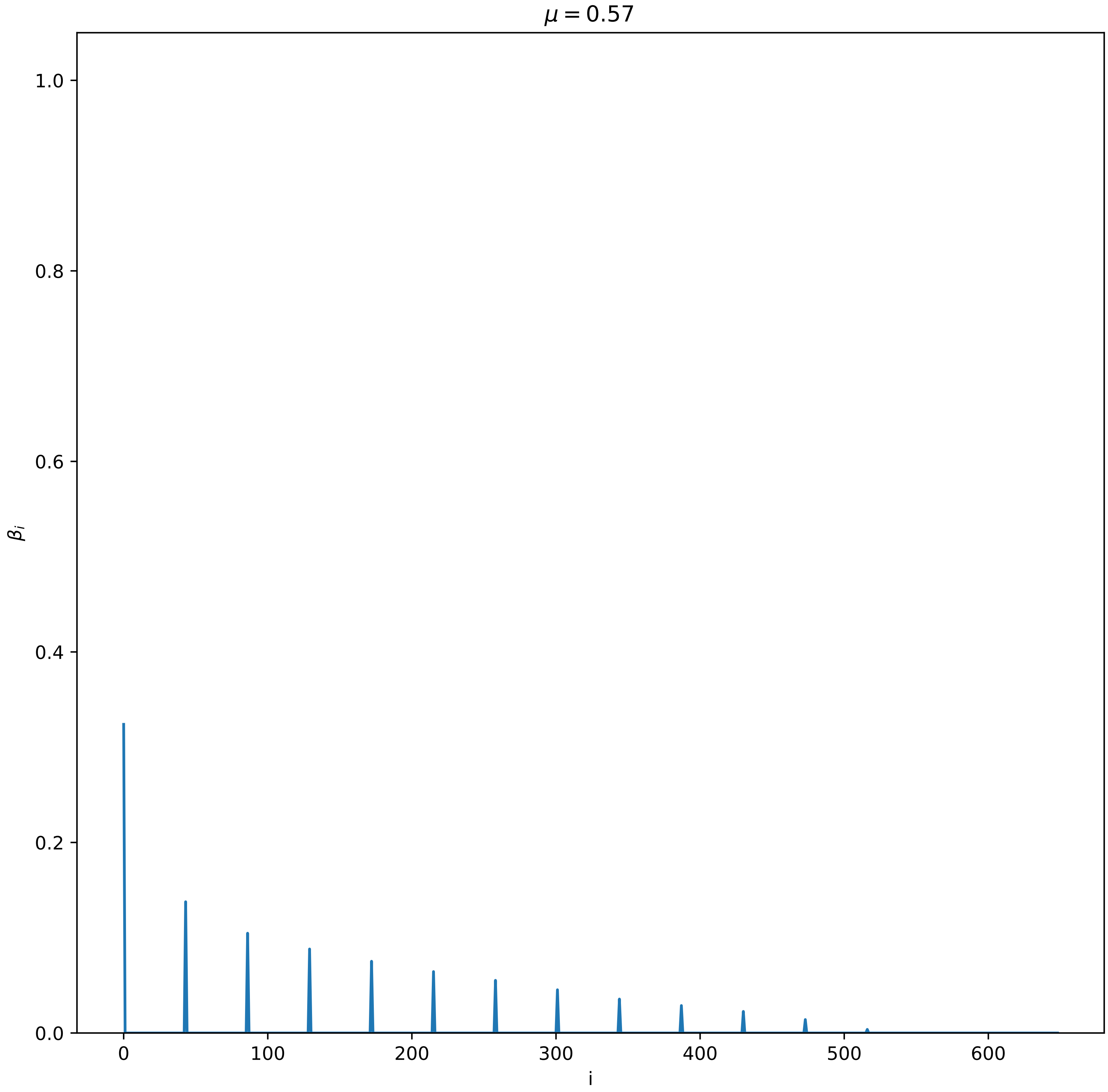
$\mu = 0.53$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



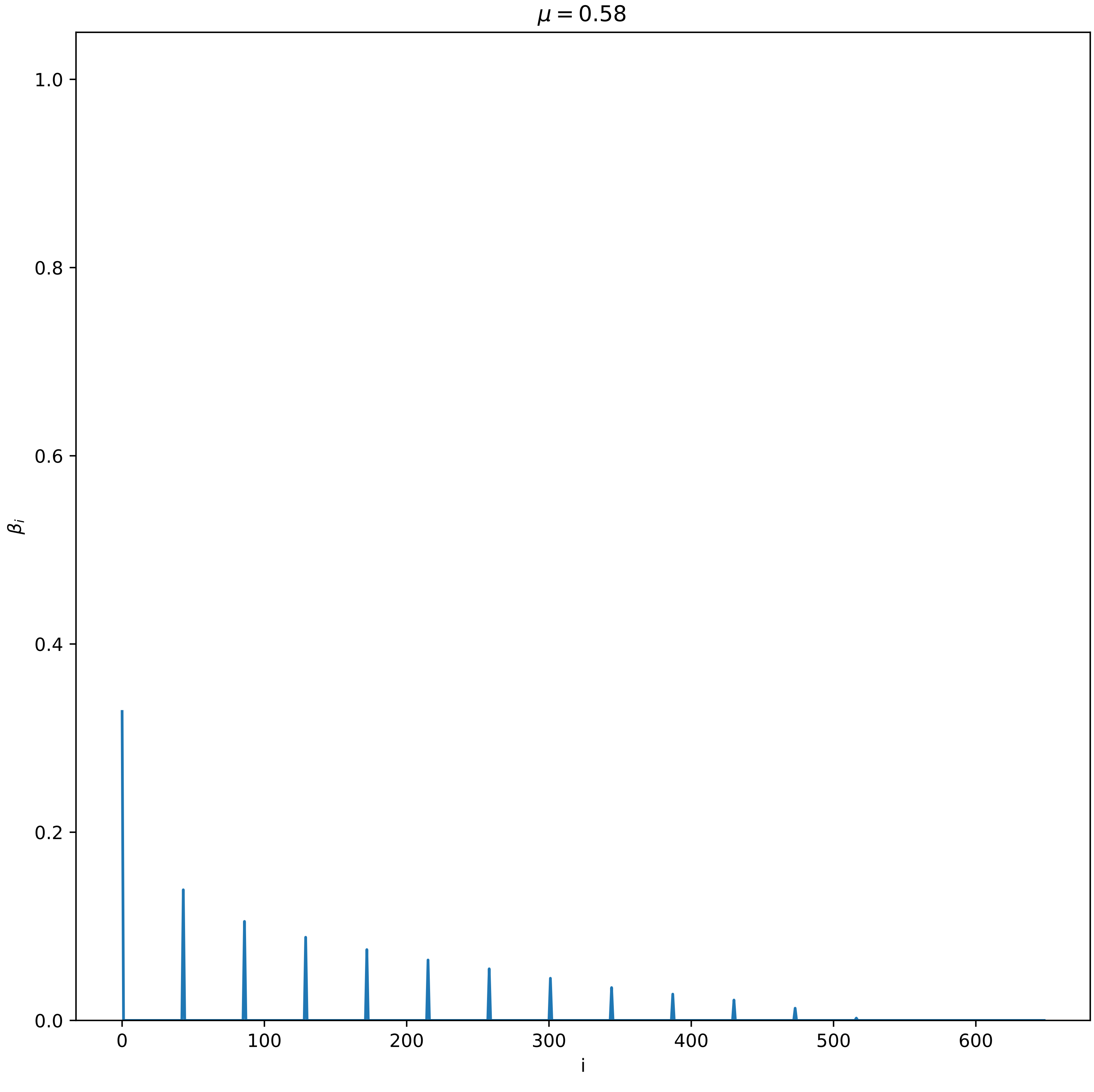
$\mu = 0.54$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 0.55$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

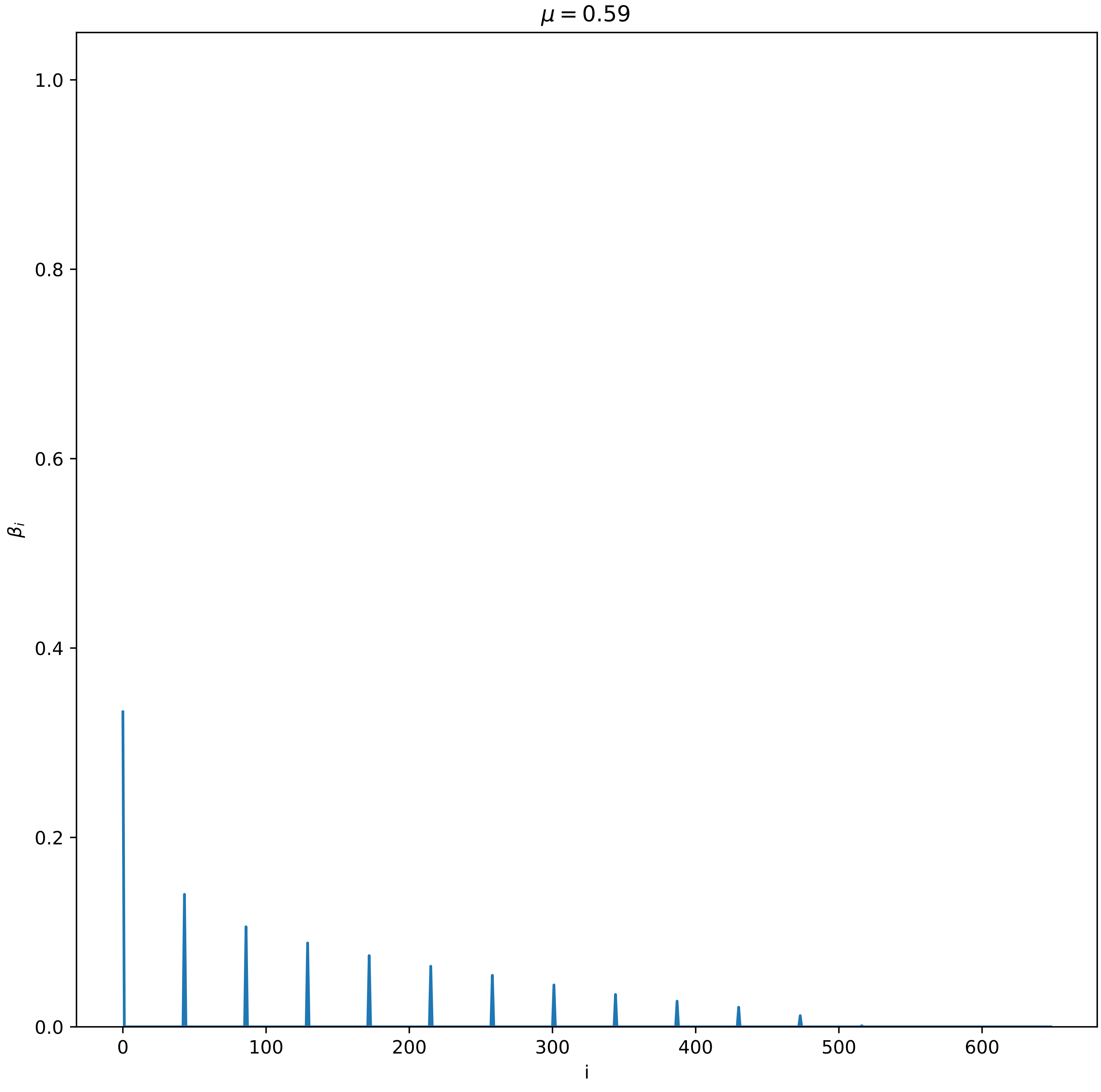

$\mu = 0.56$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



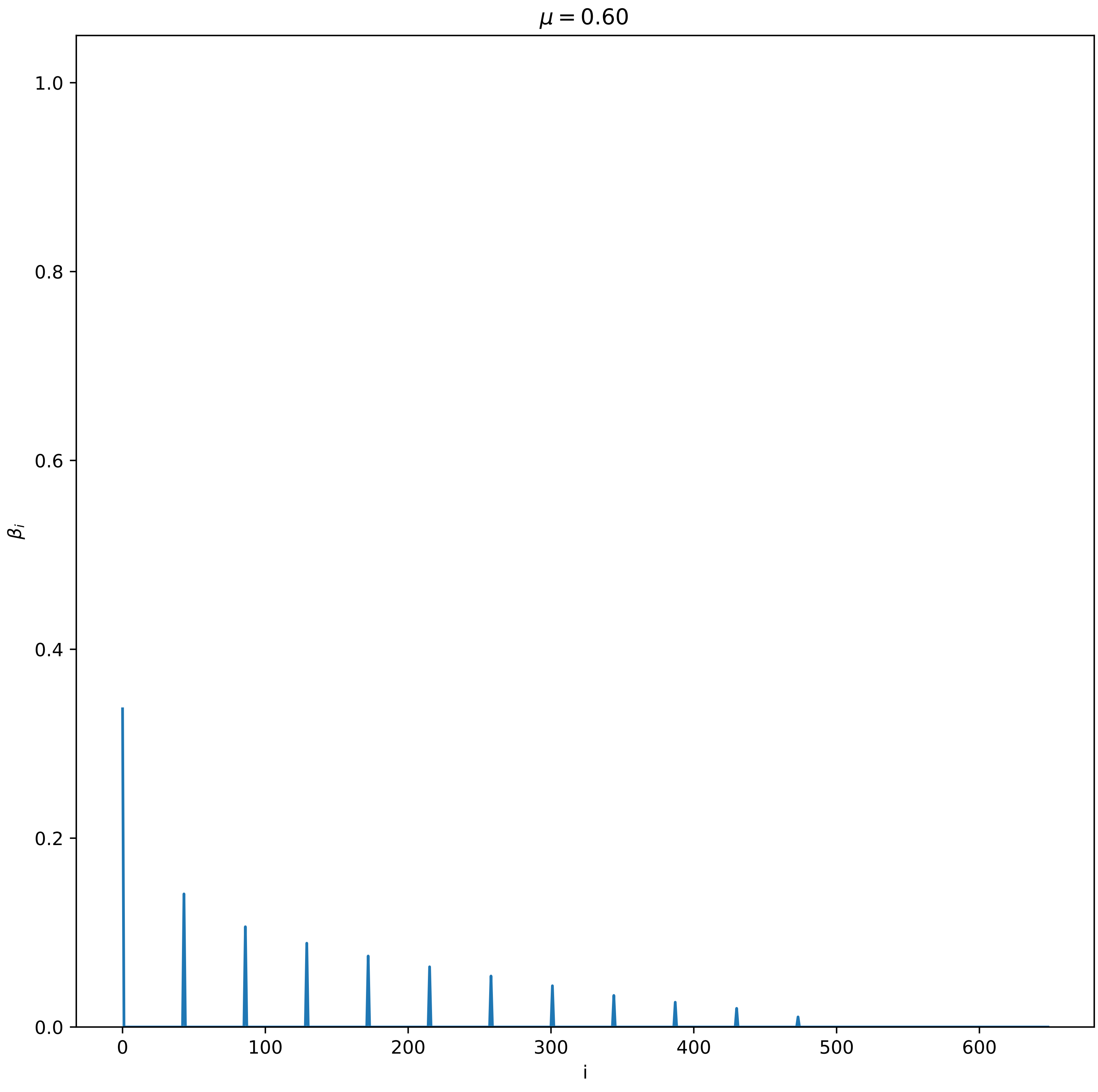
$\mu = 0.57$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



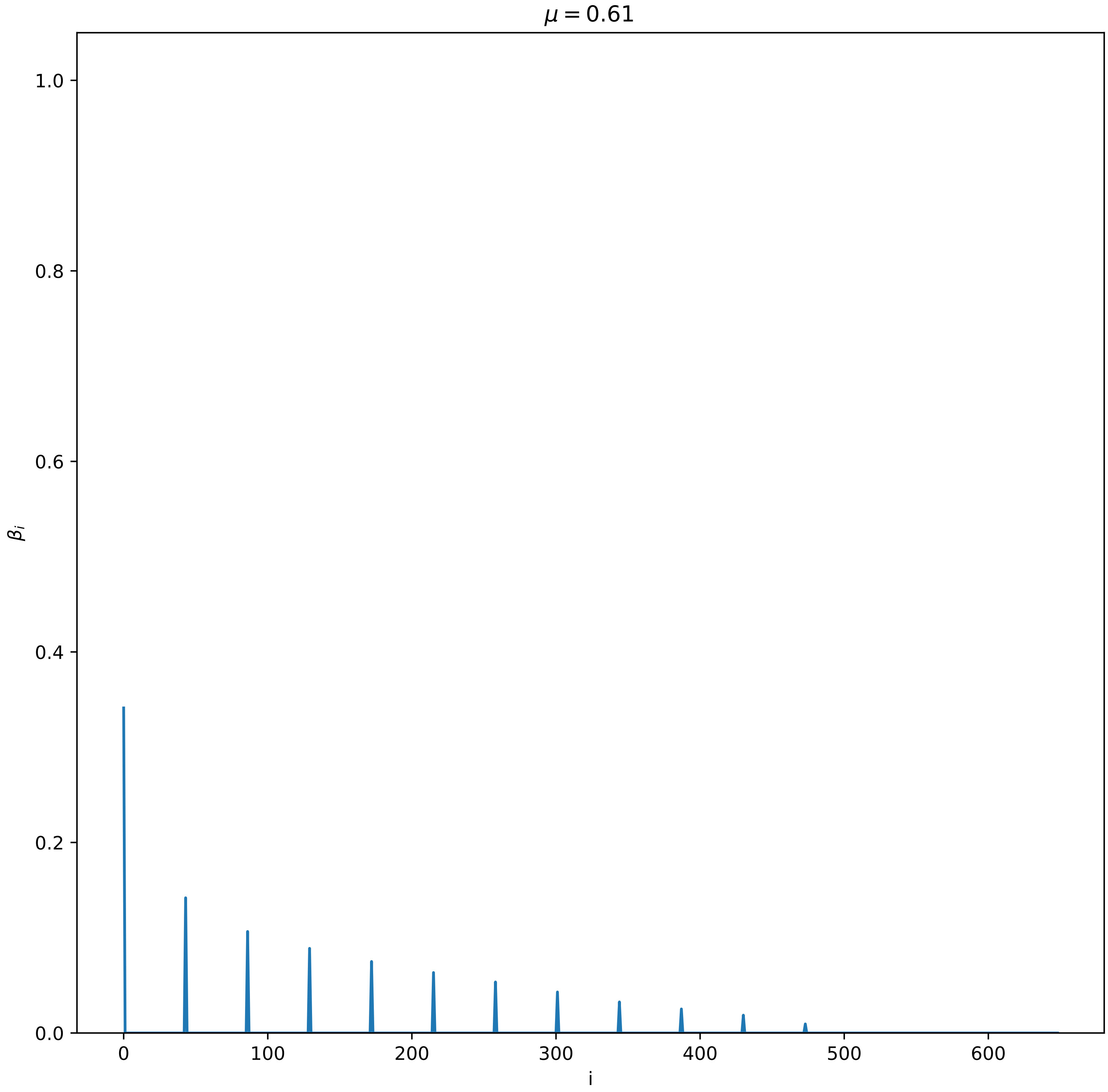
$\mu = 0.58$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



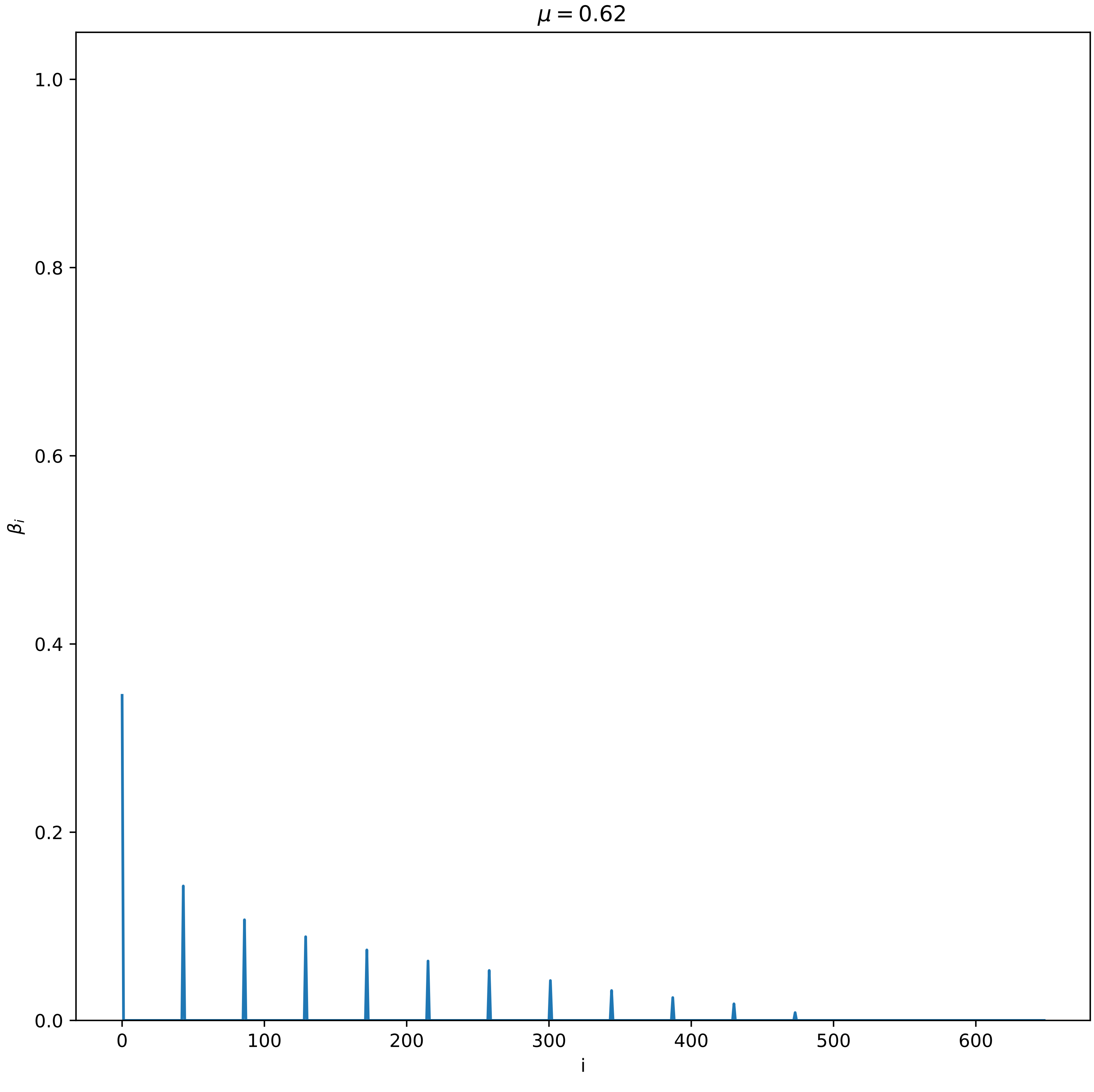
$\mu = 0.59$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



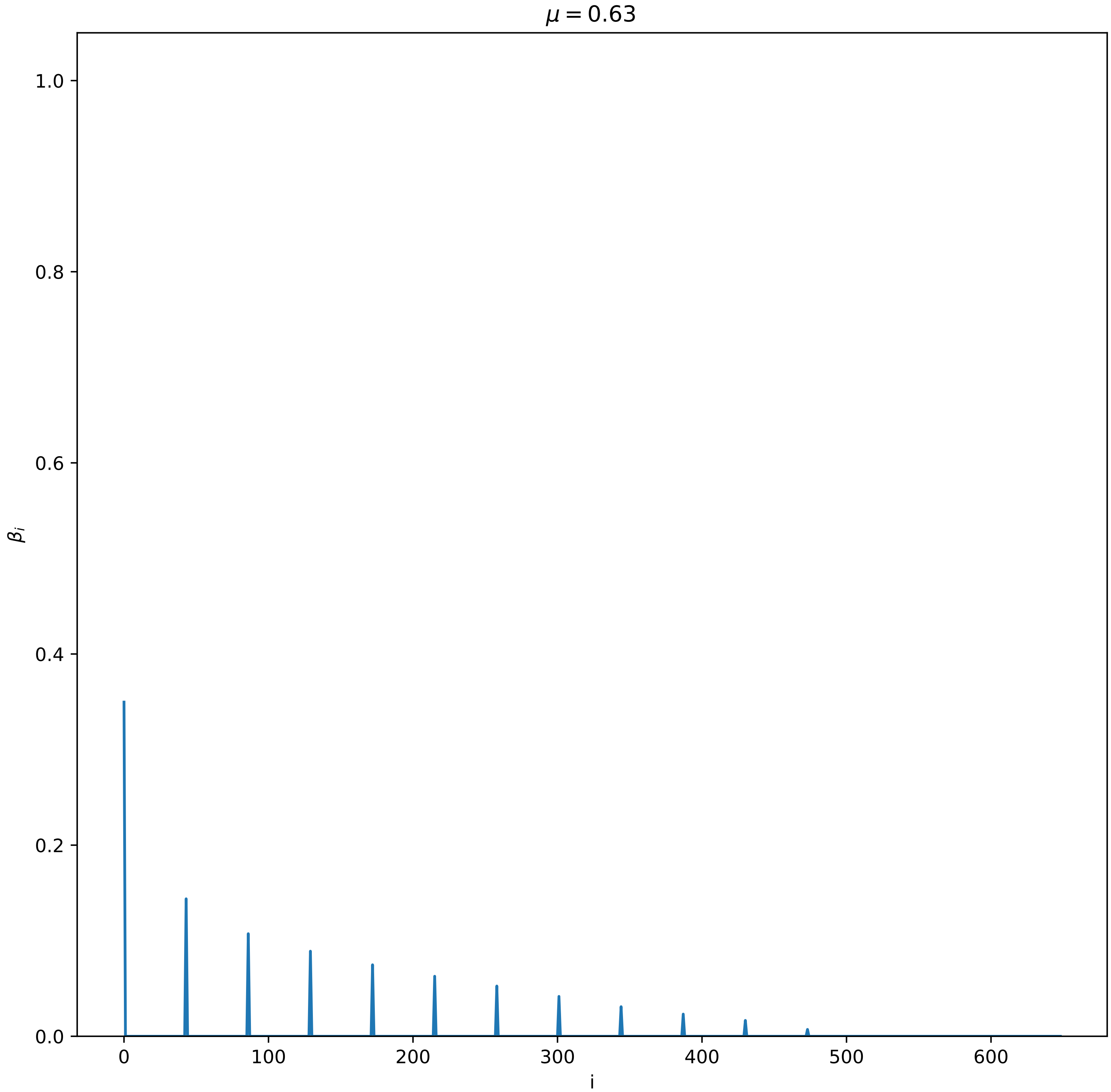
$\mu = 0.60$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



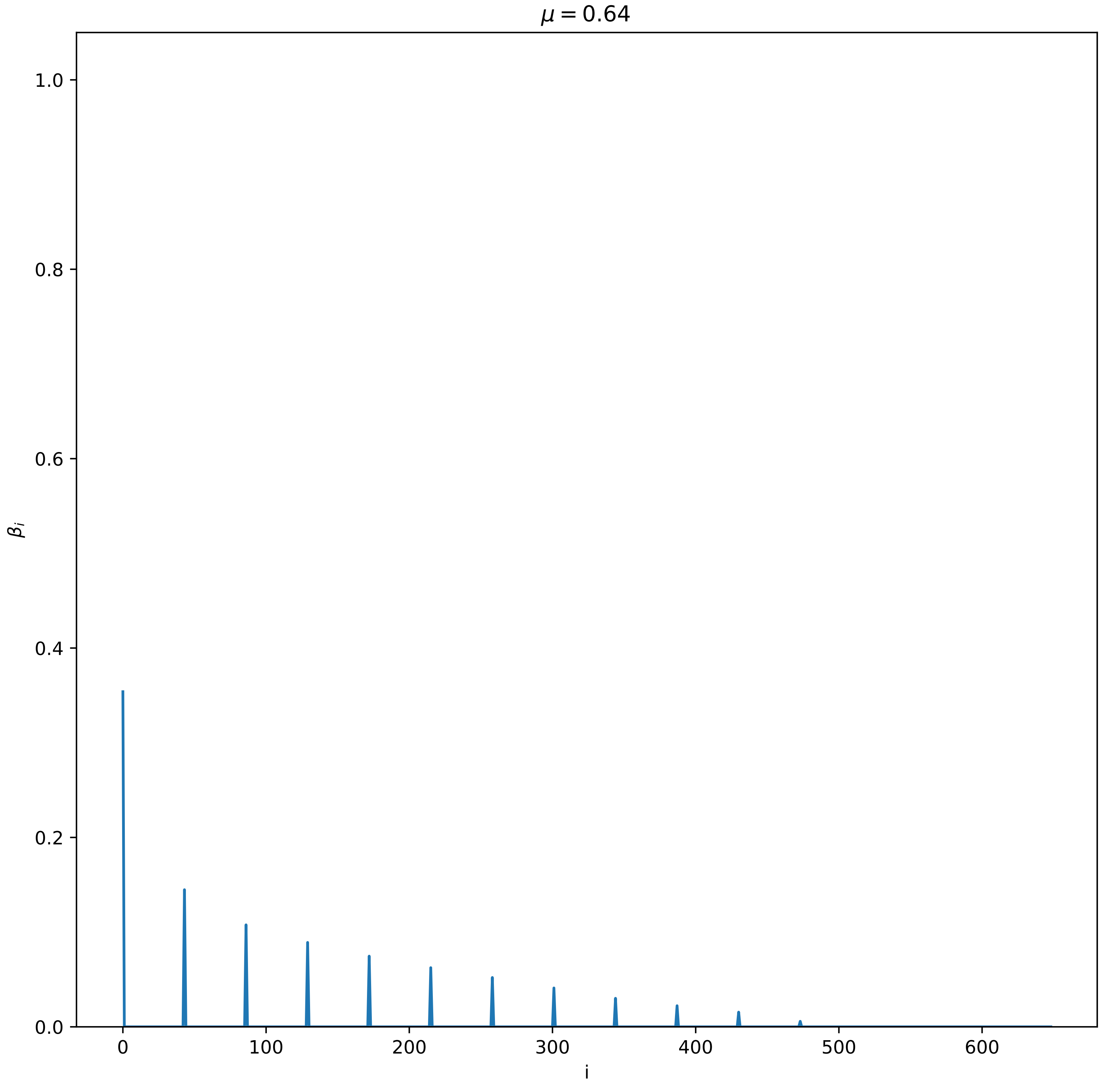
$\mu = 0.61$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



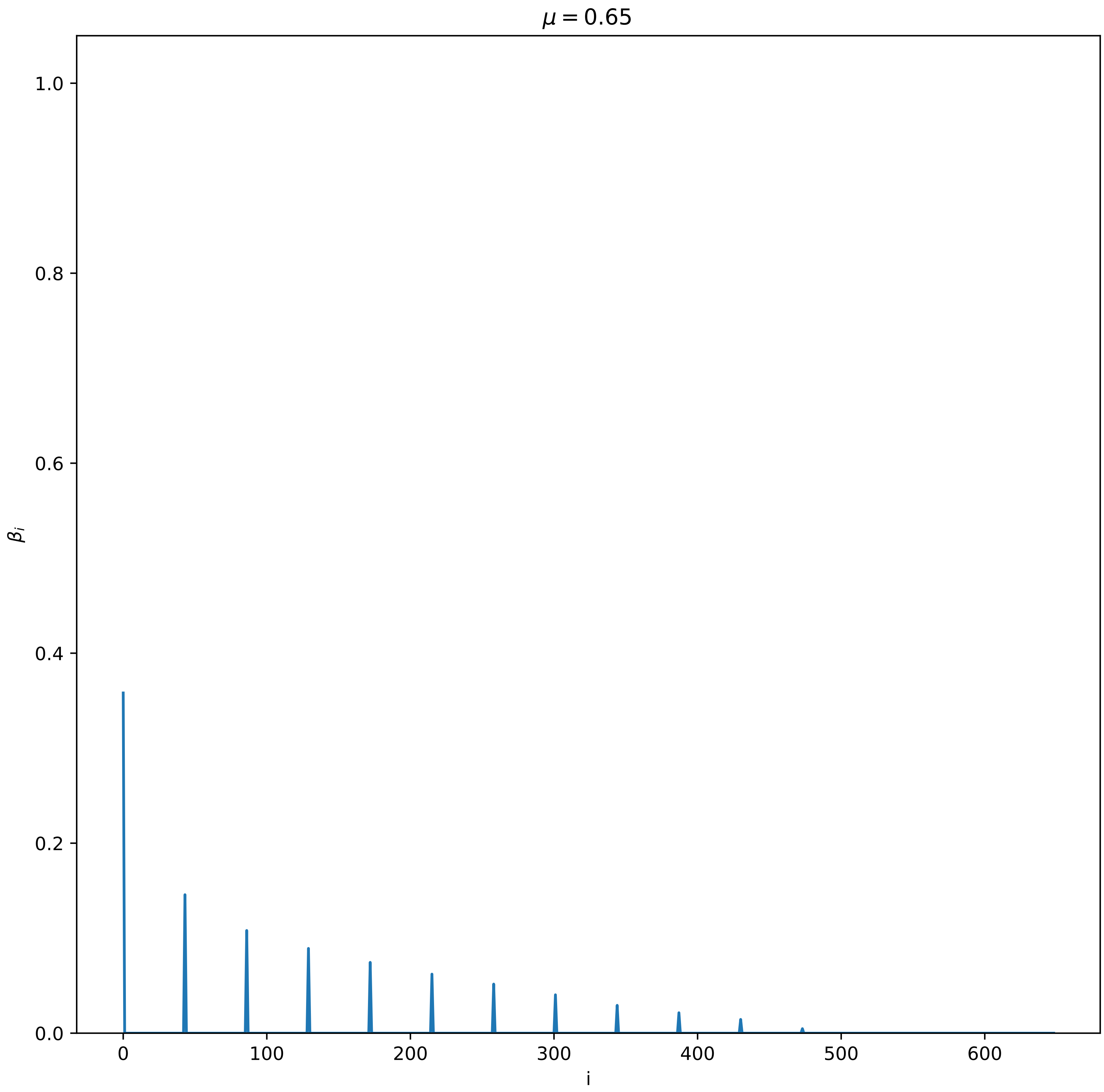
$\mu = 0.62$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



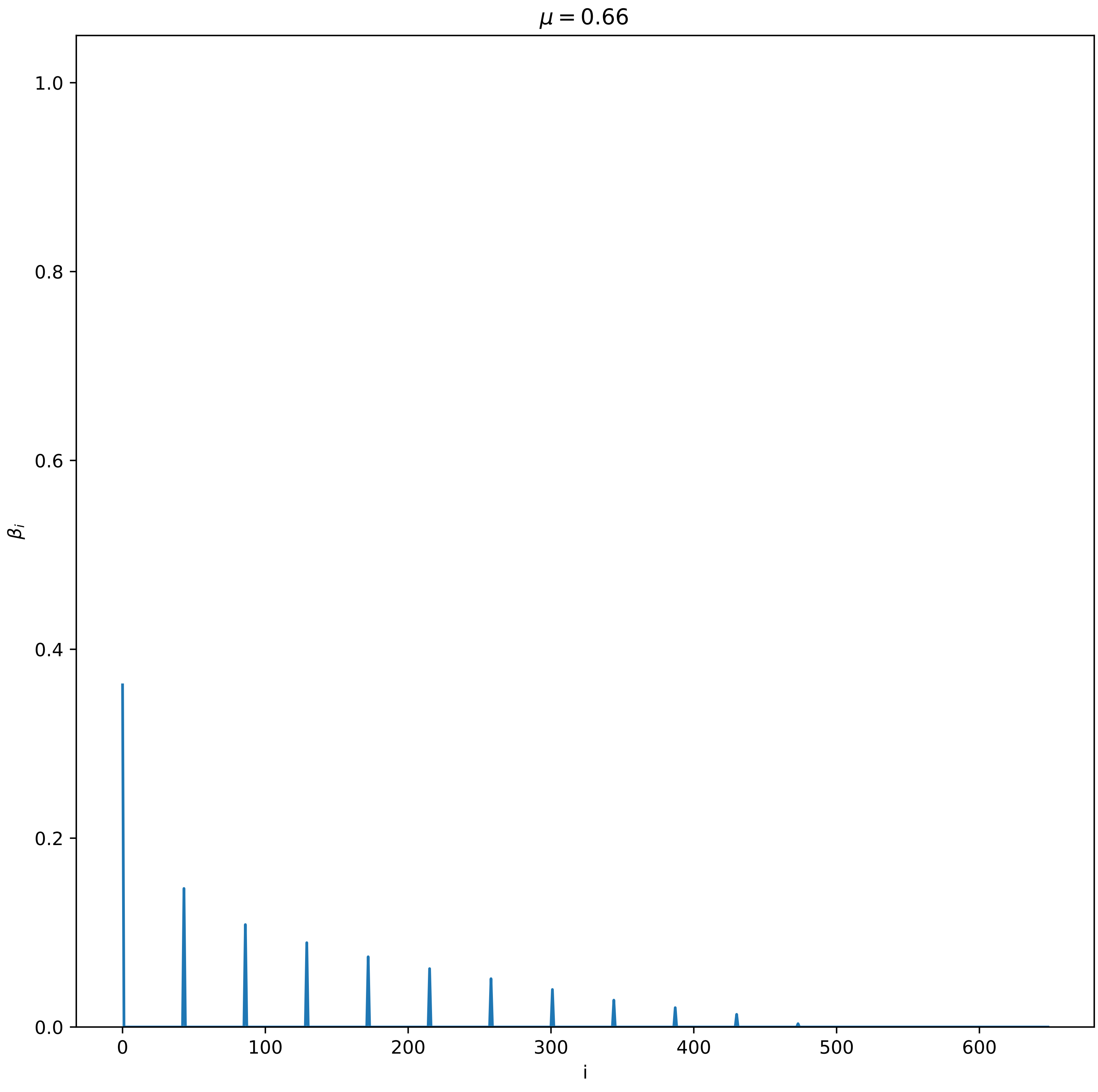
$\mu = 0.63$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



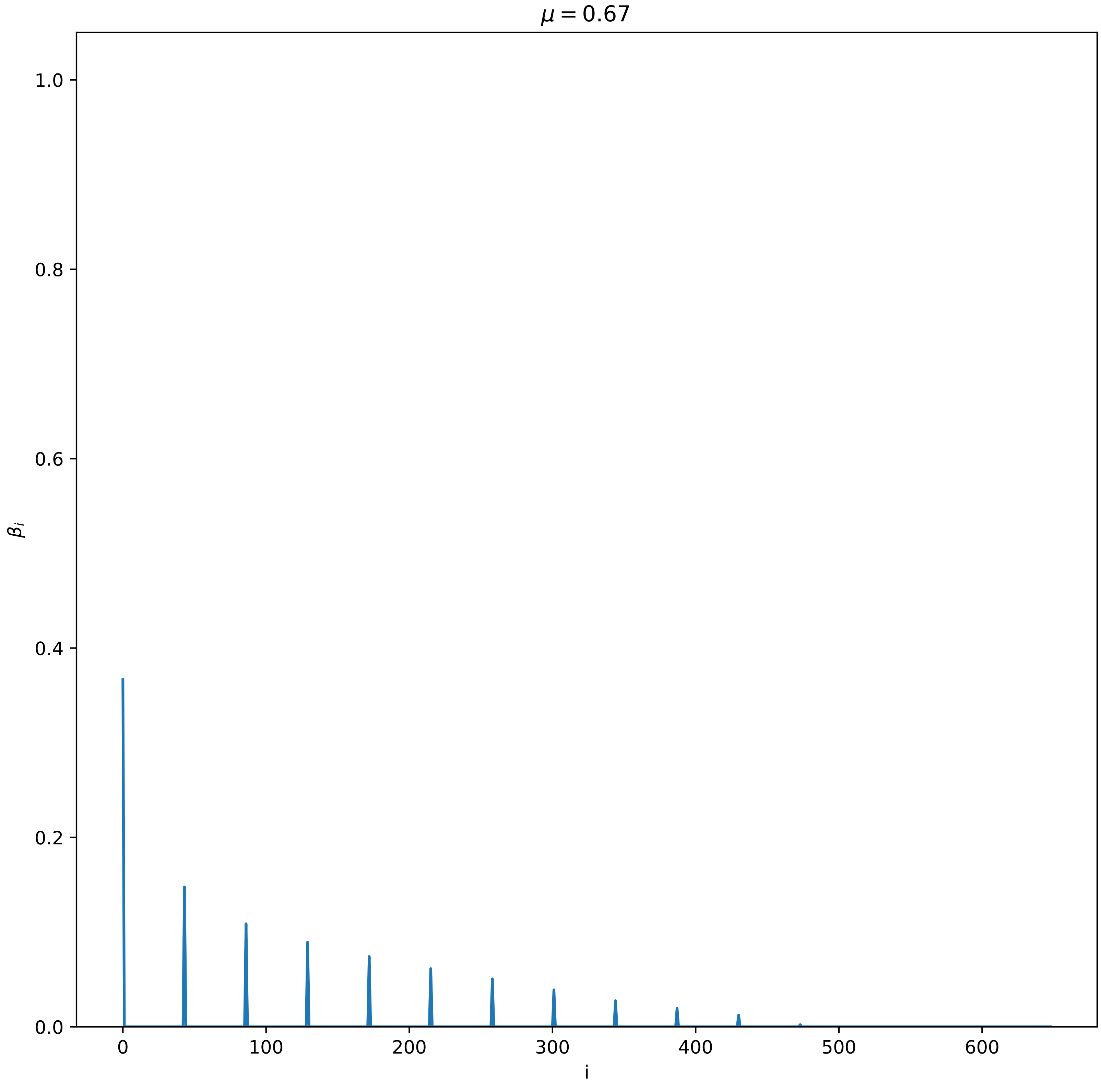
$\mu = 0.64$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



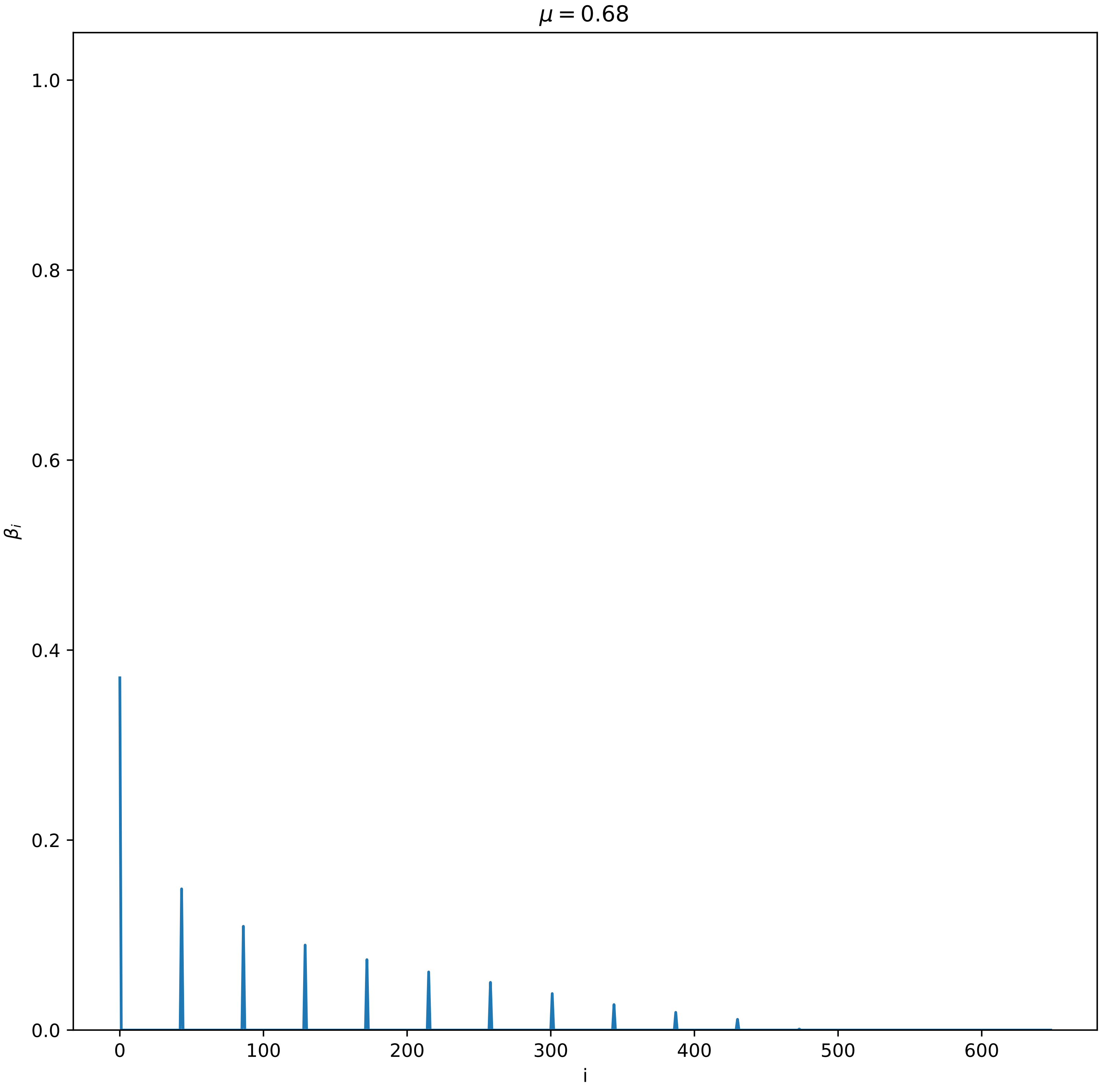
$\mu = 0.65$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



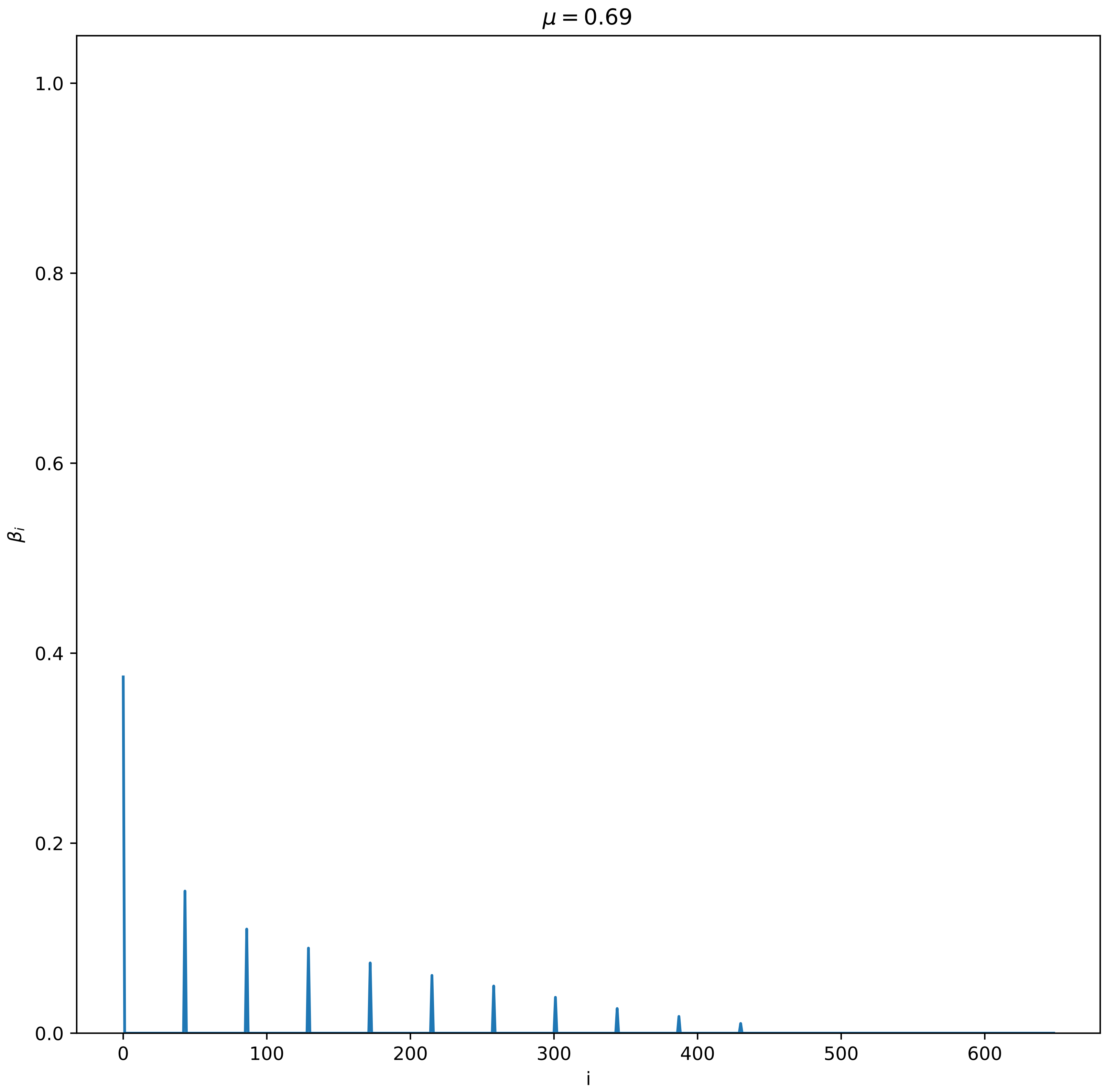
$\mu = 0.66$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



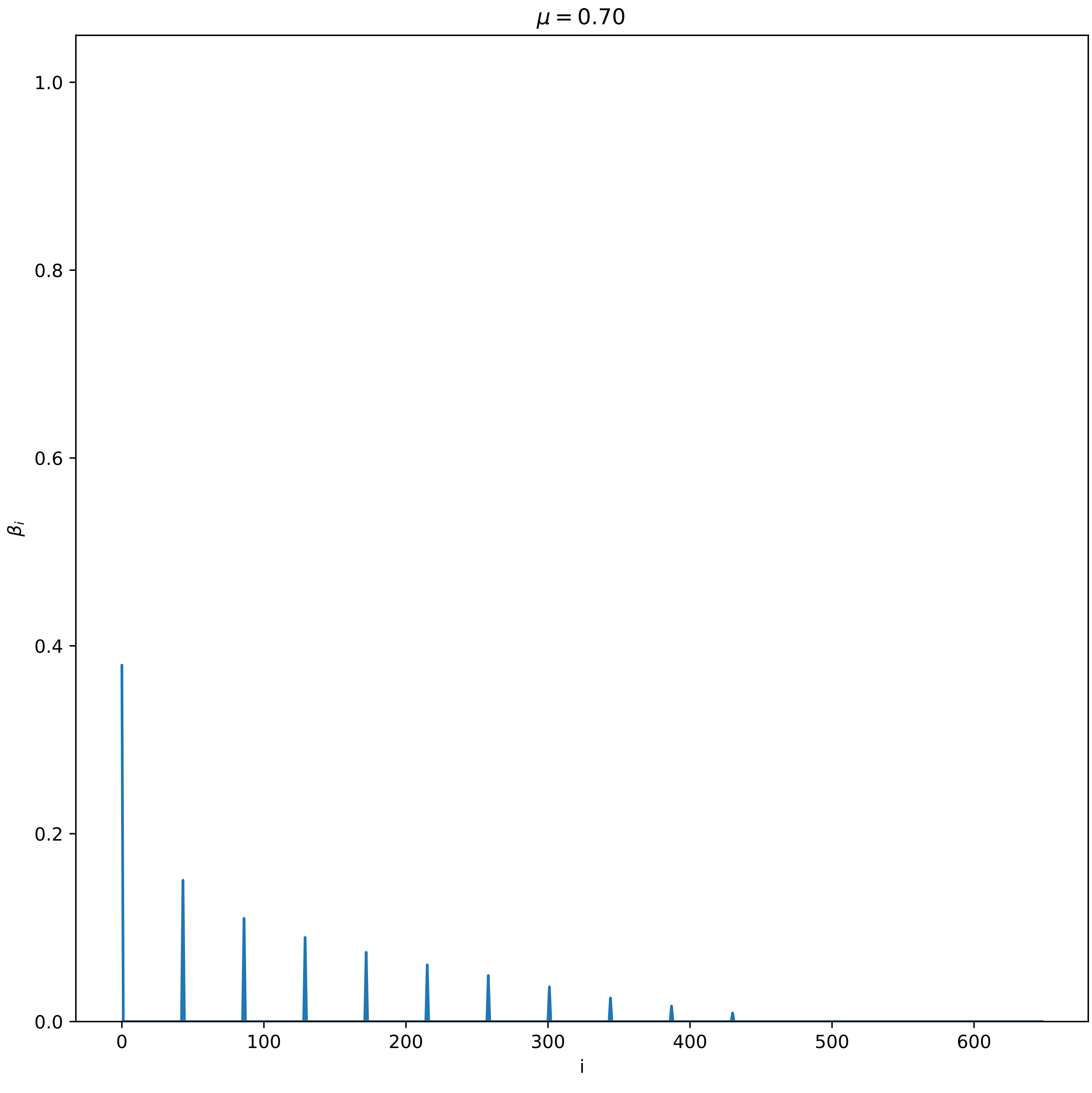
$\mu = 0.67$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



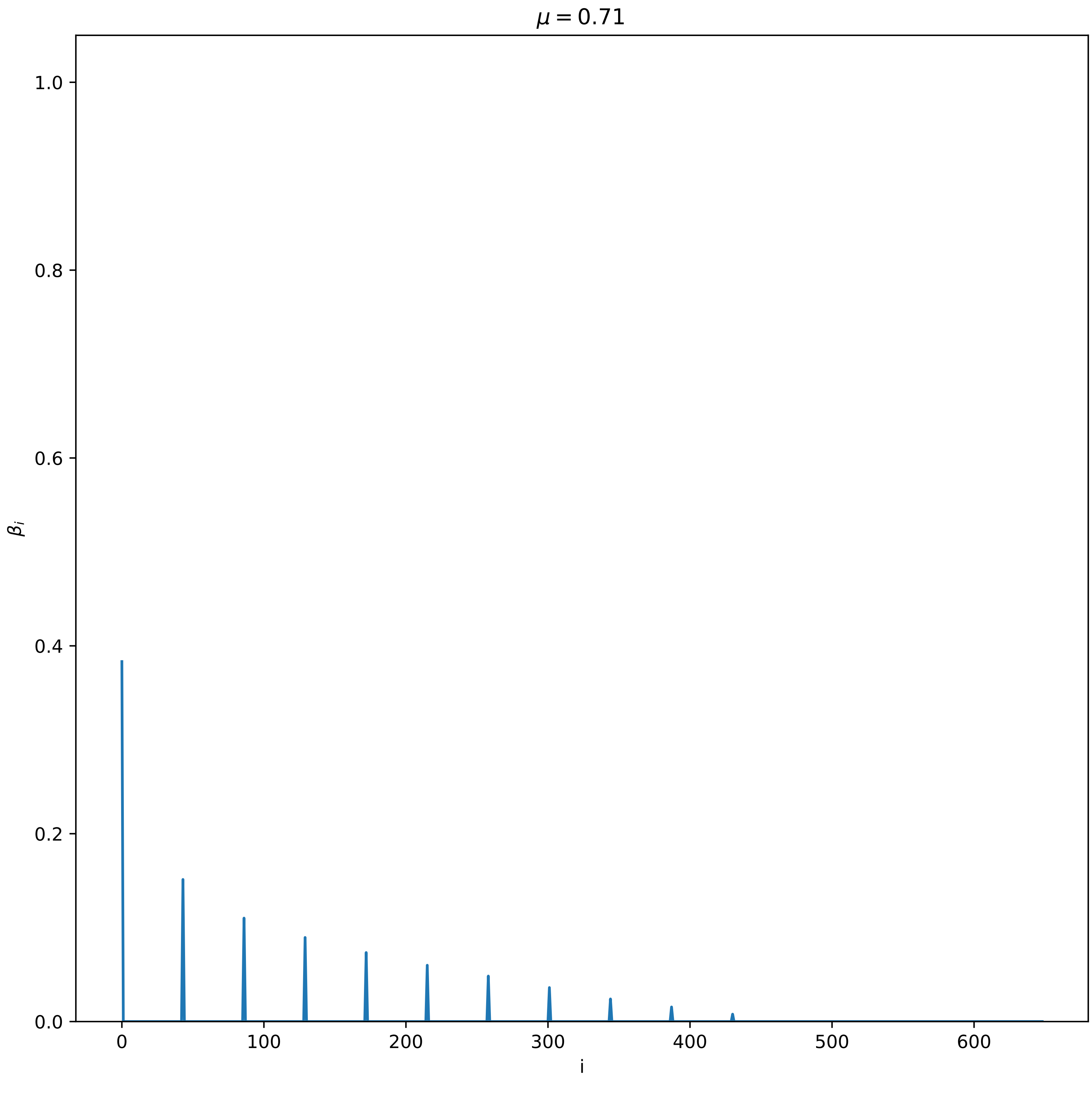
$\mu = 0.68$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



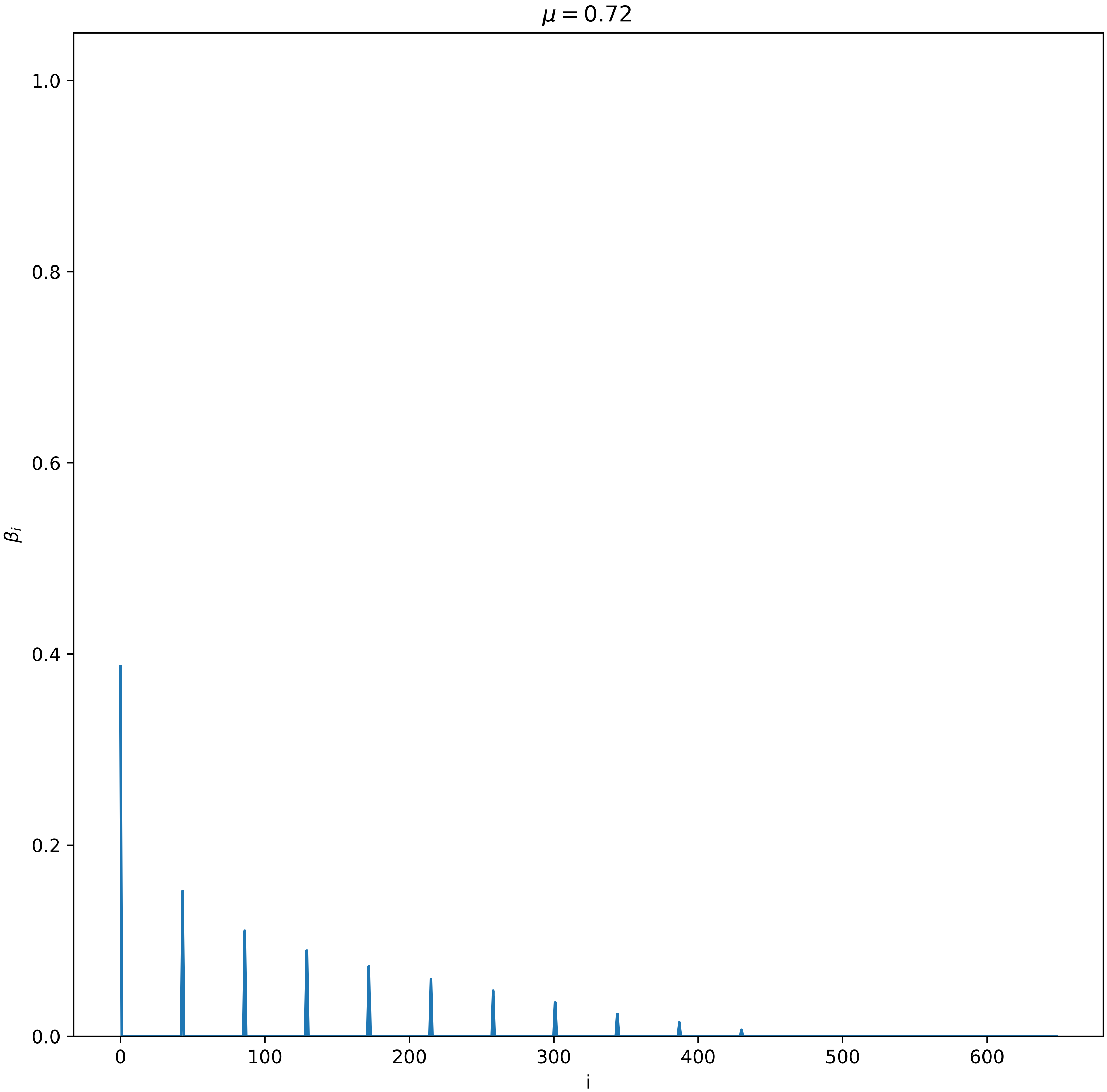
$\mu = 0.69$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



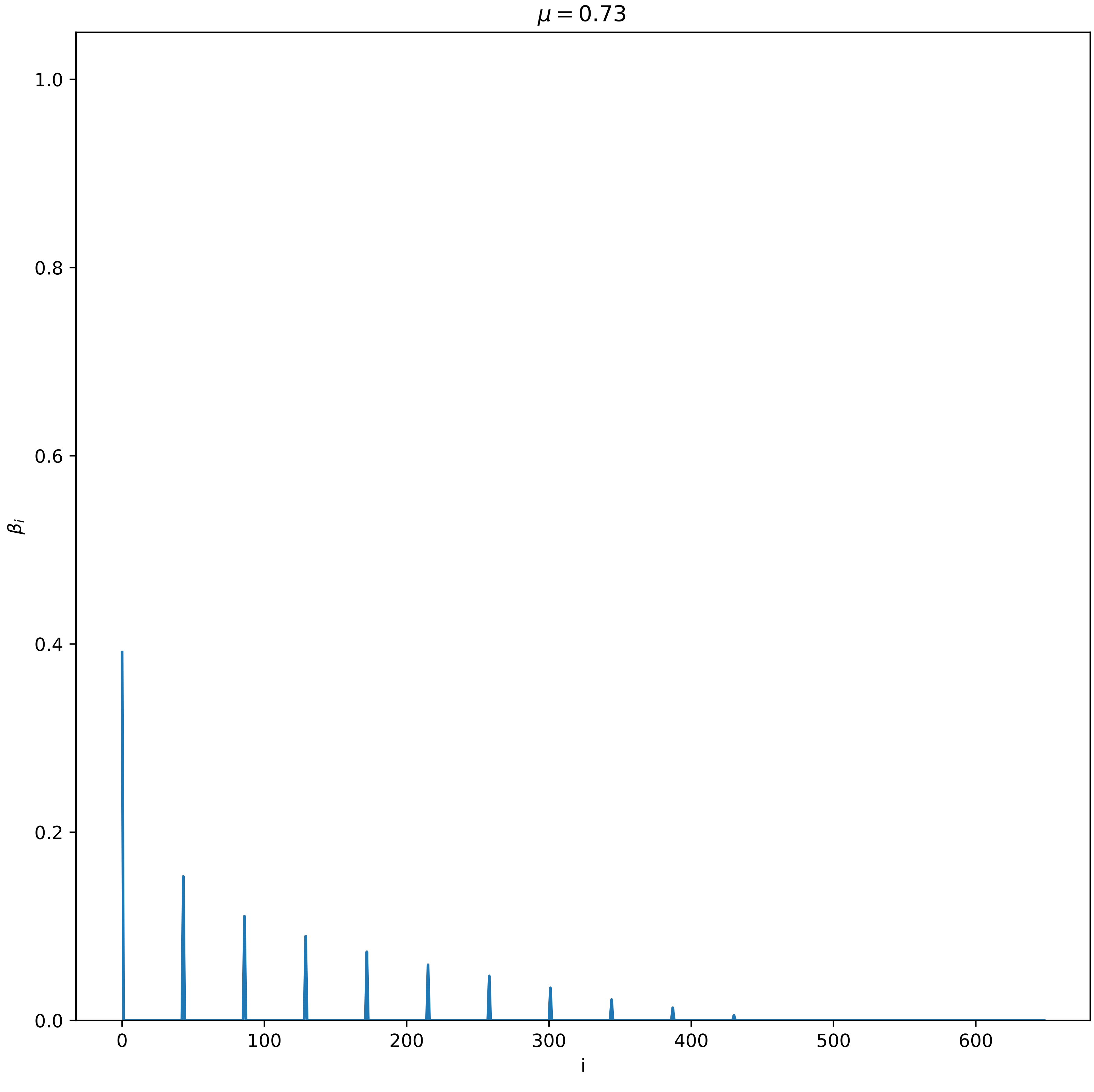
$\mu = 0.70$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



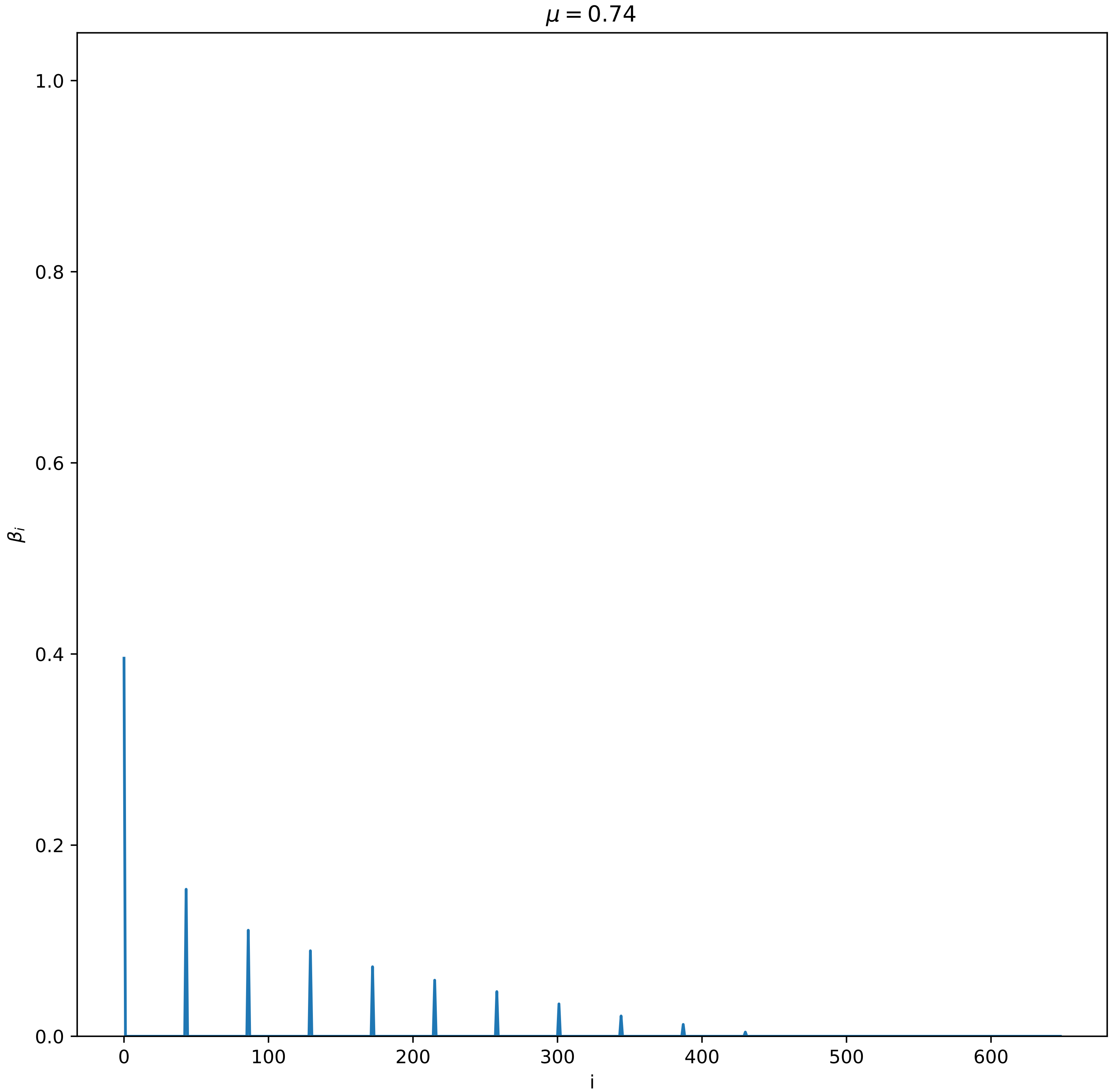
$\mu = 0.71$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

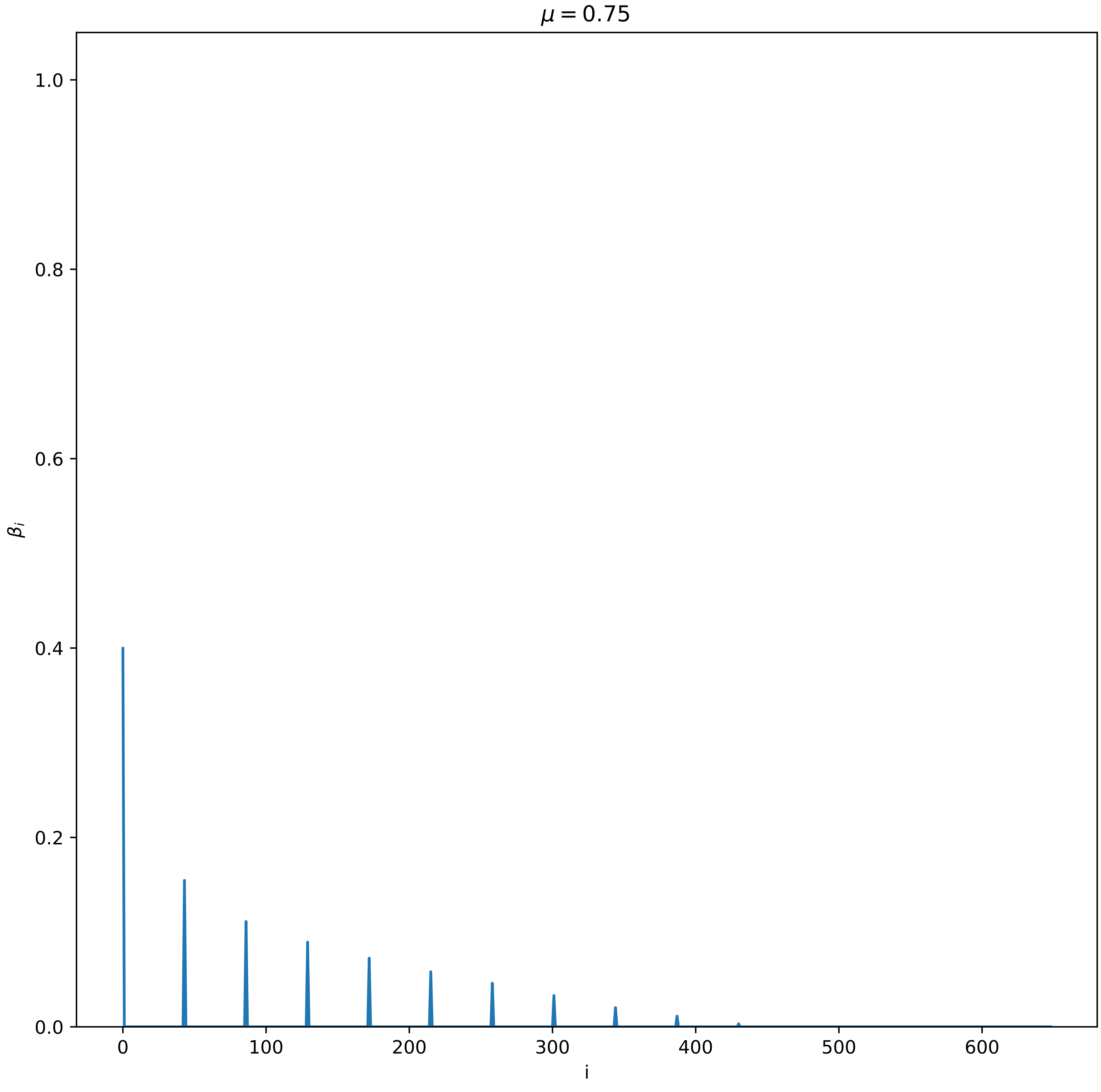


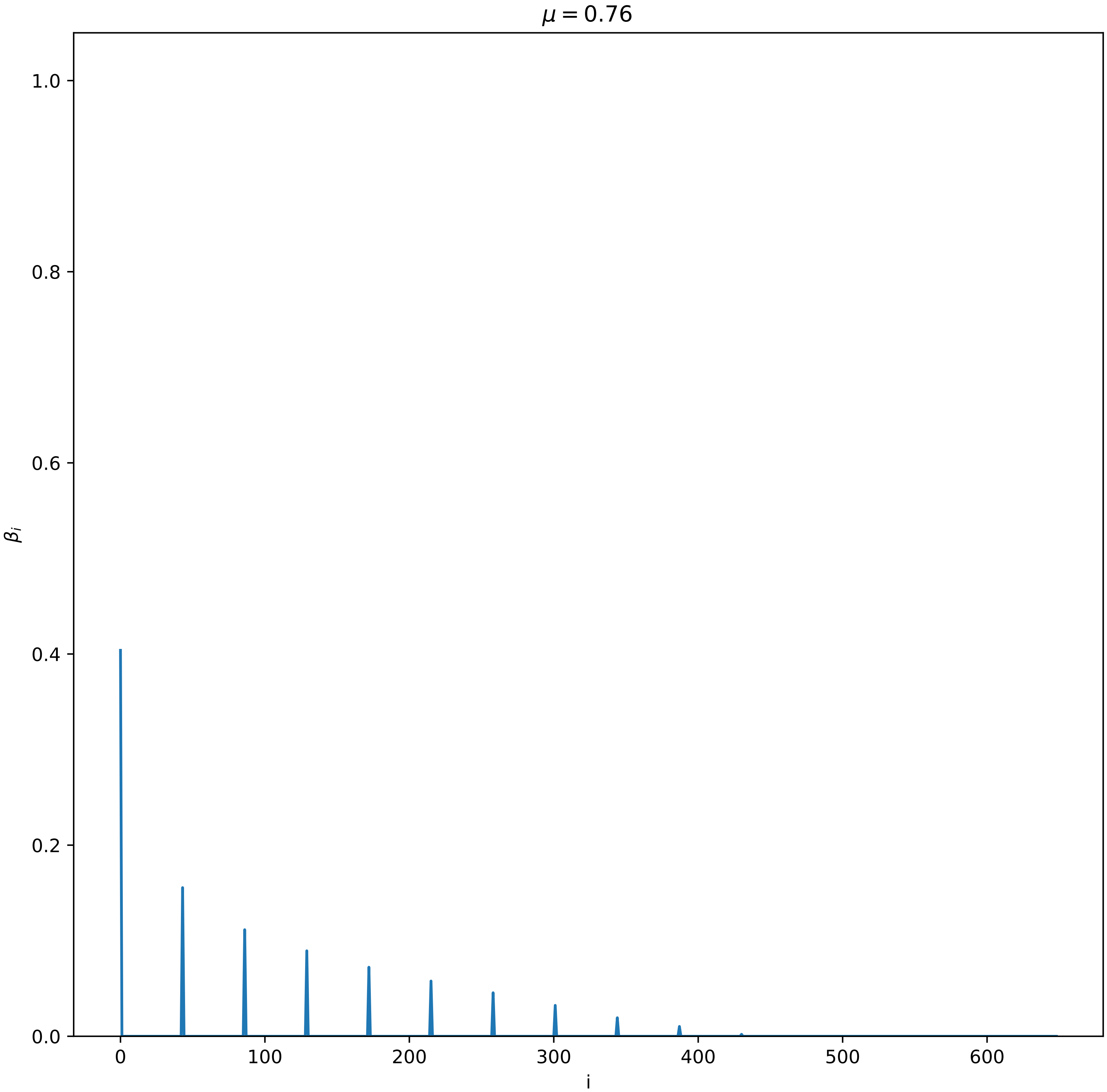
$\mu = 0.72$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

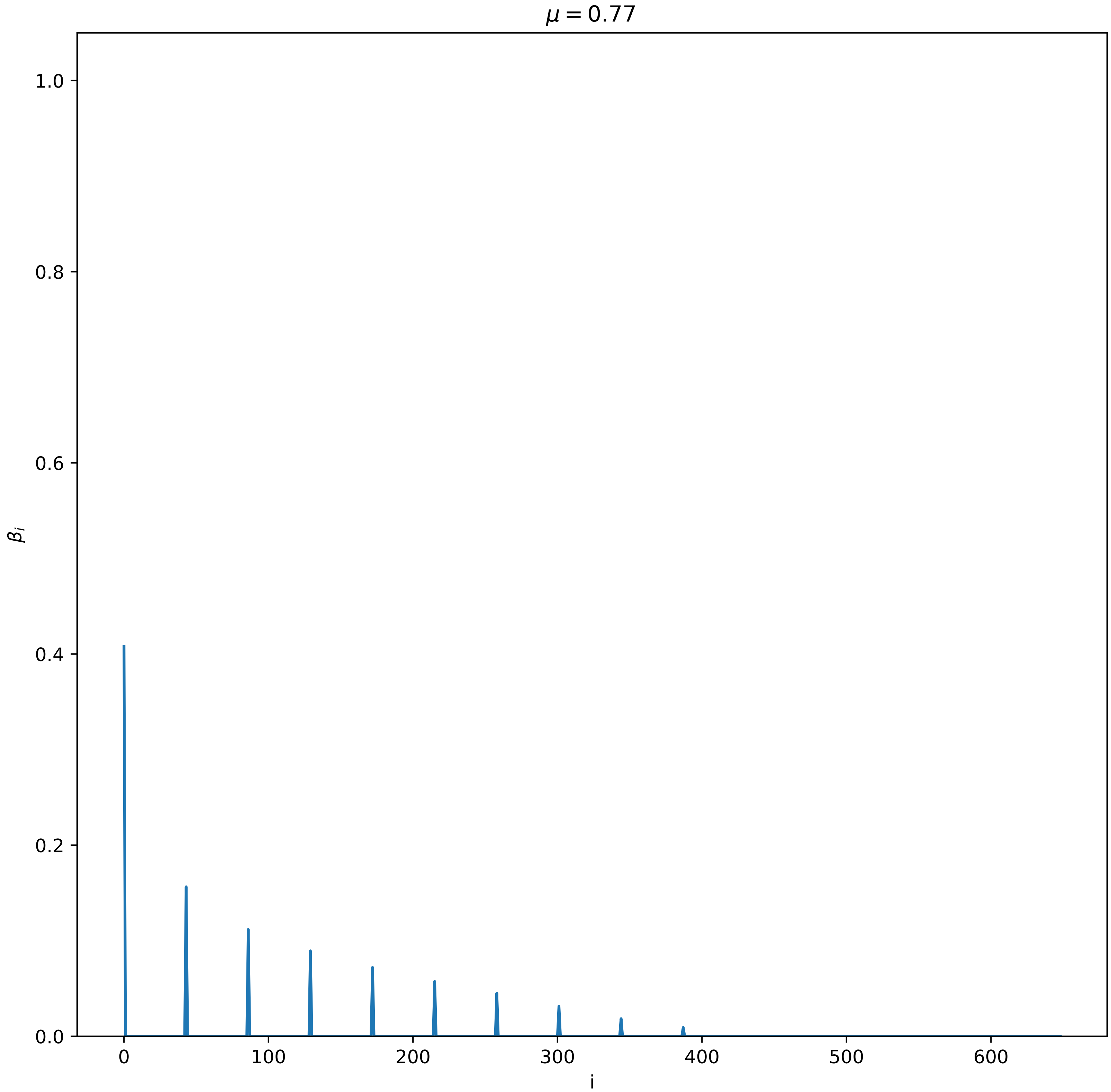


$\mu = 0.73$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

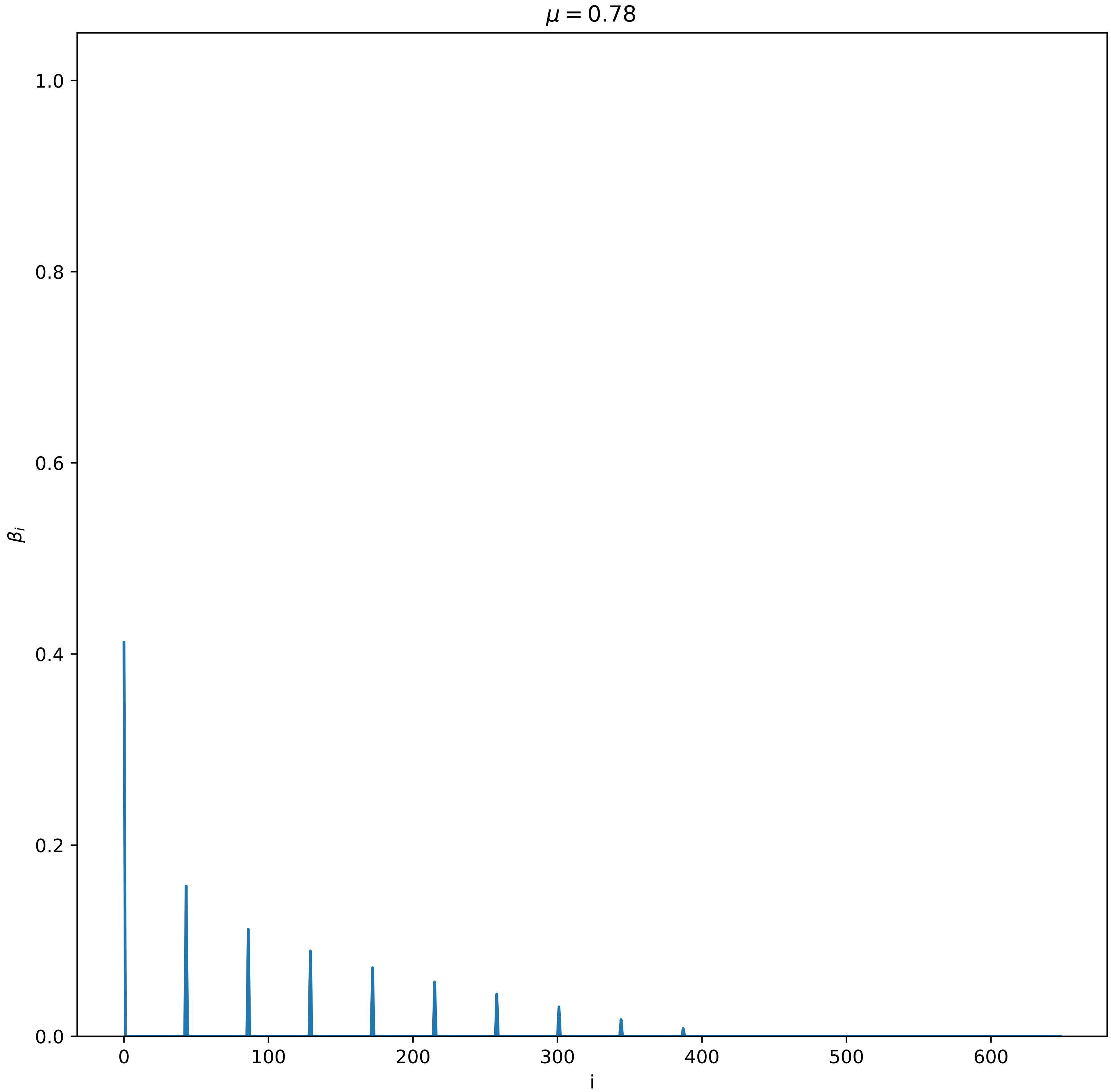

$\mu = 0.74$
β_i
i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600


$\mu = 0.75$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

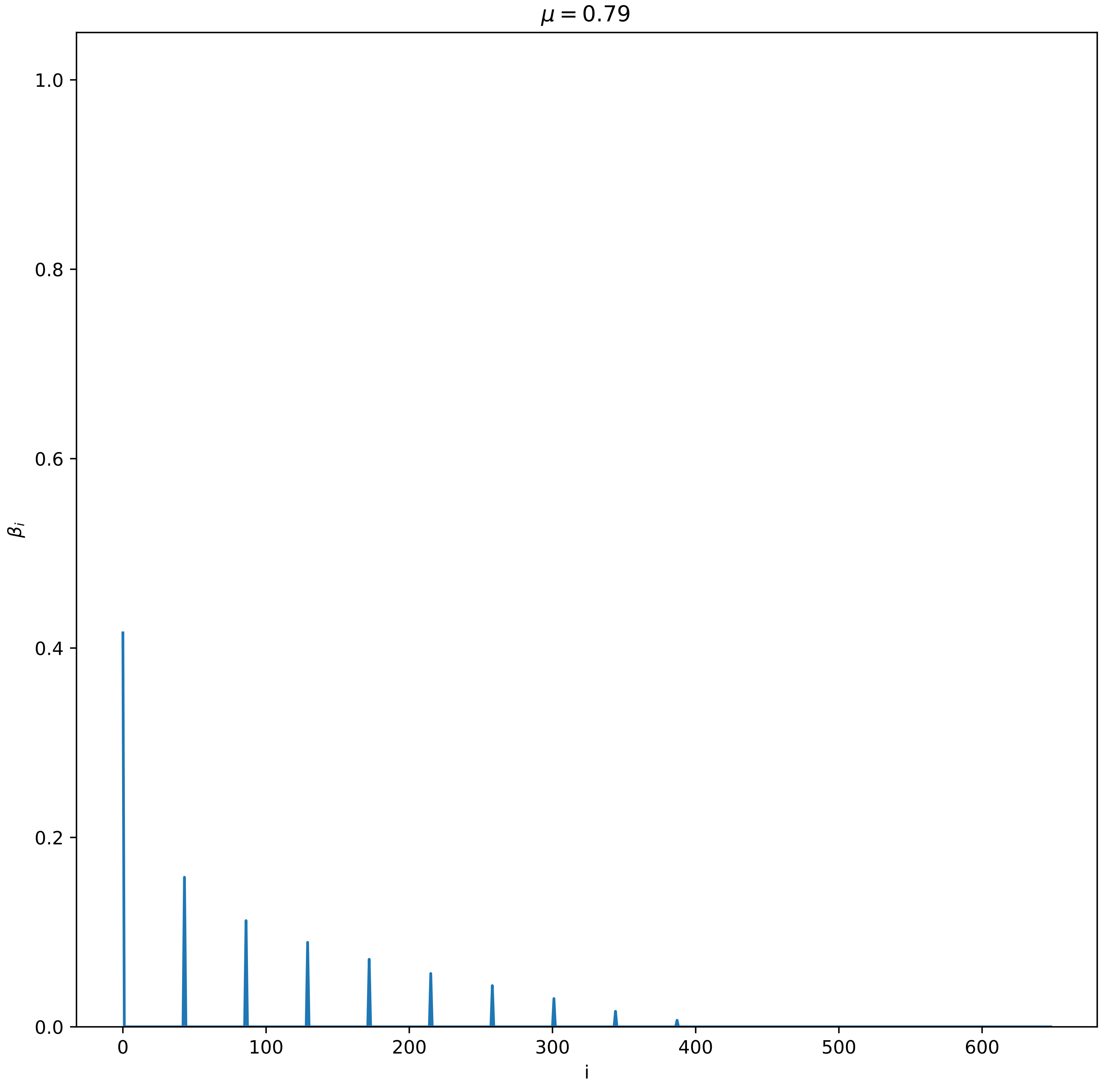

$\mu = 0.76$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



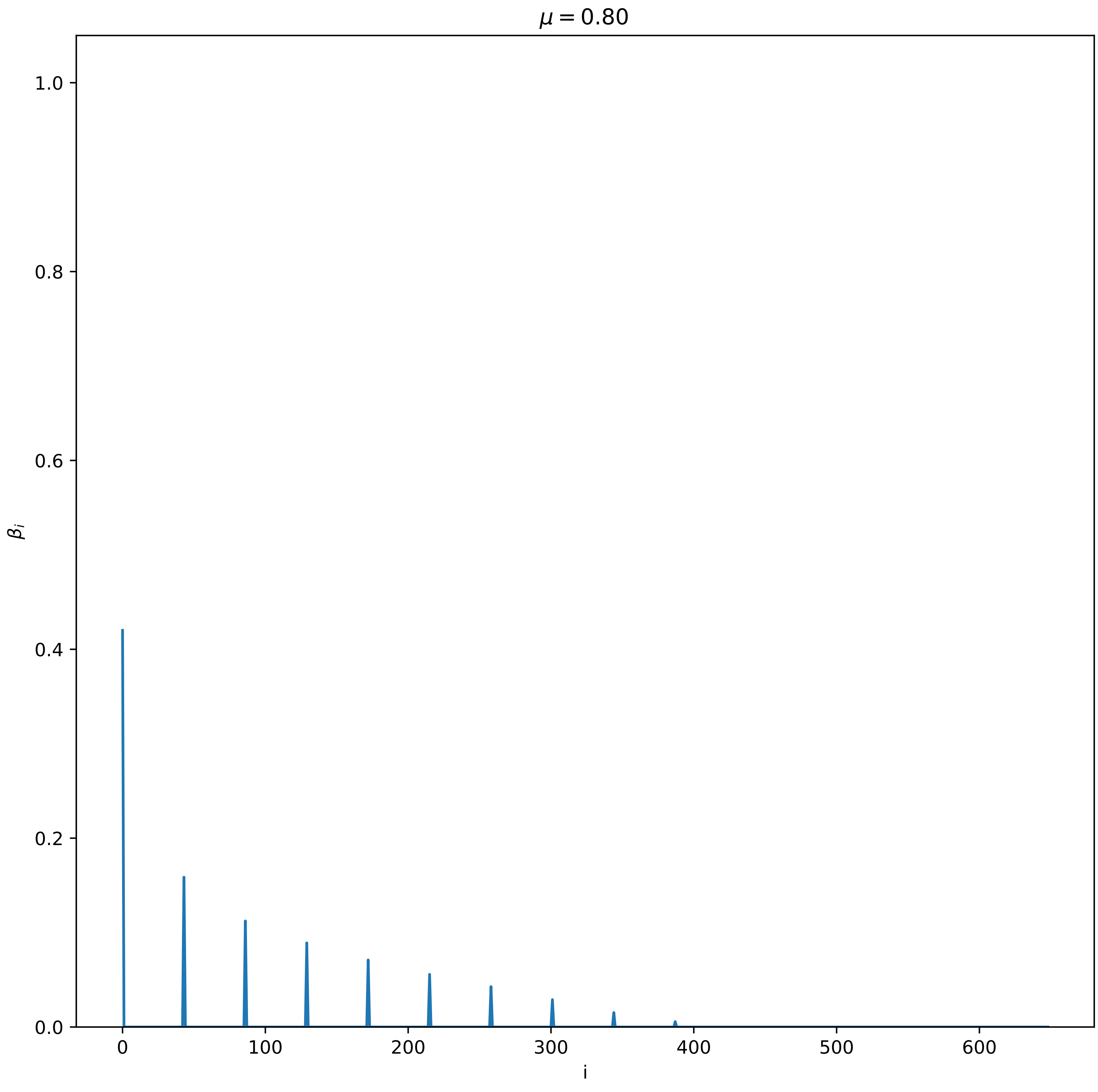
$\mu = 0.77$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



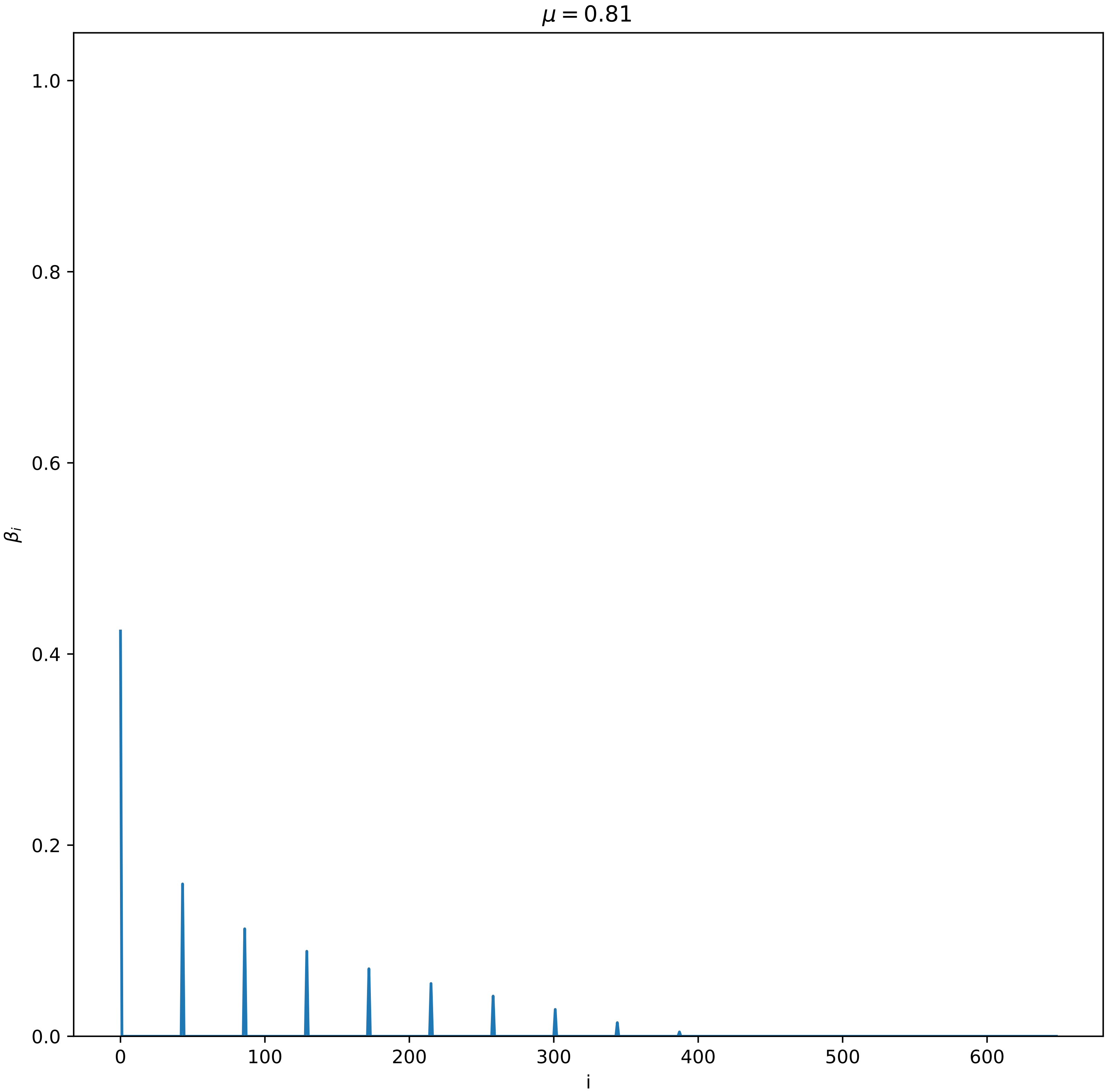
$\mu = 0.78$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

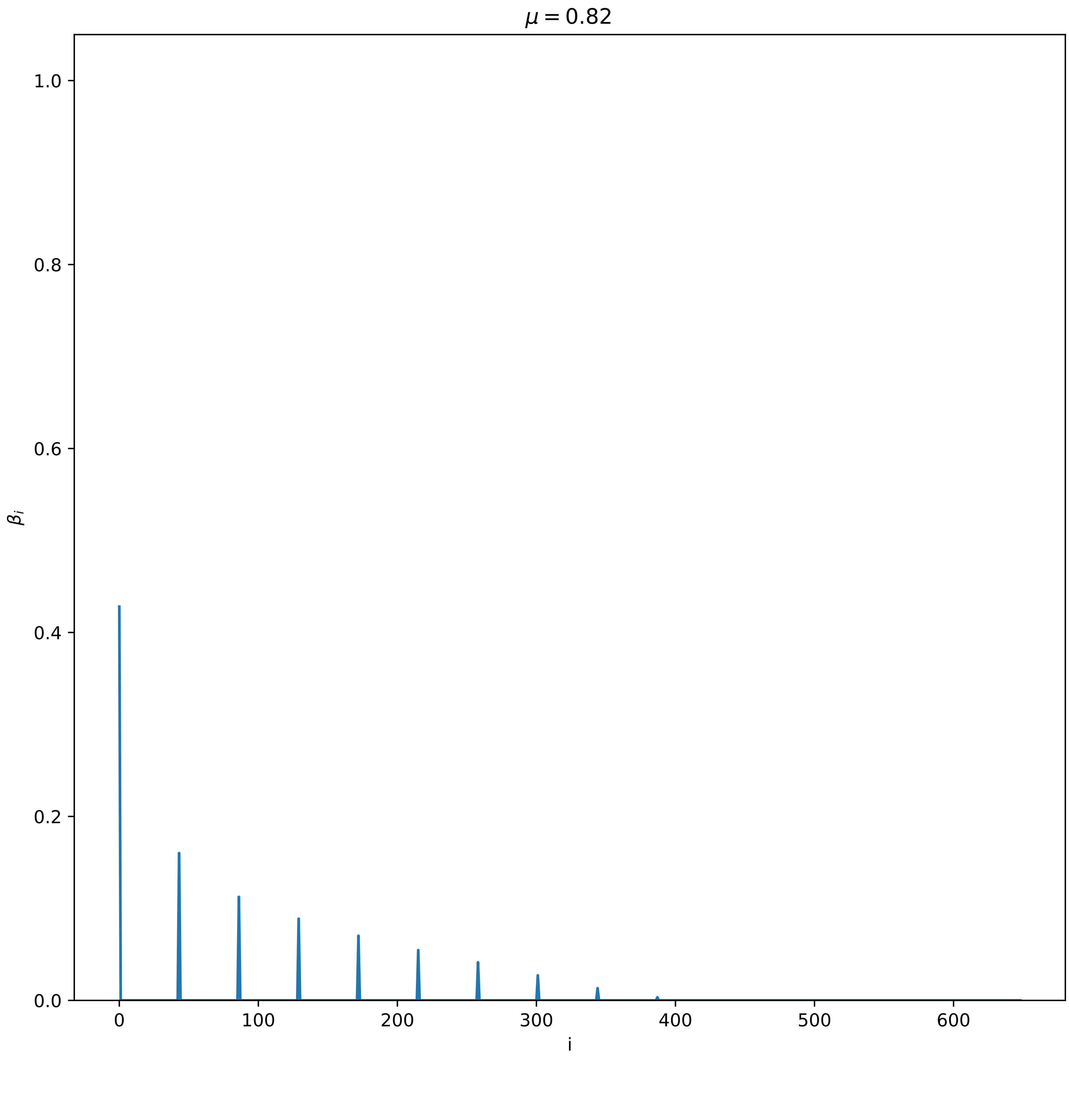


$\mu = 0.79$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

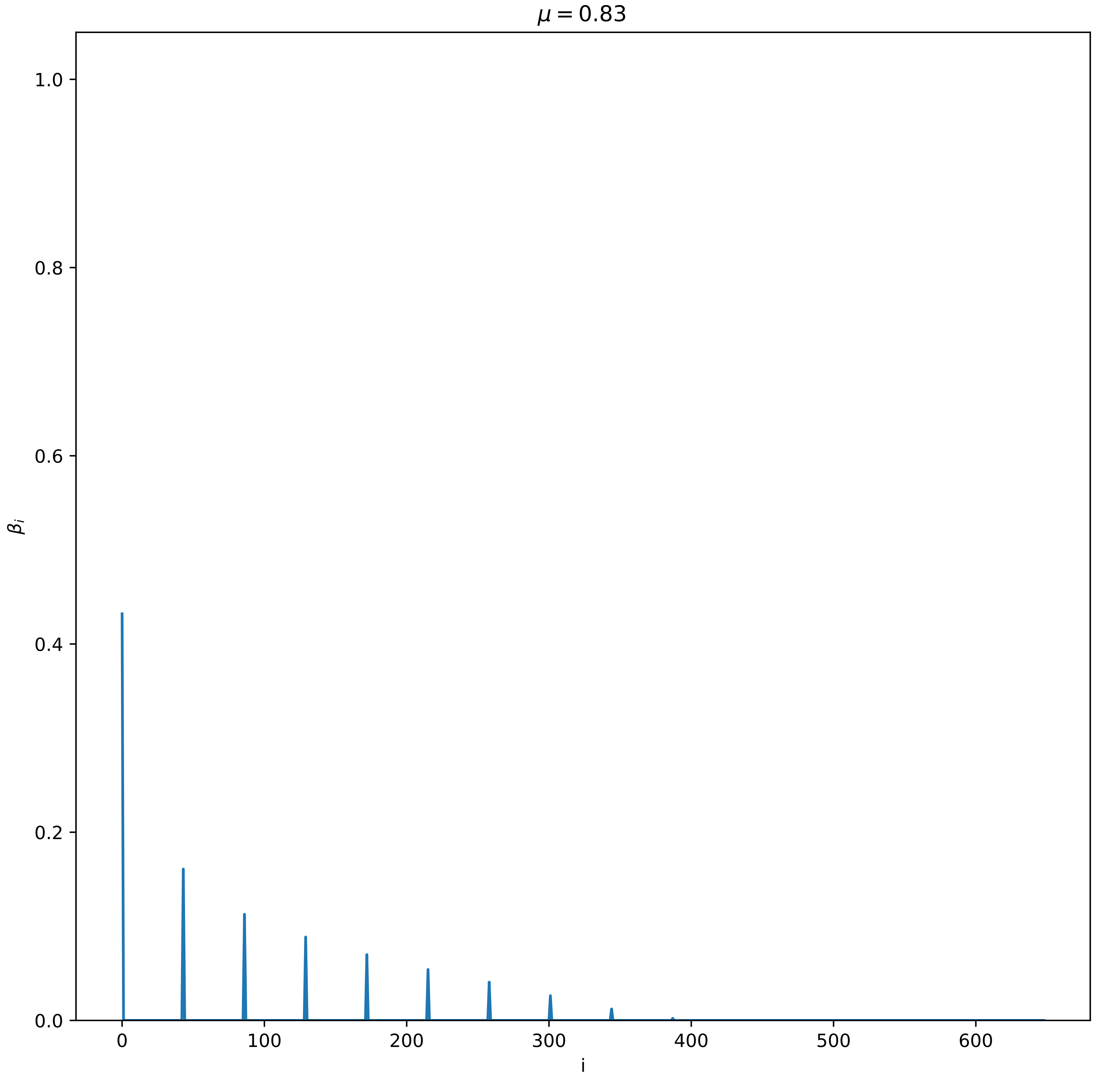


$\mu = 0.80$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

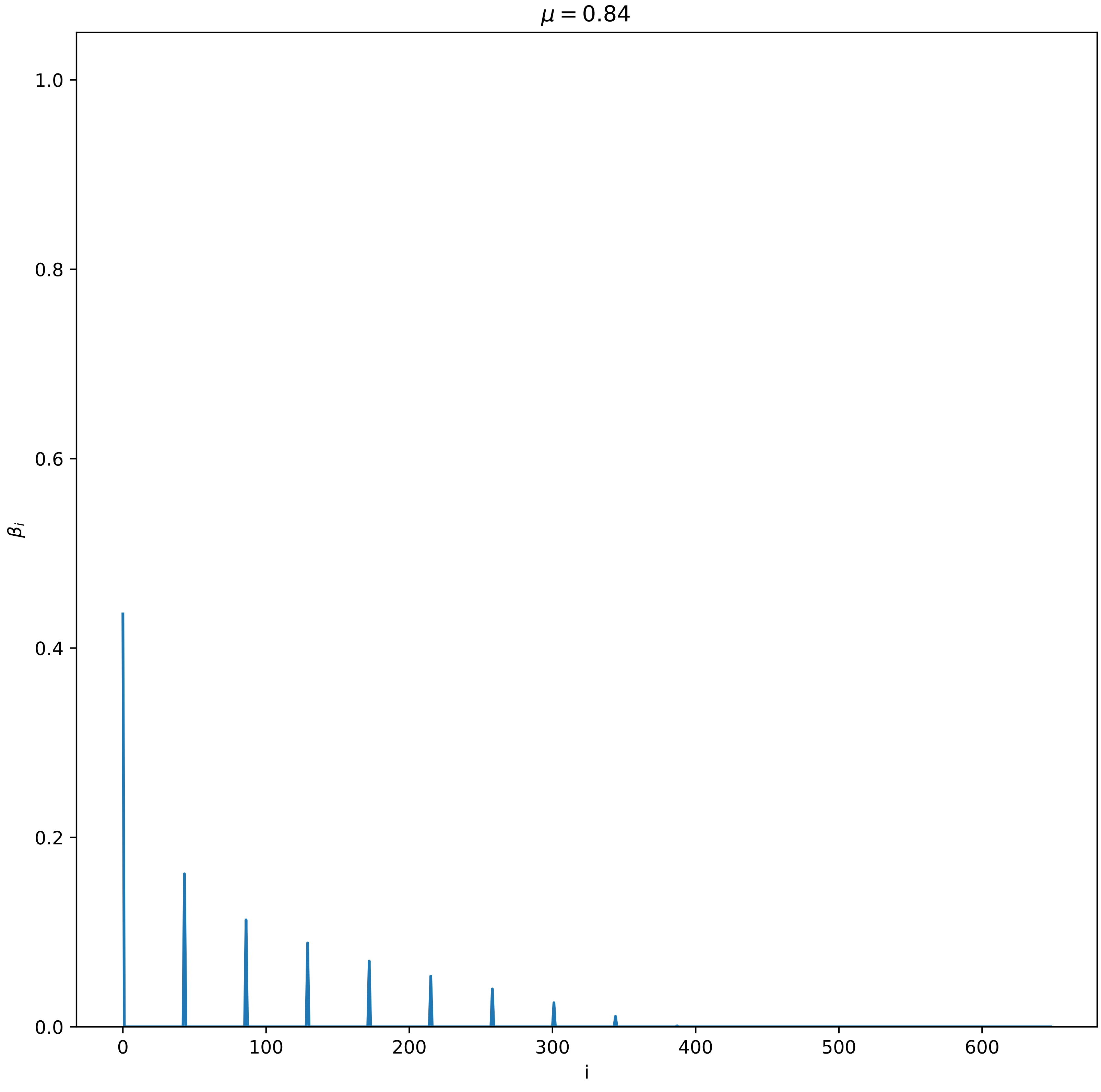

$\mu = 0.81$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



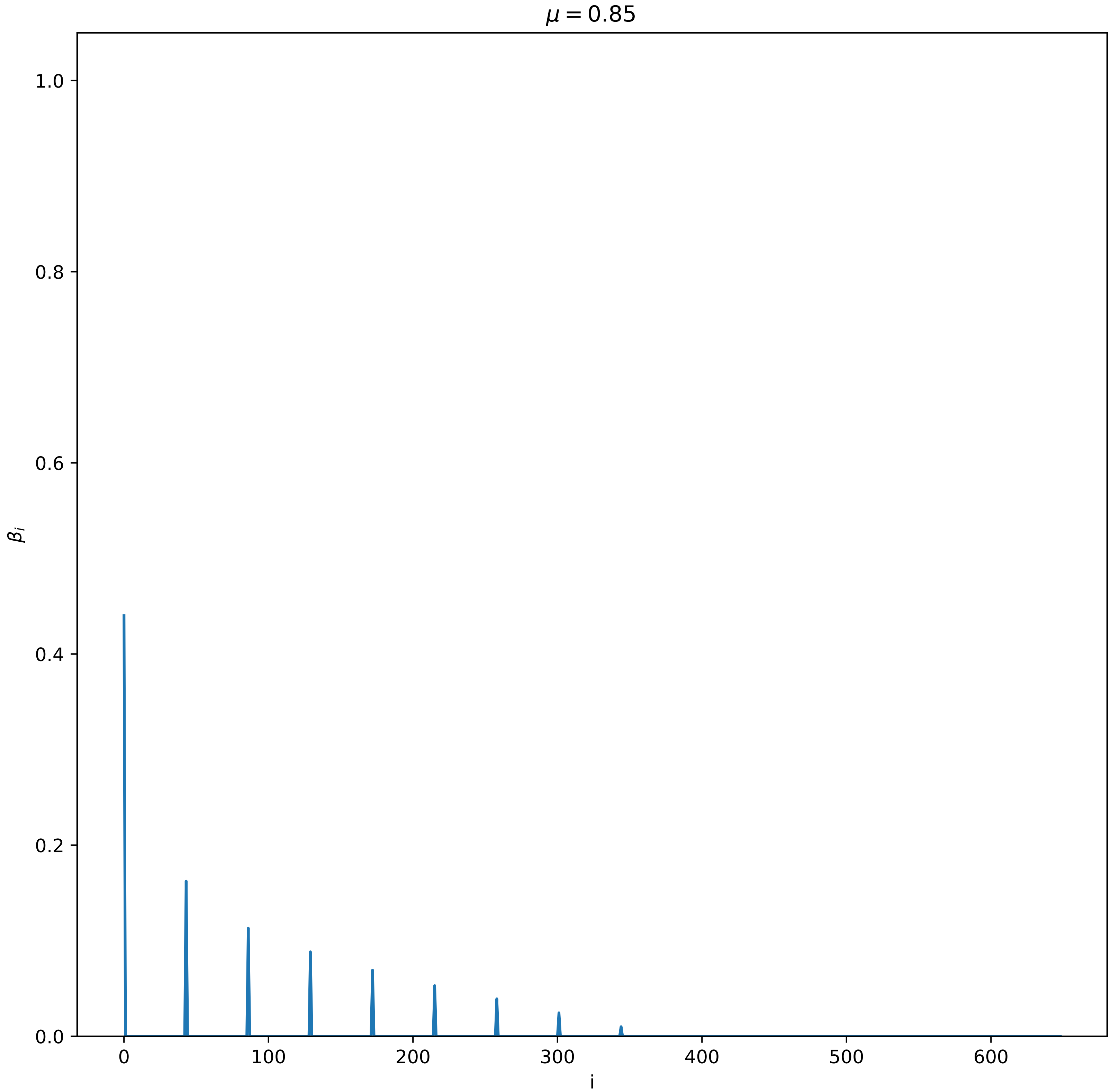
$\mu = 0.82$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



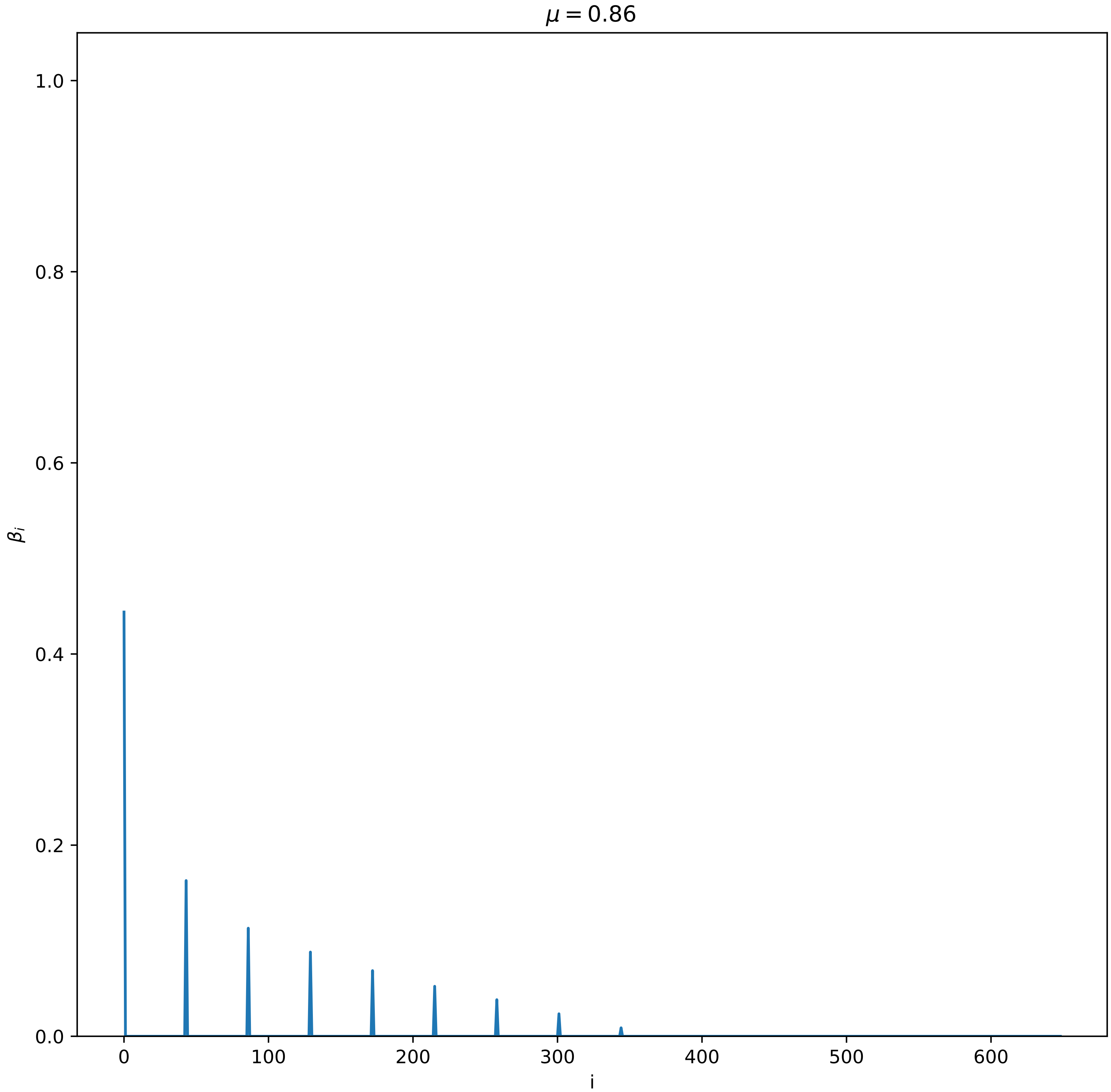
$\mu = 0.83$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

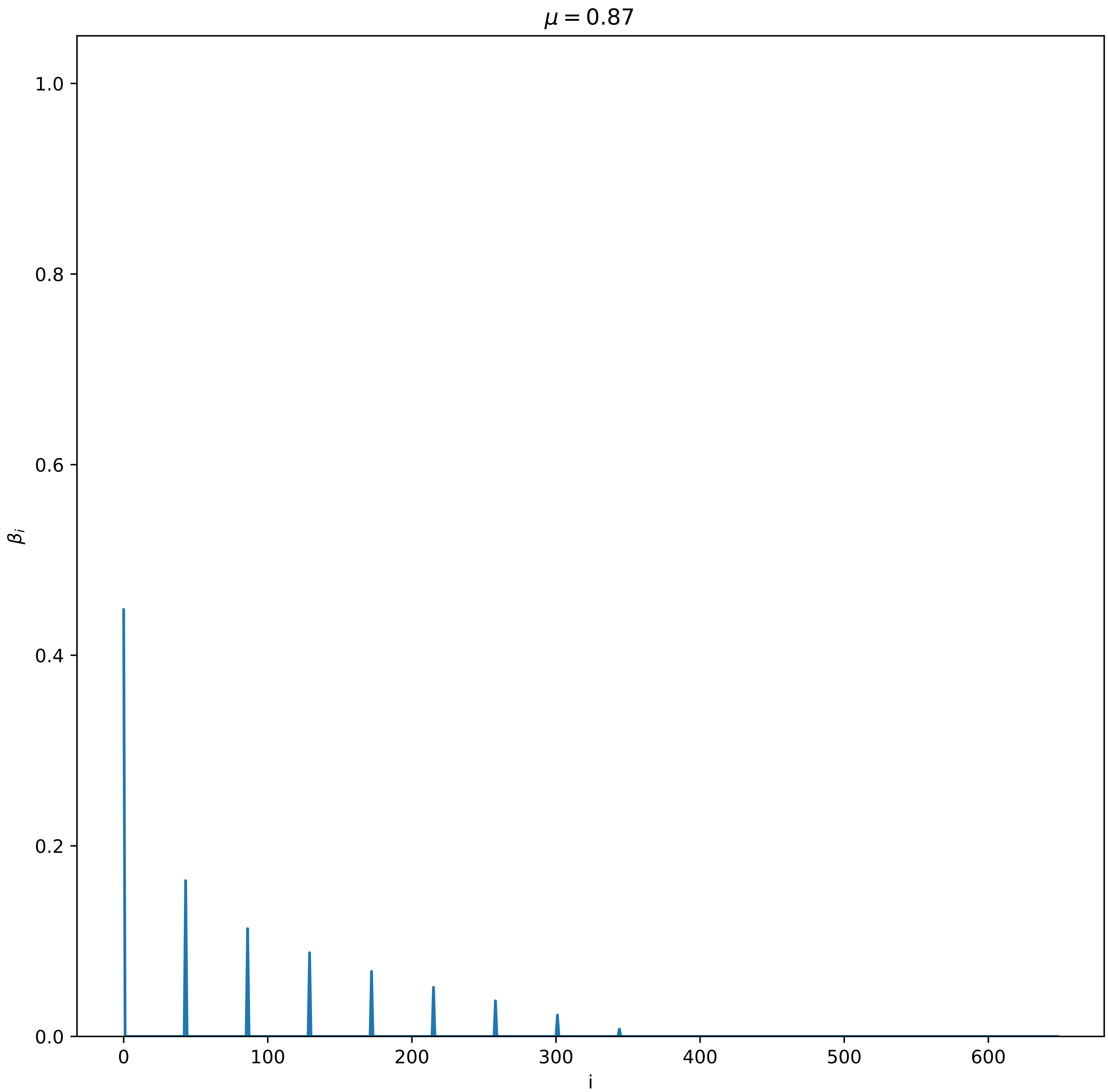


$\mu = 0.84$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

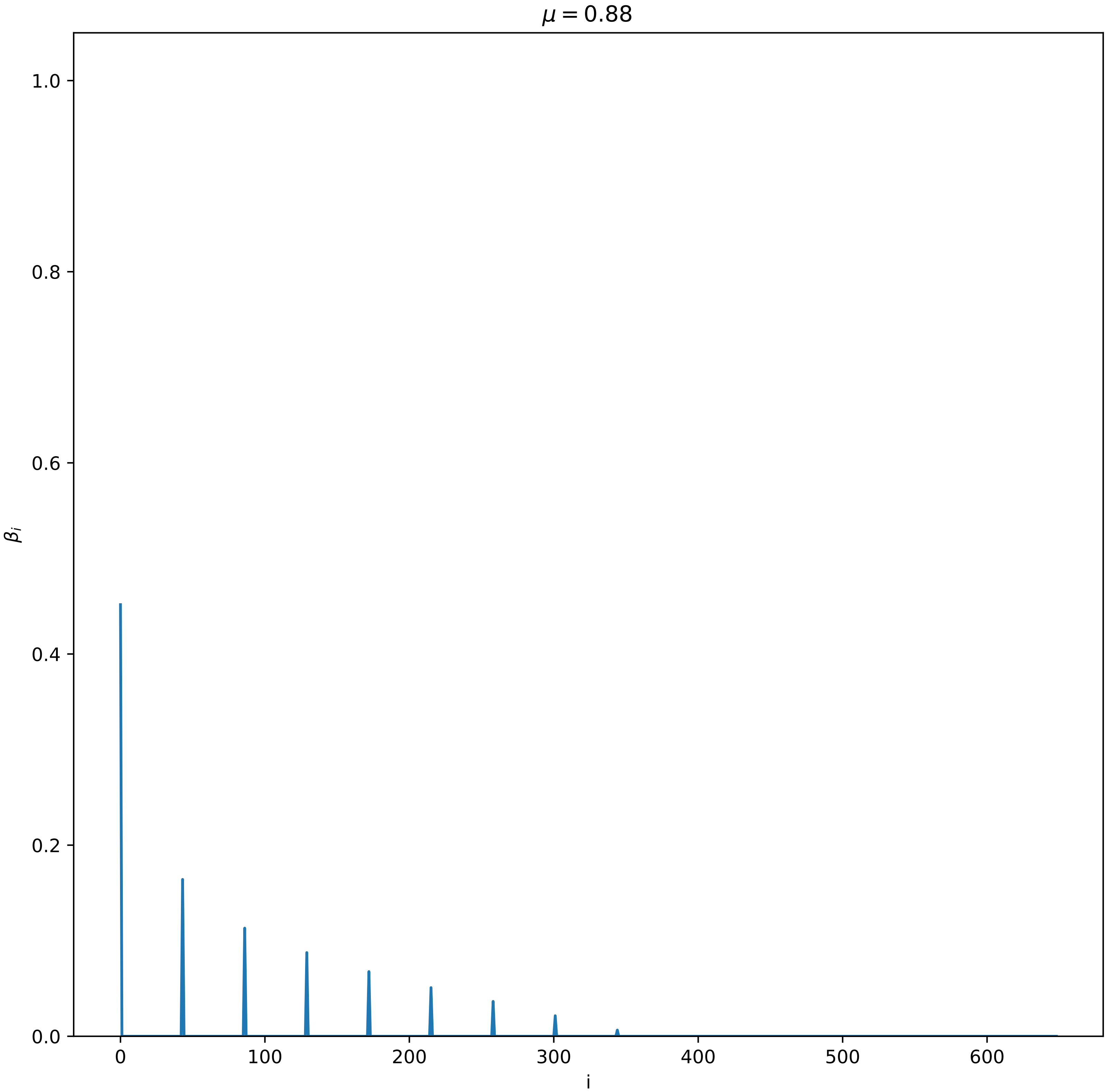


$\mu = 0.85$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

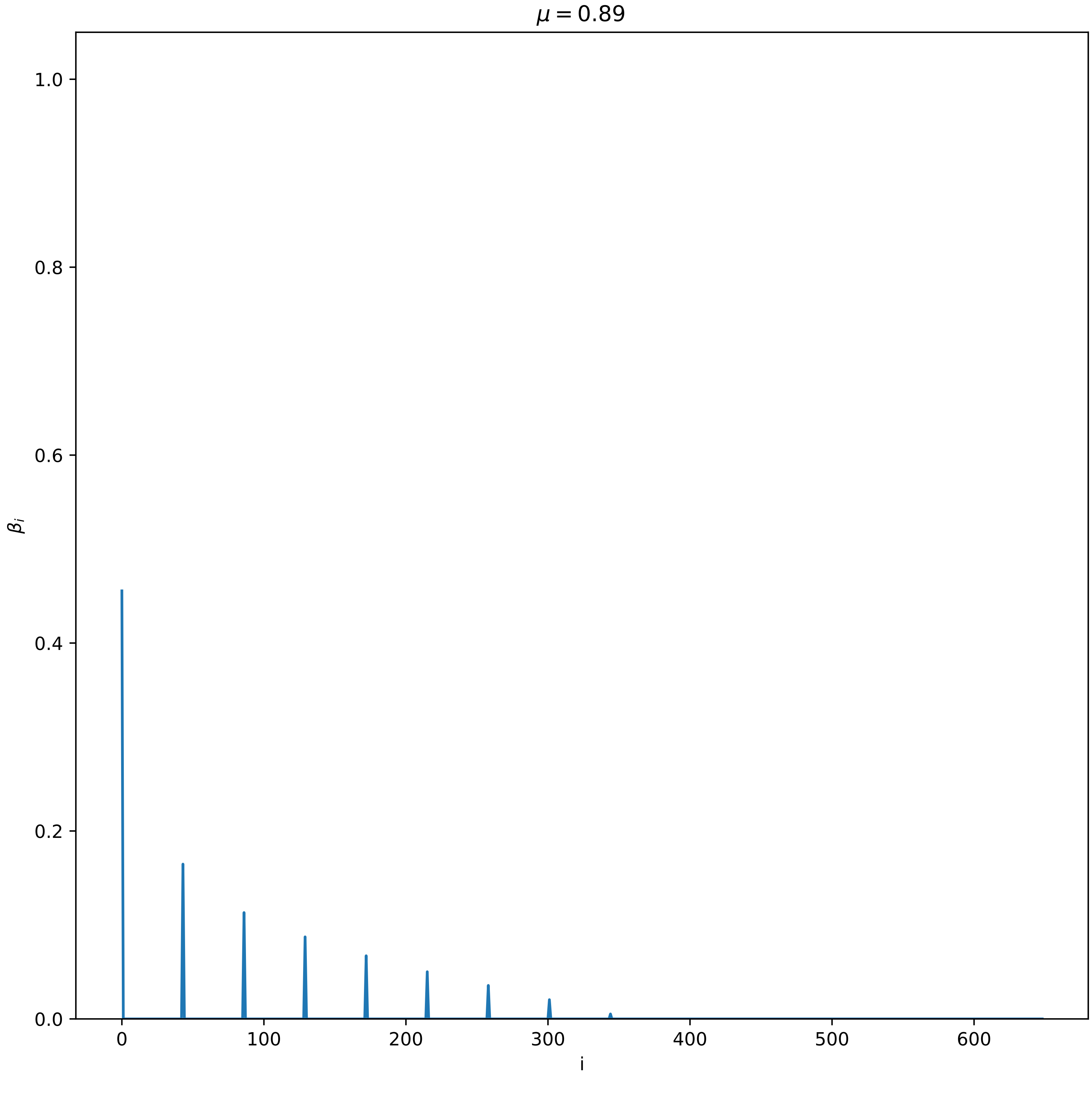

$\mu = 0.86$
β_i
i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600



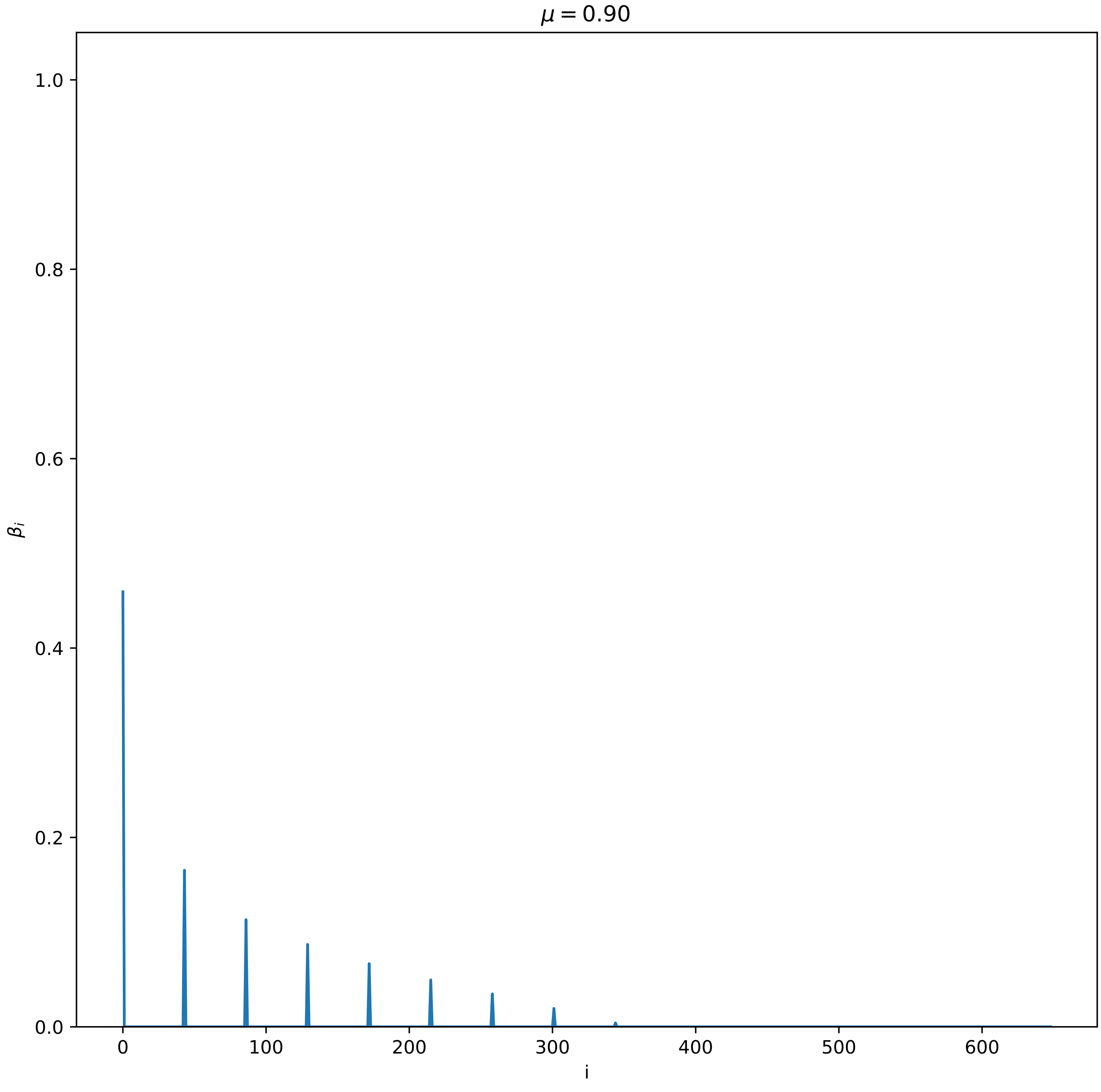
$\mu = 0.87$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

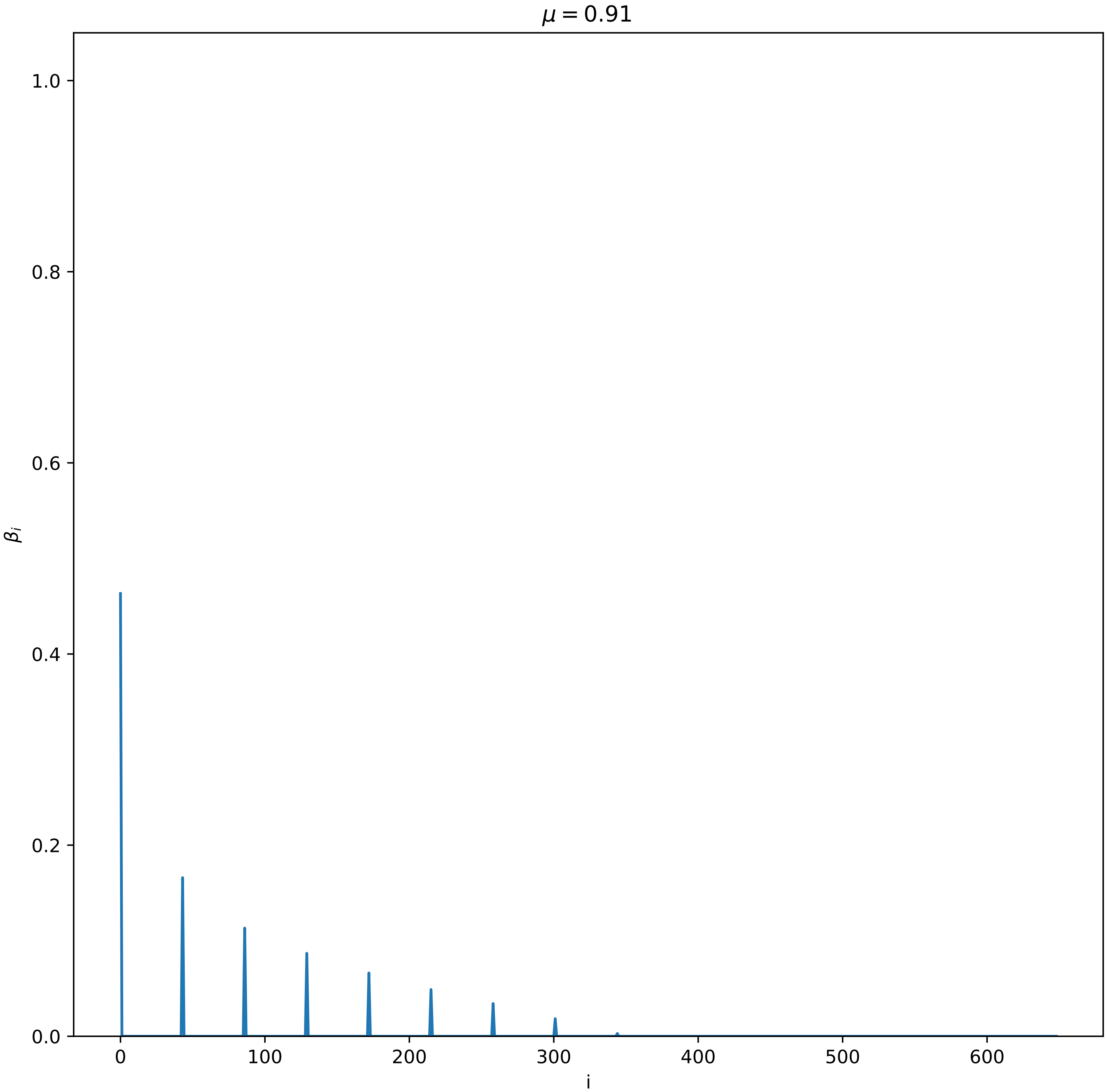


$\mu = 0.88$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

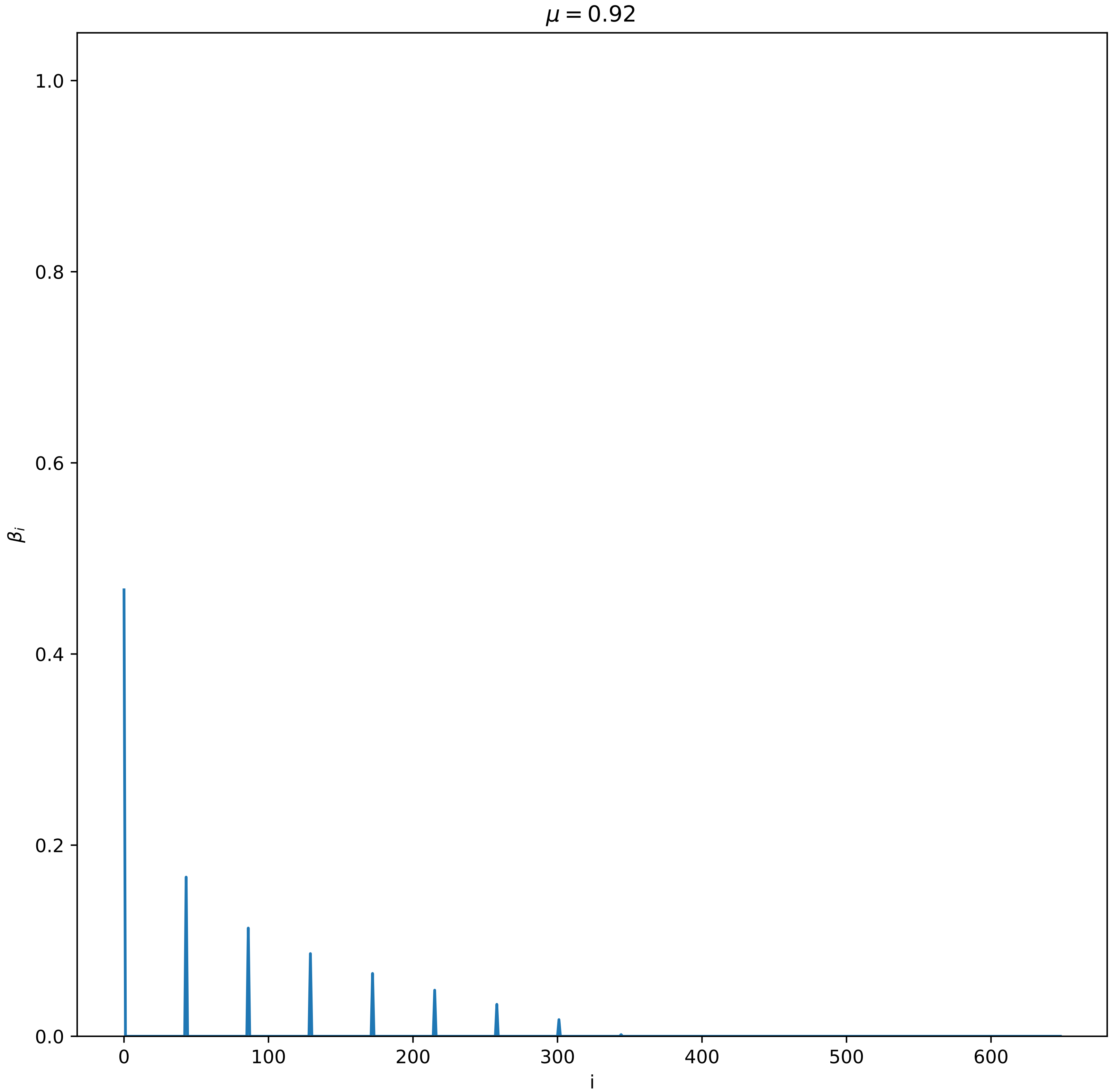


$\mu = 0.89$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

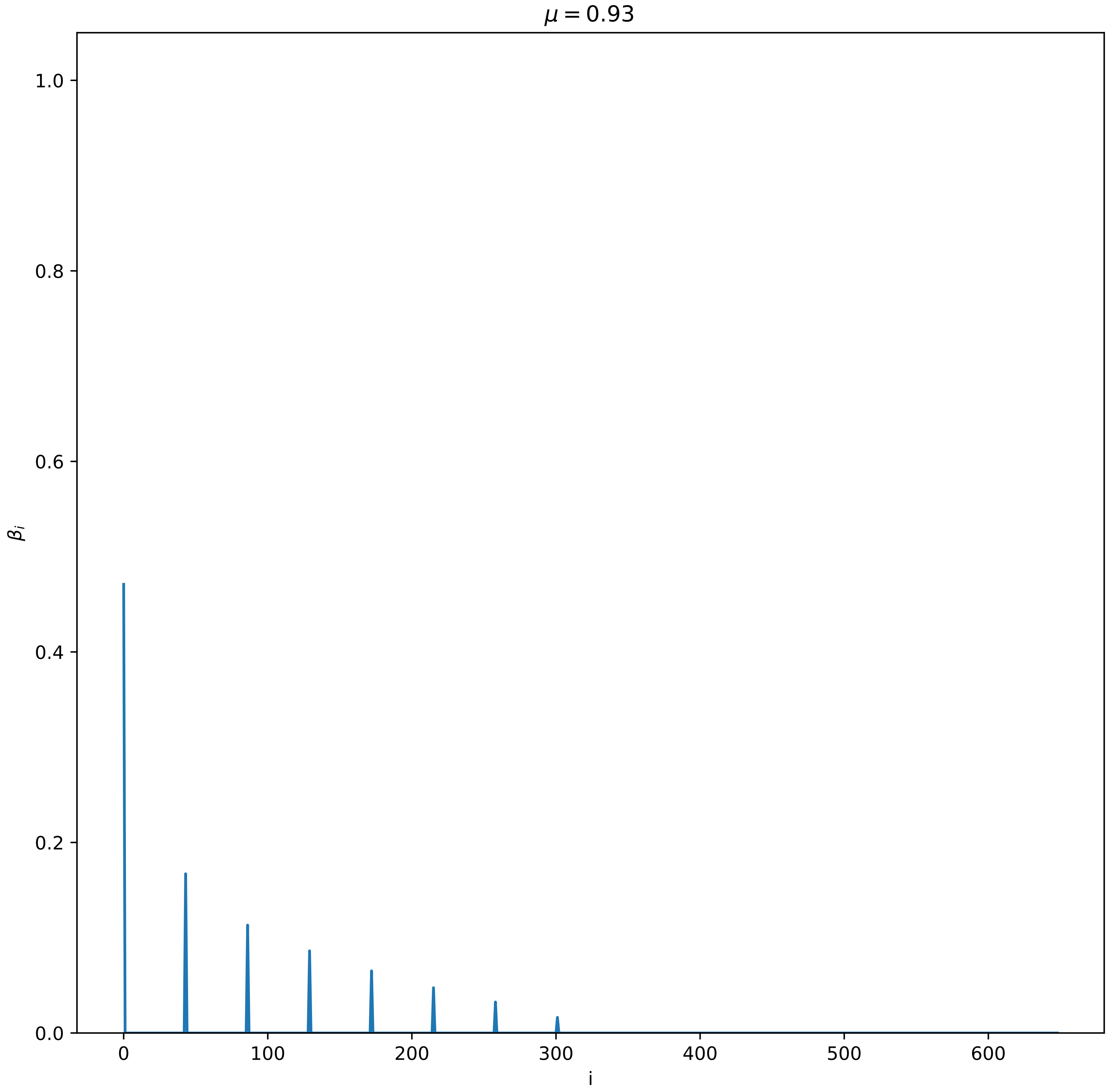

$\mu = 0.90$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



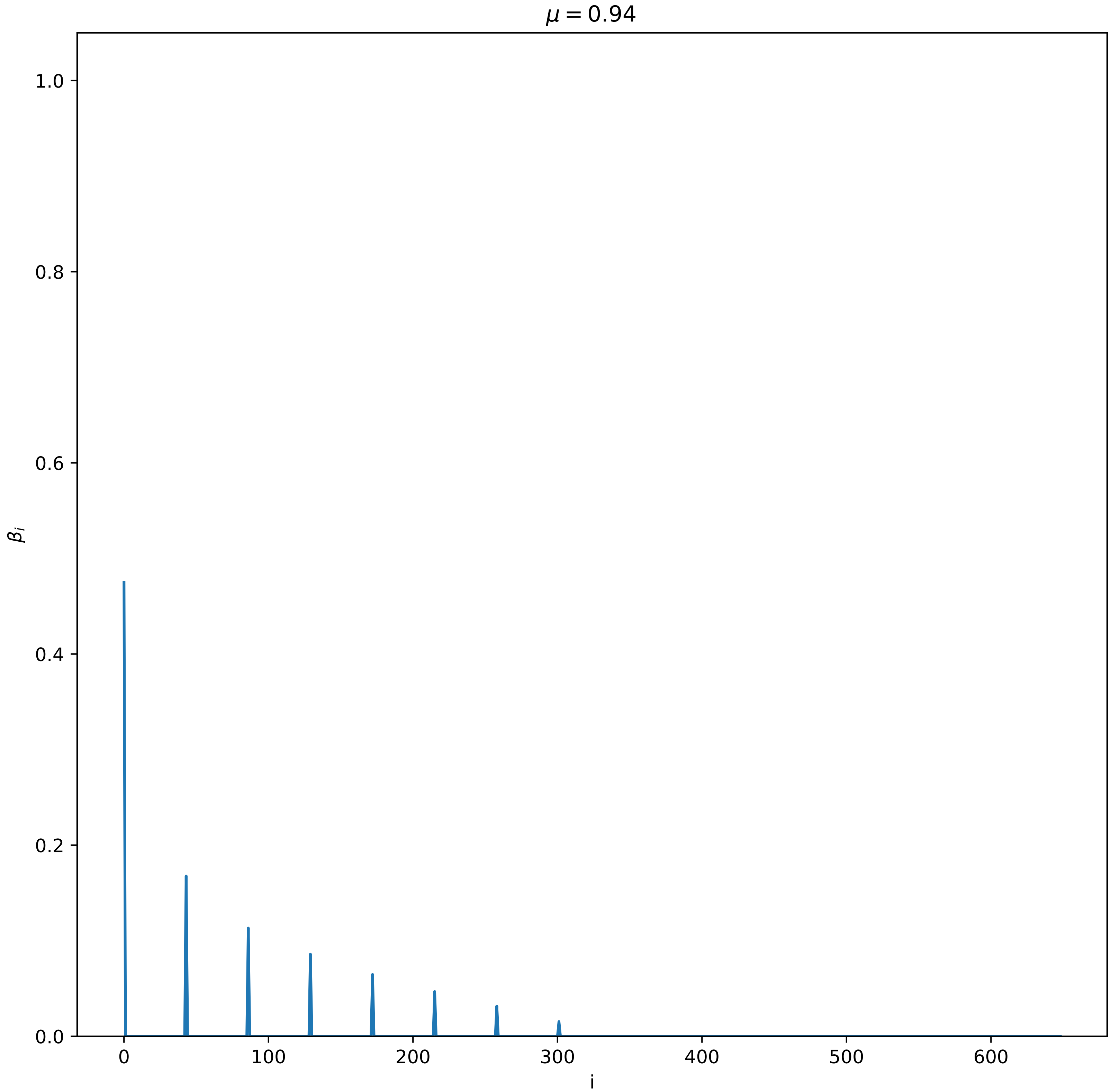
$\mu = 0.91$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



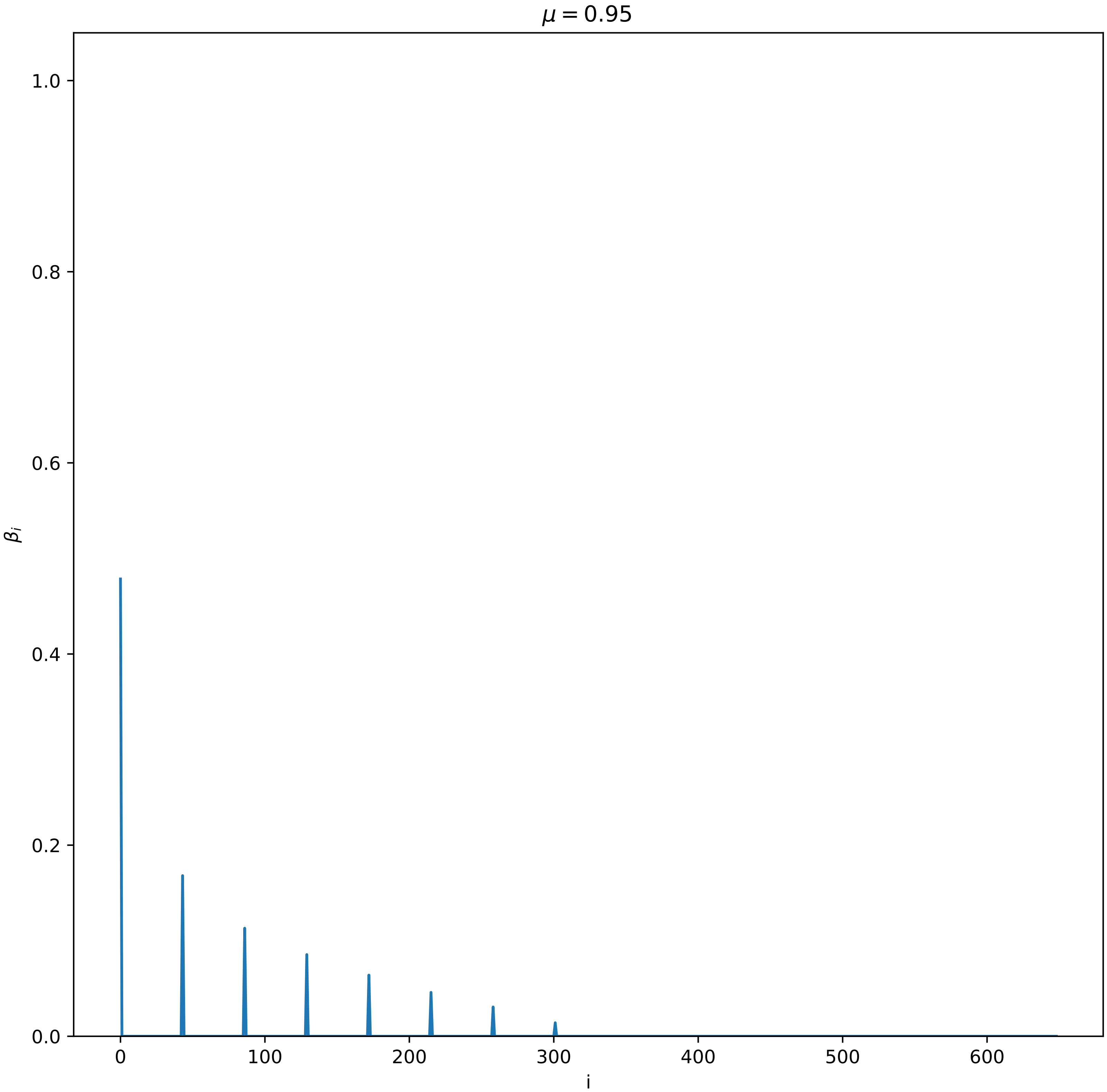
$\mu = 0.92$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



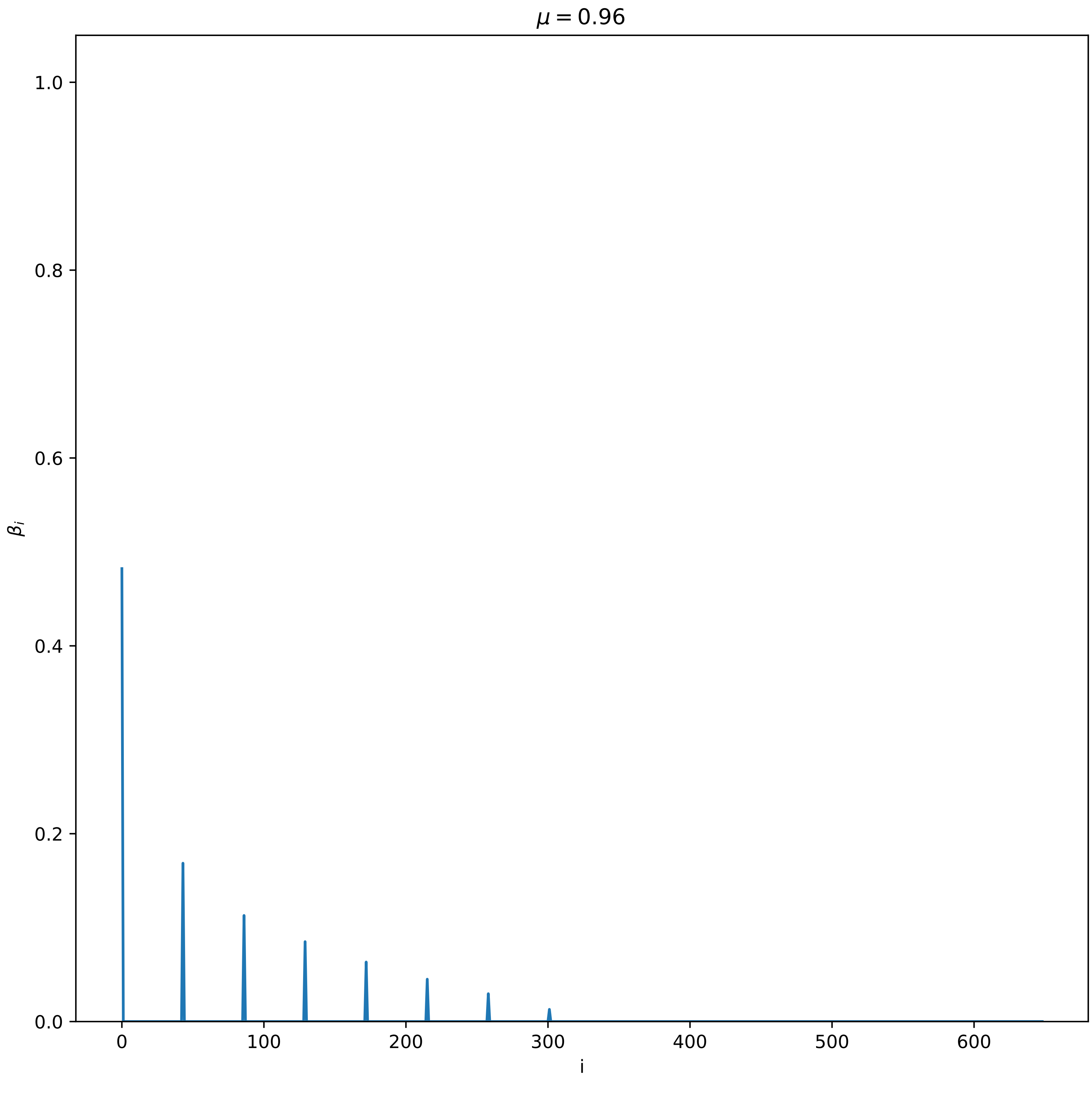
$\mu = 0.93$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



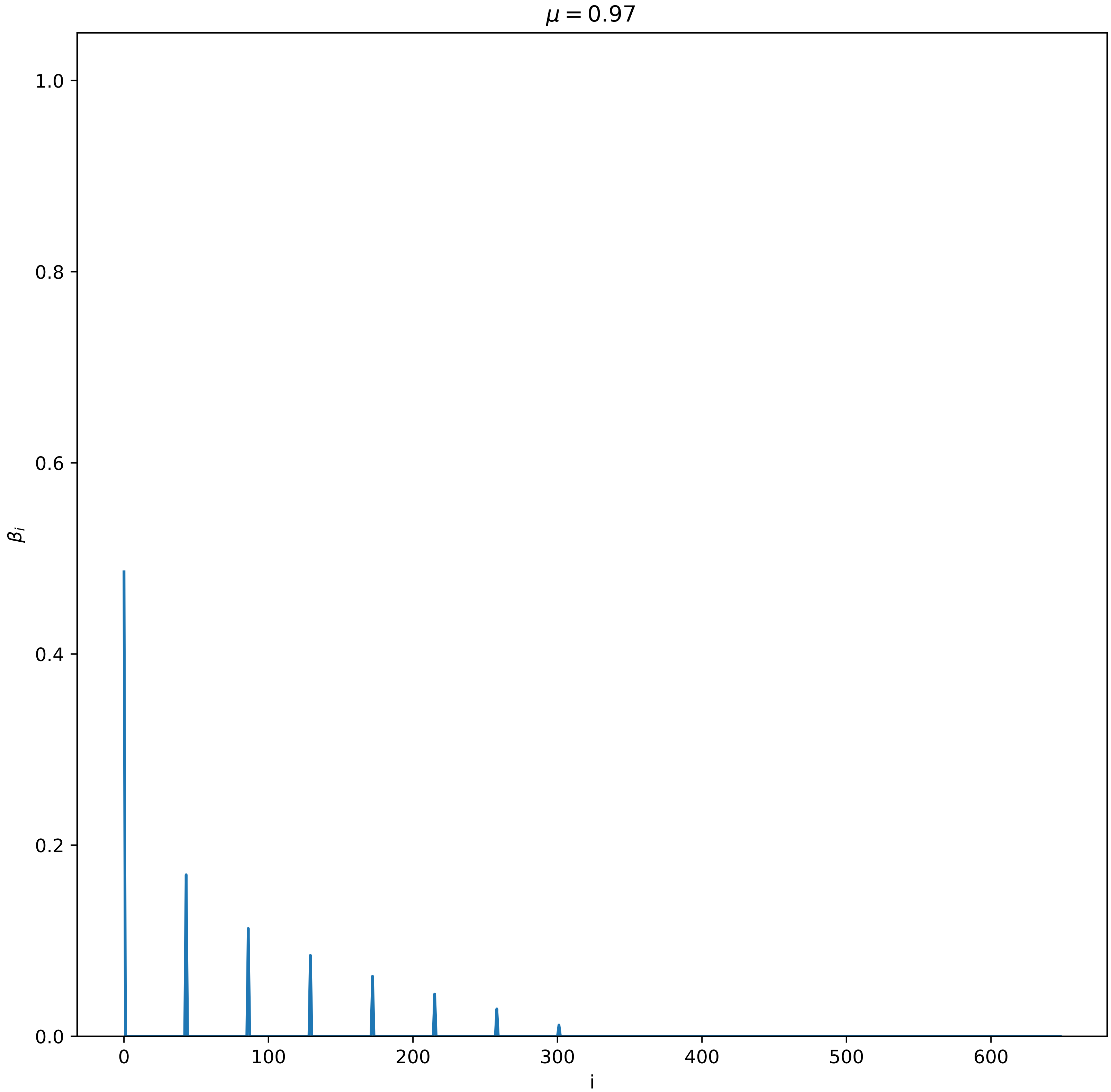
$\mu = 0.94$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

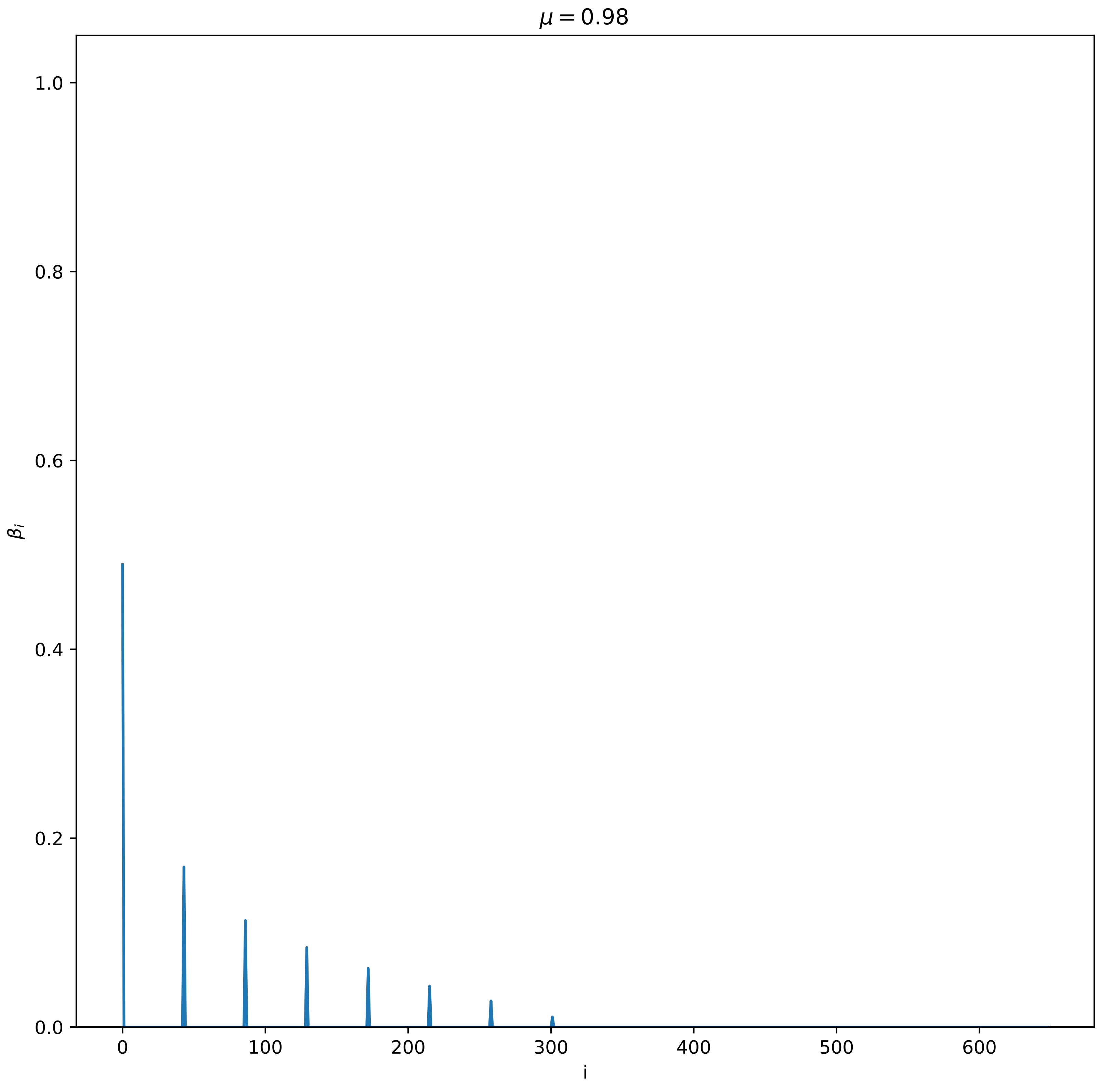


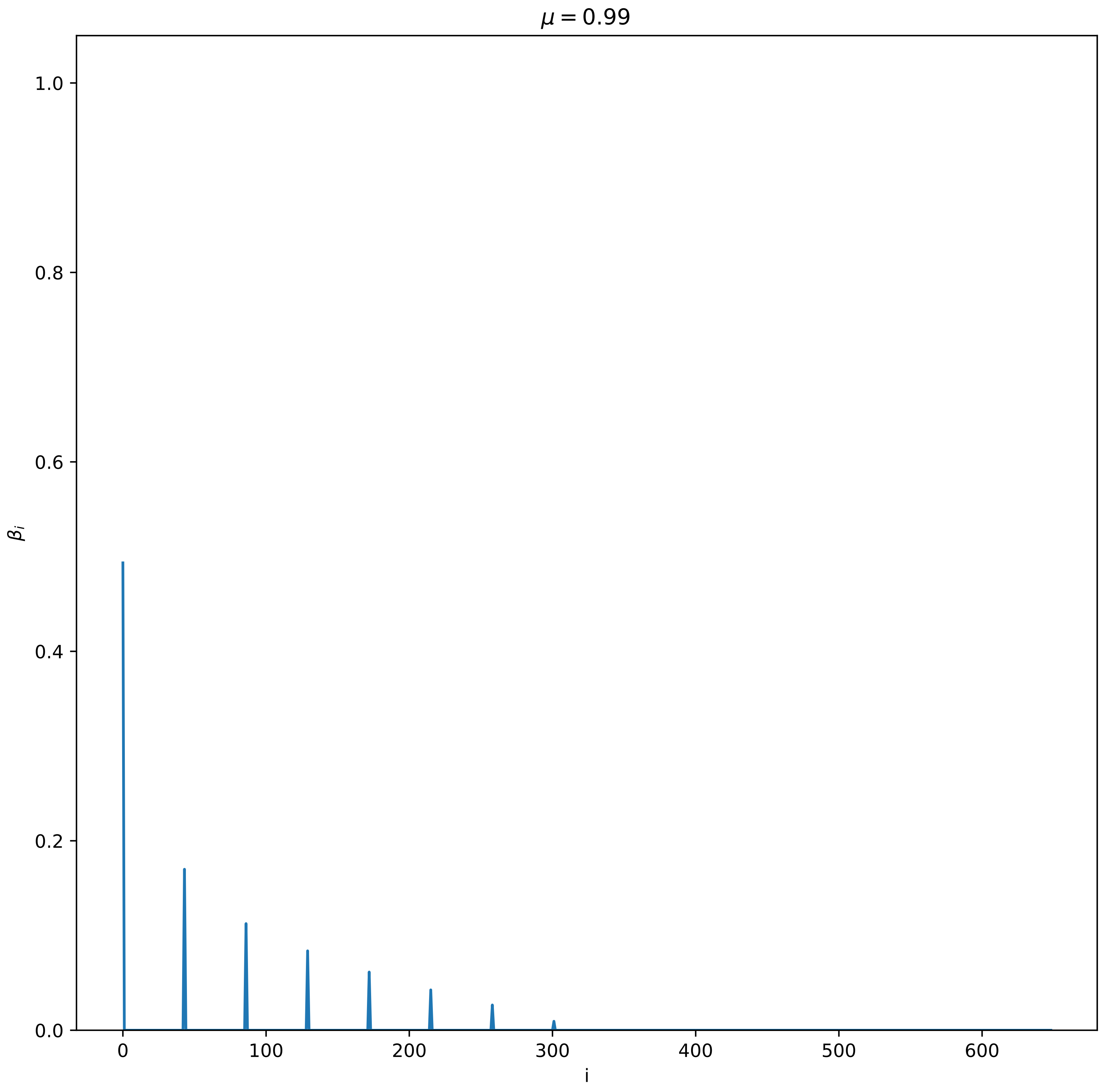
$\mu = 0.95$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



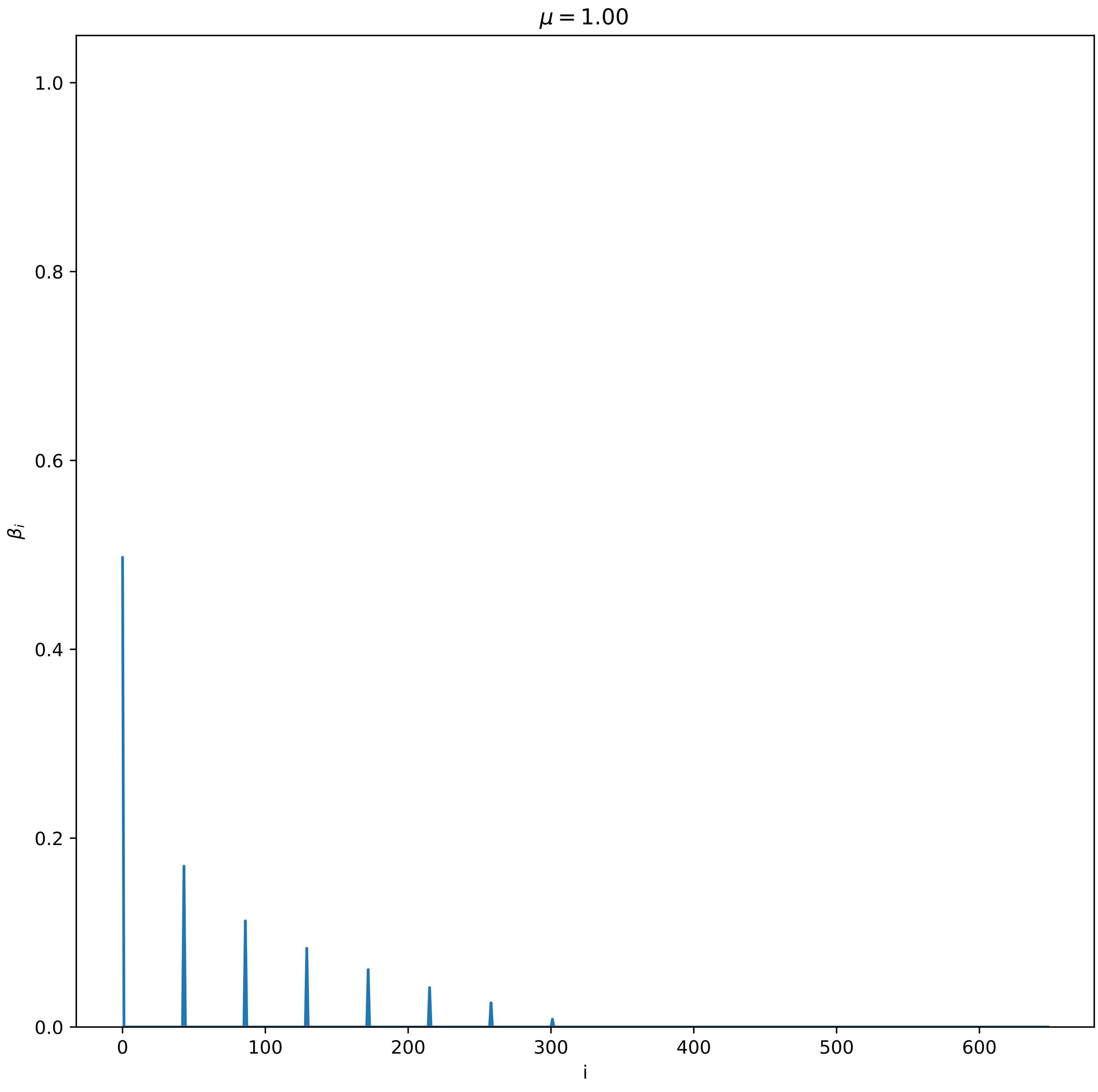
$\mu = 0.96$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

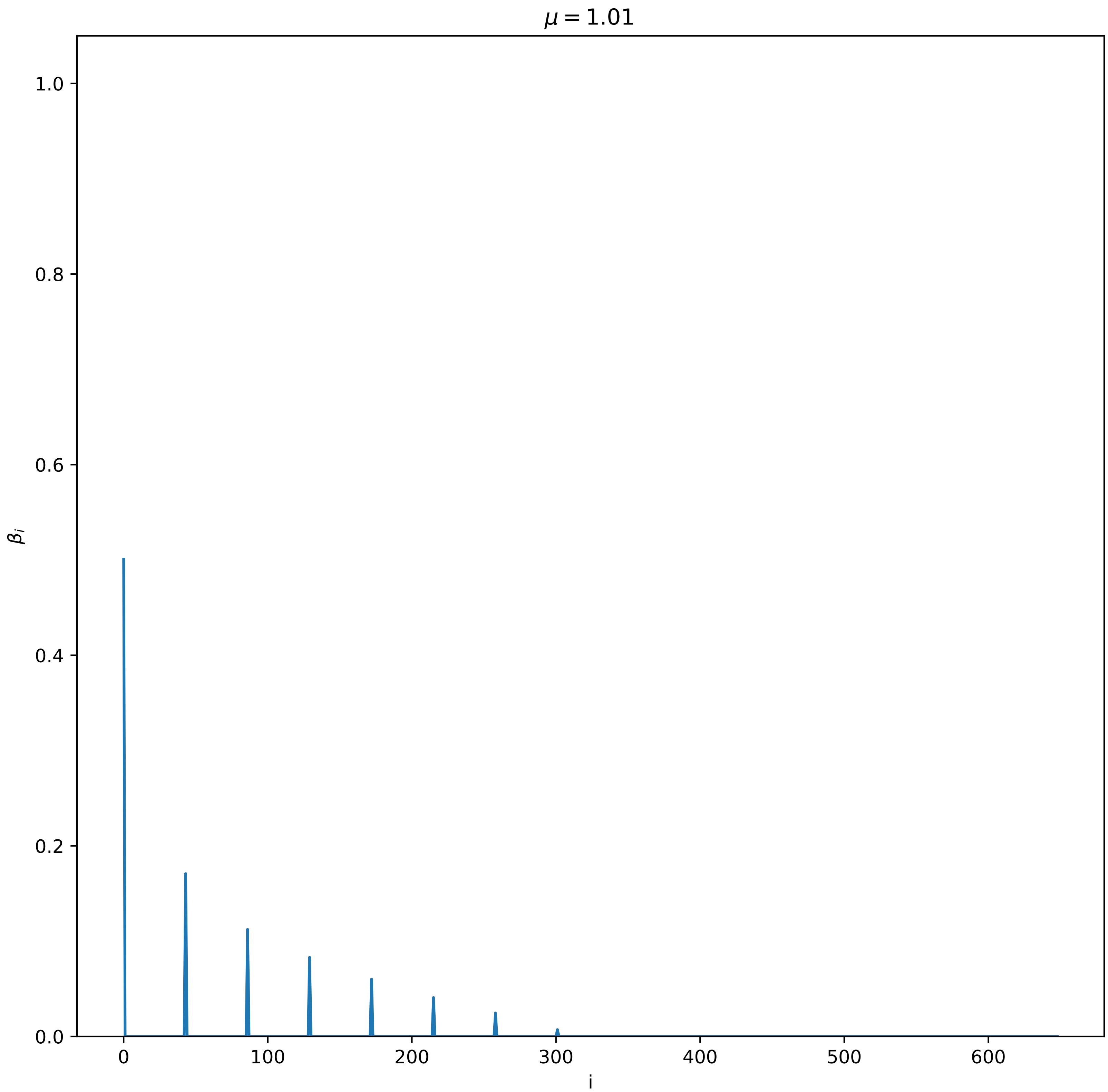

$\mu = 0.97$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

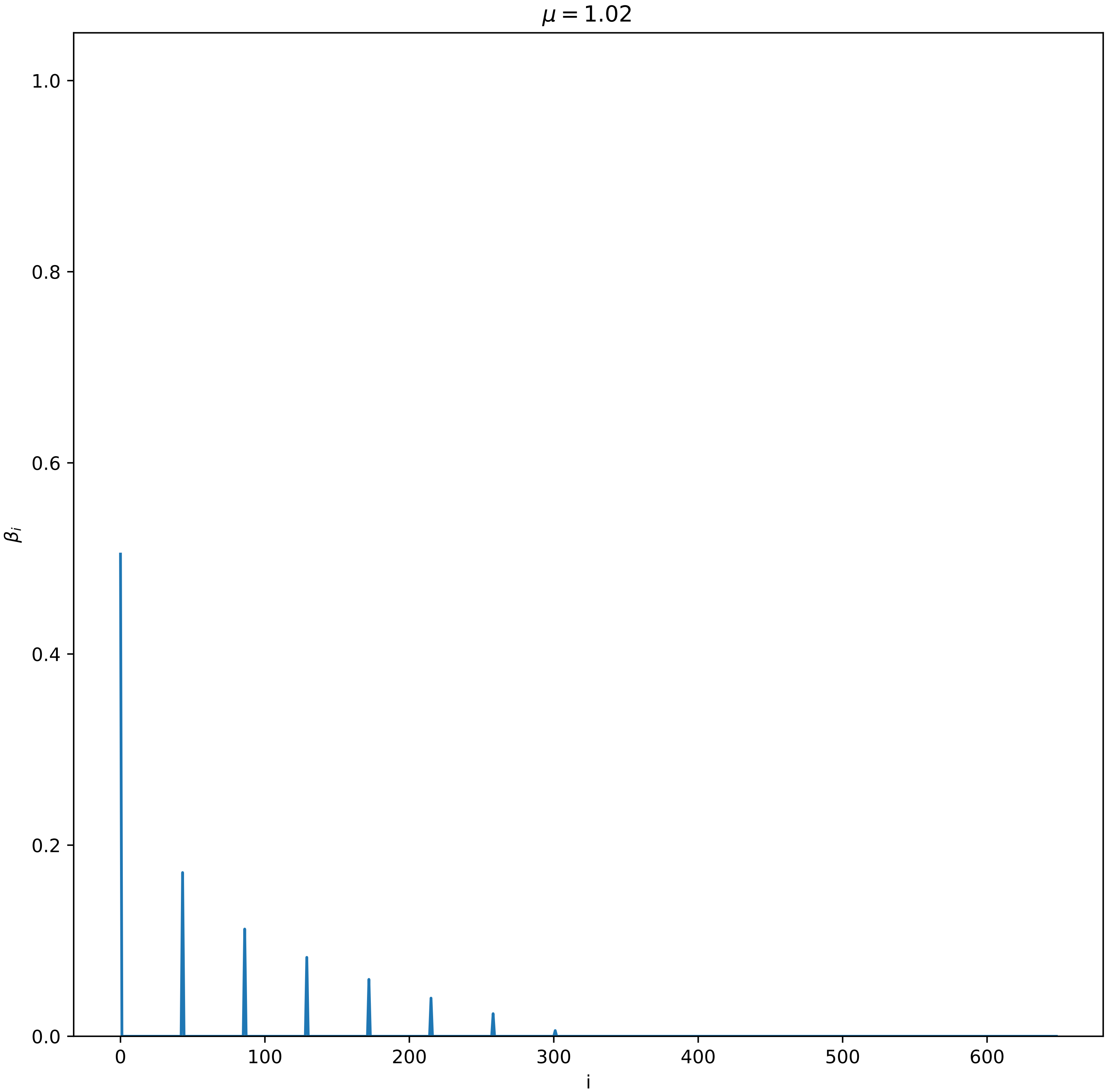

$\mu = 0.98$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



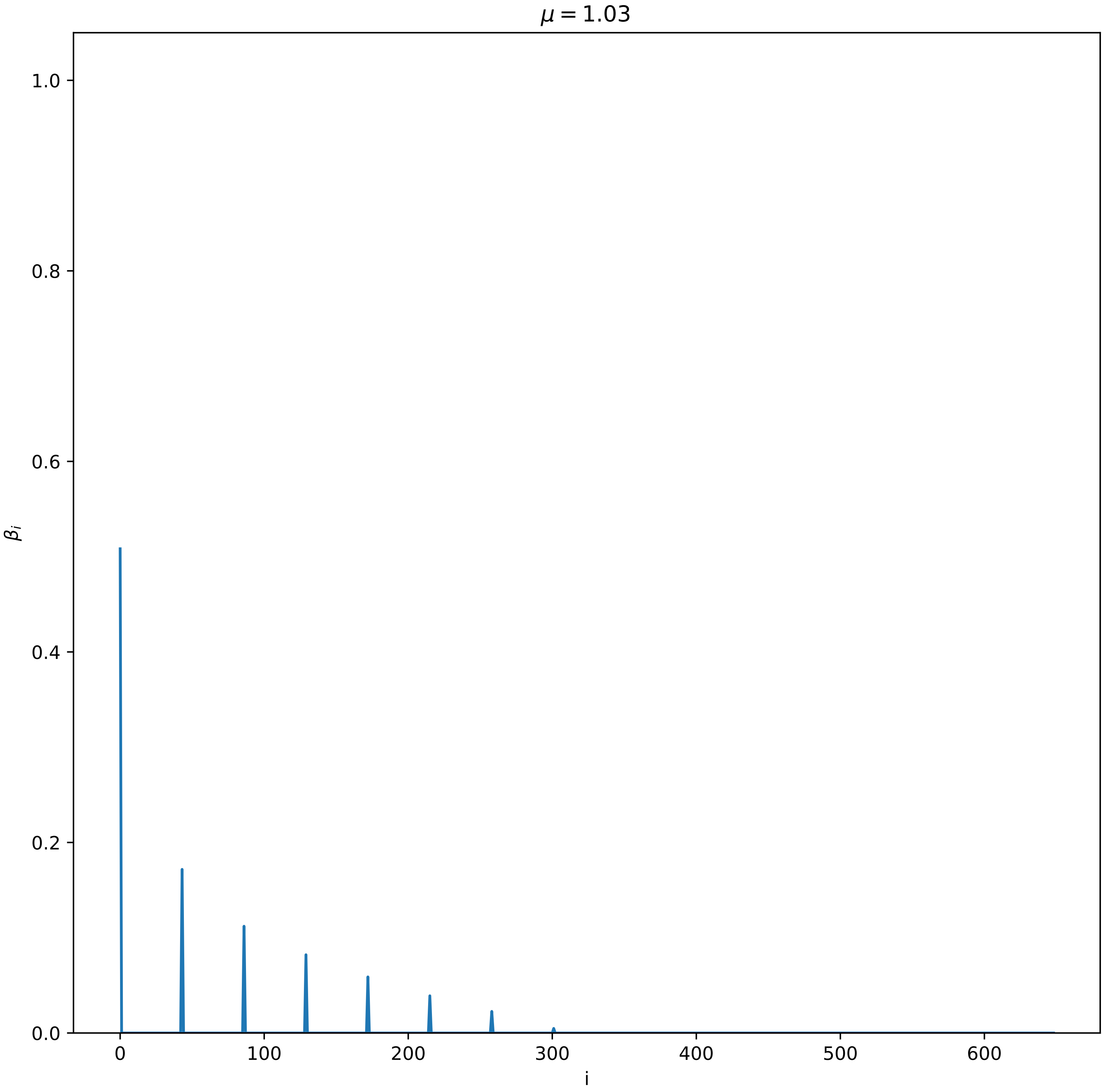
$\mu = 0.99$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.00$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

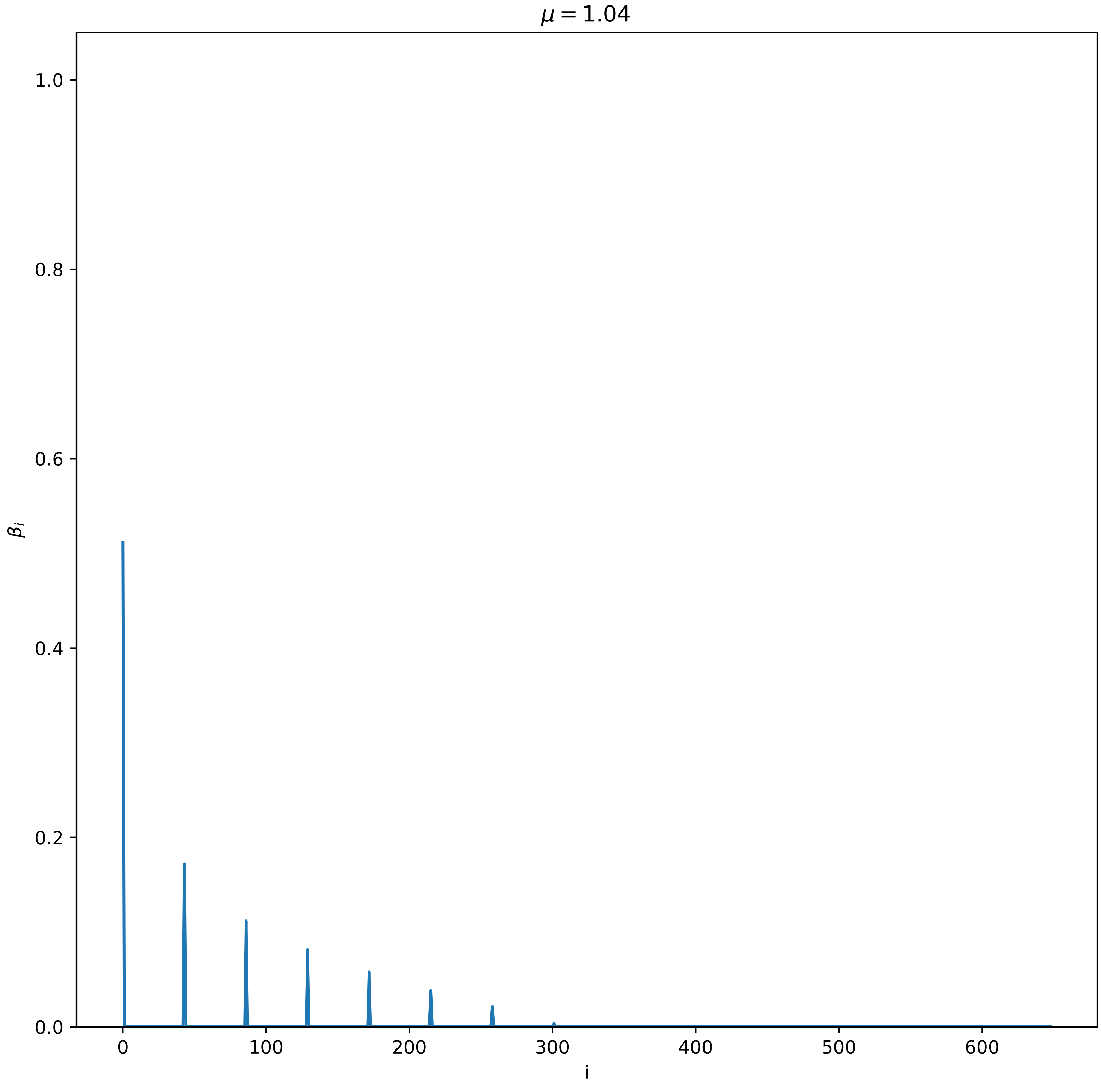

$\mu = 1.01$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



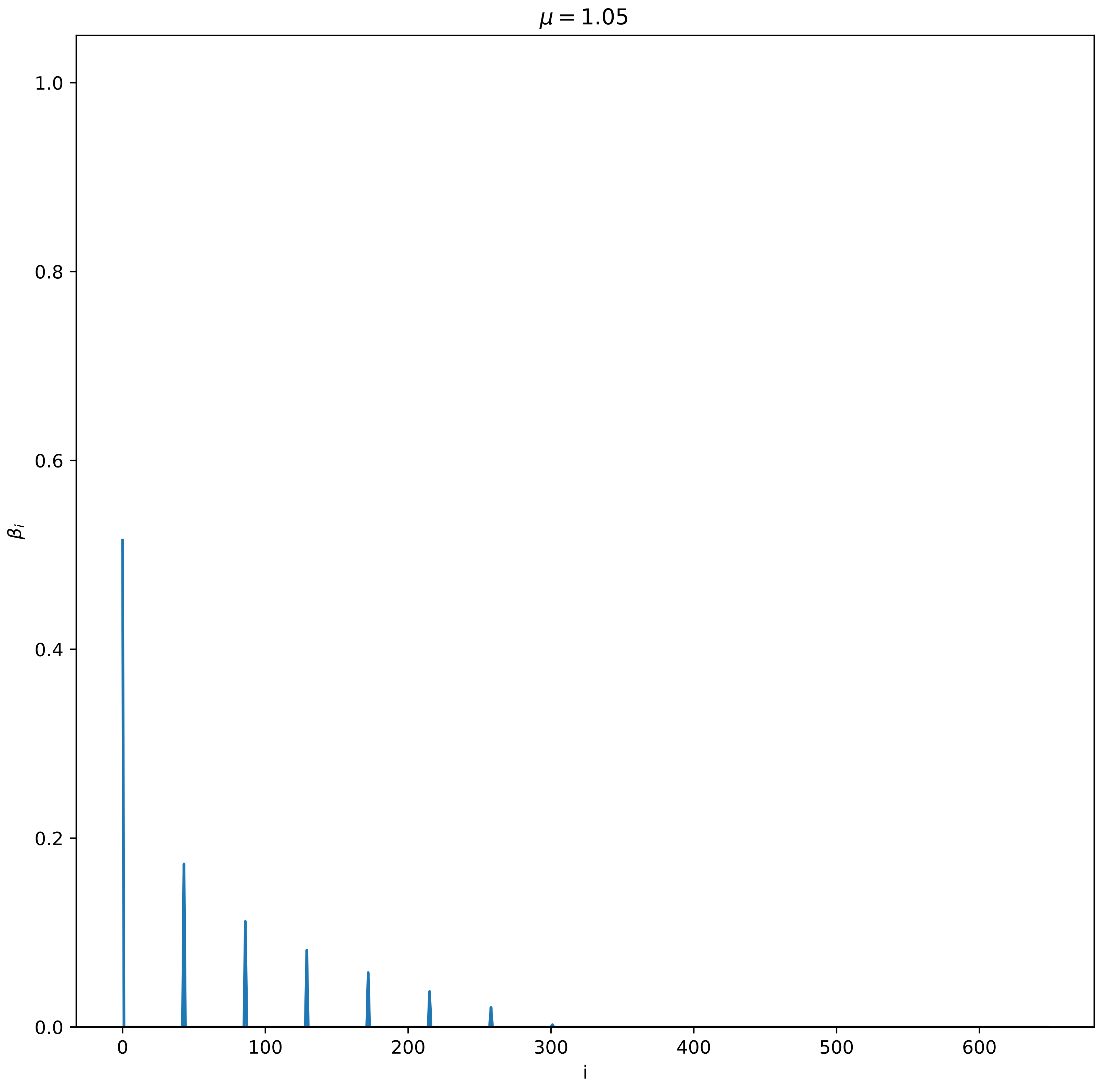
$\mu = 1.02$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



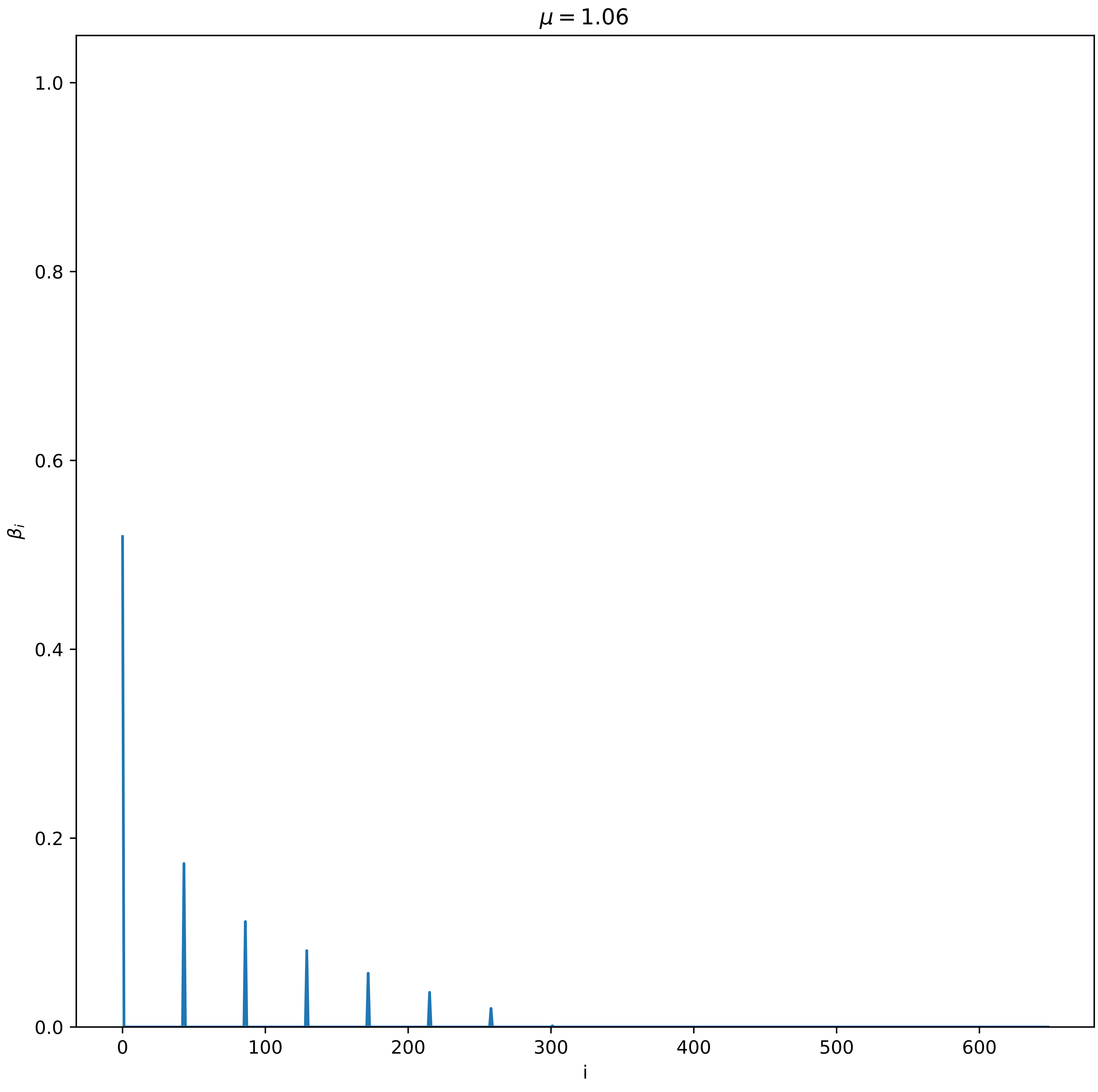
$\mu = 1.03$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

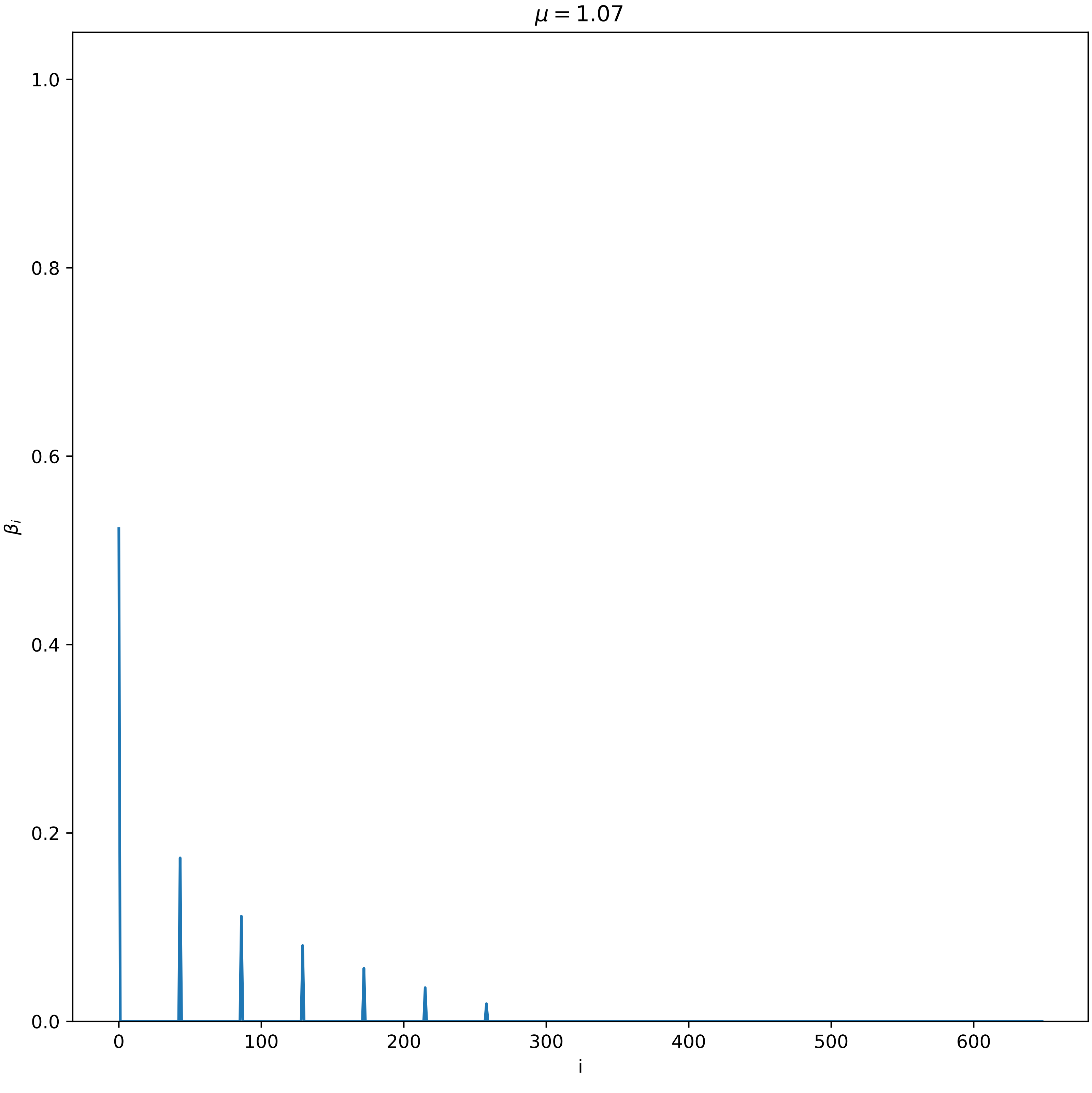


$\mu = 1.04$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

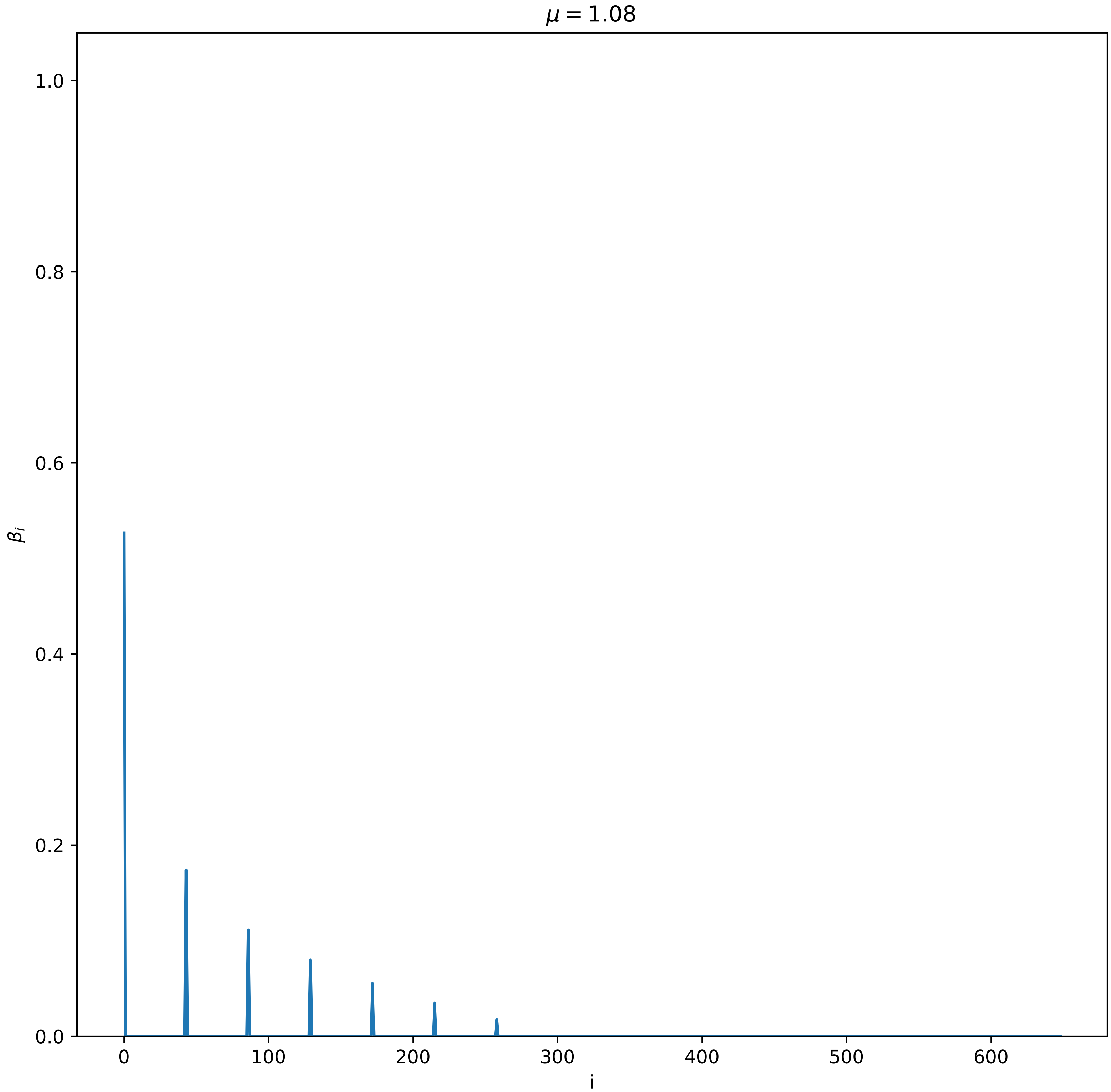


$\mu = 1.05$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

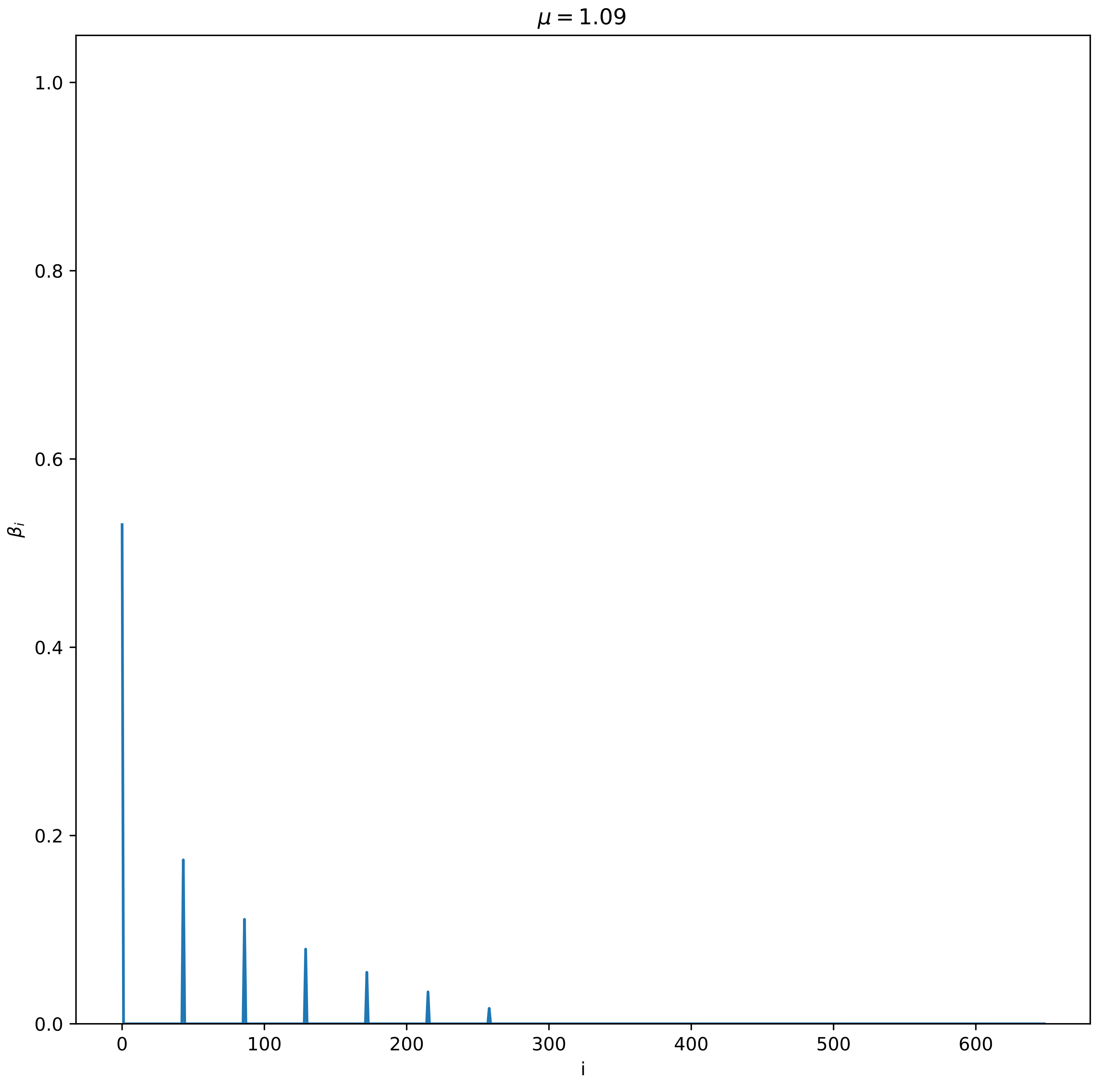

$\mu = 1.06$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

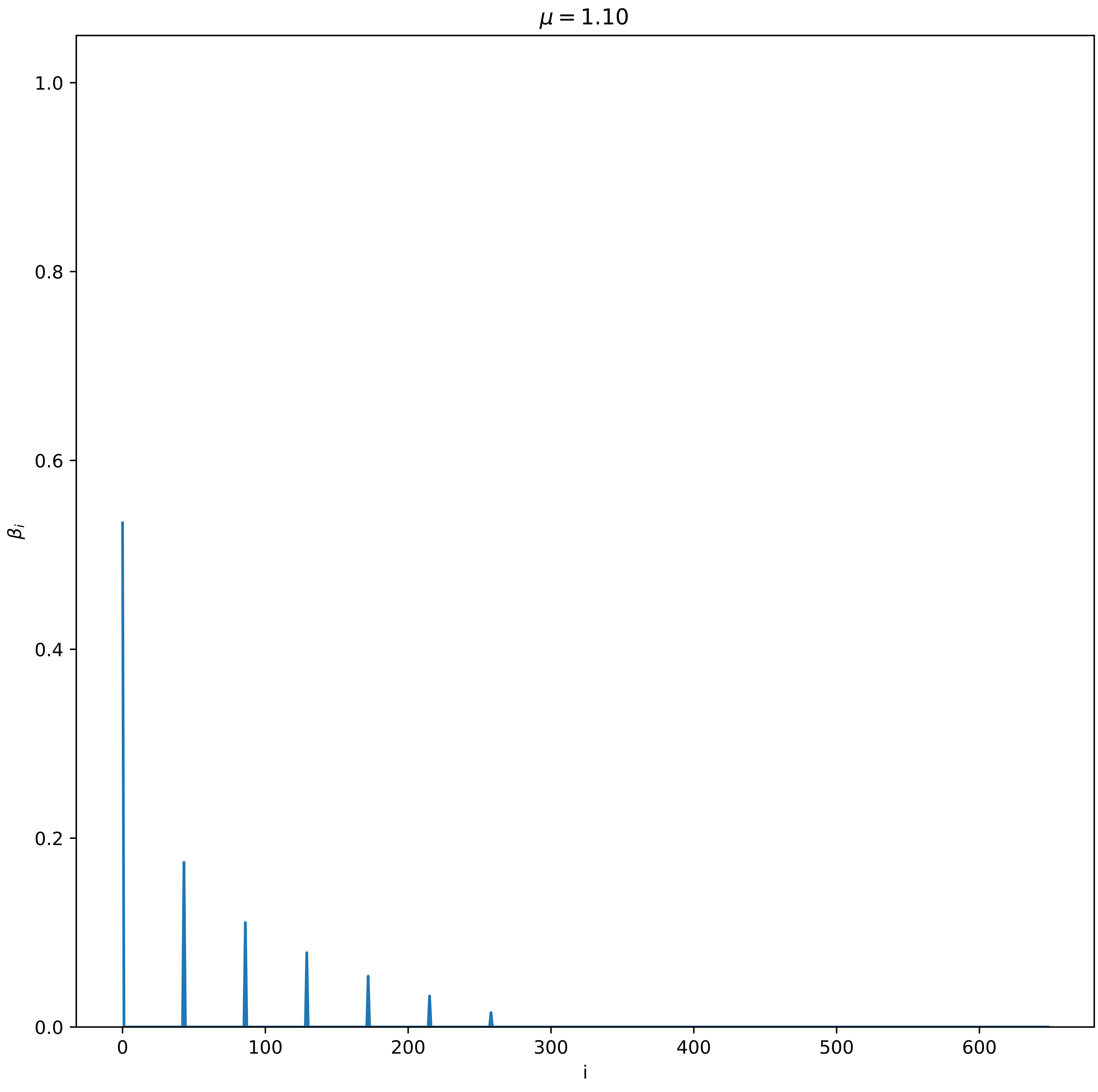


$\mu = 1.07$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

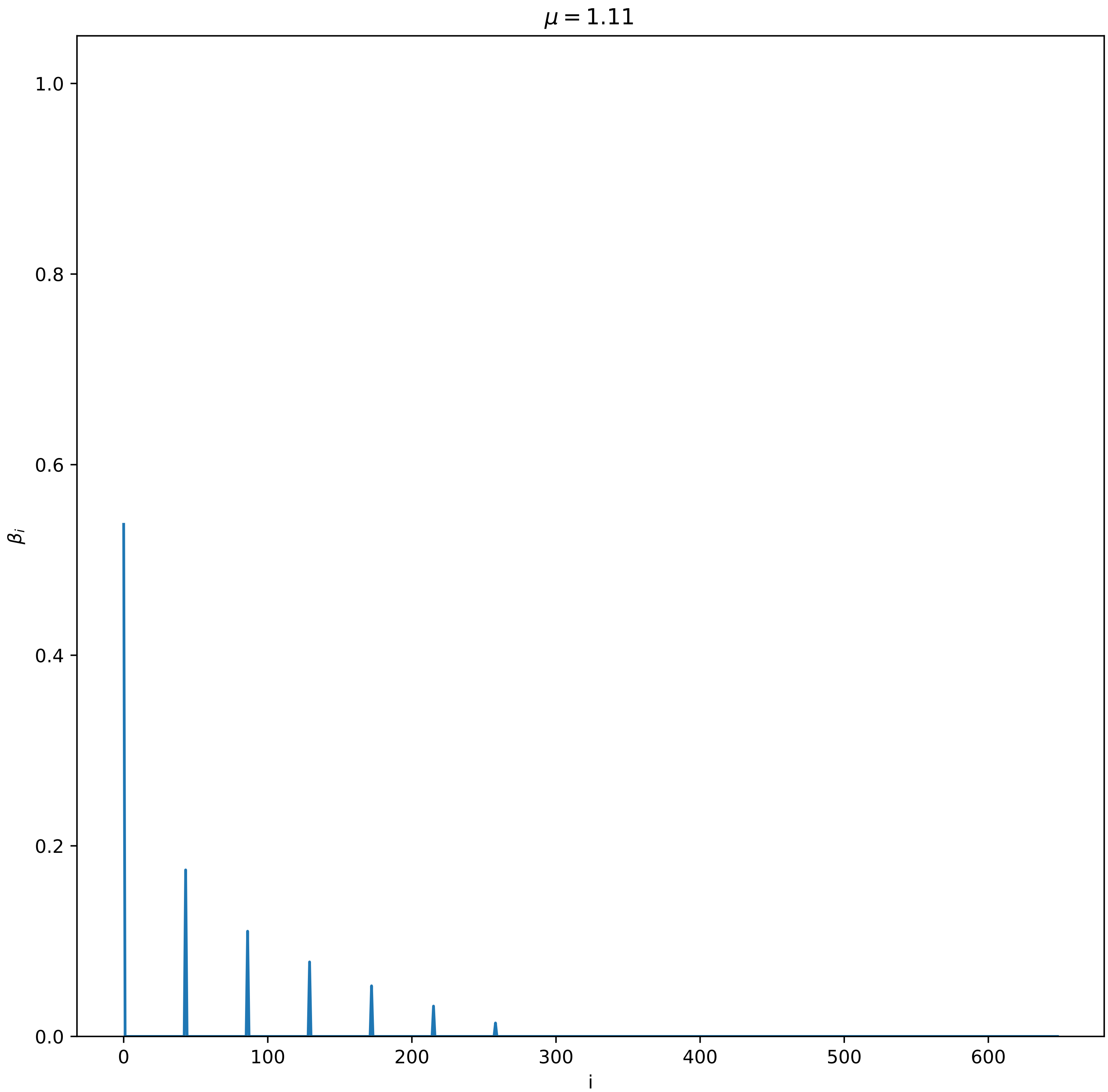


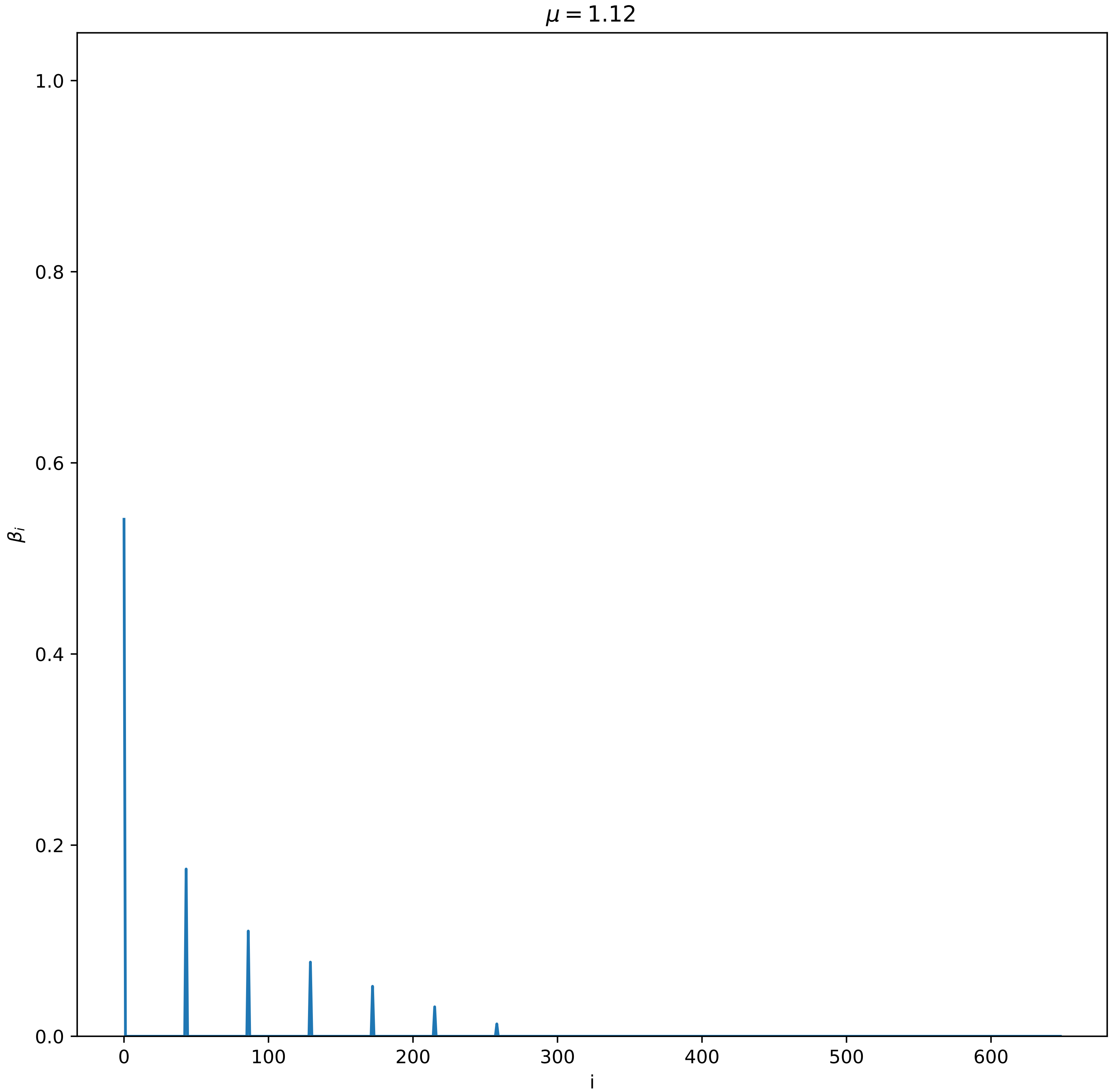
$\mu = 1.08$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.09$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

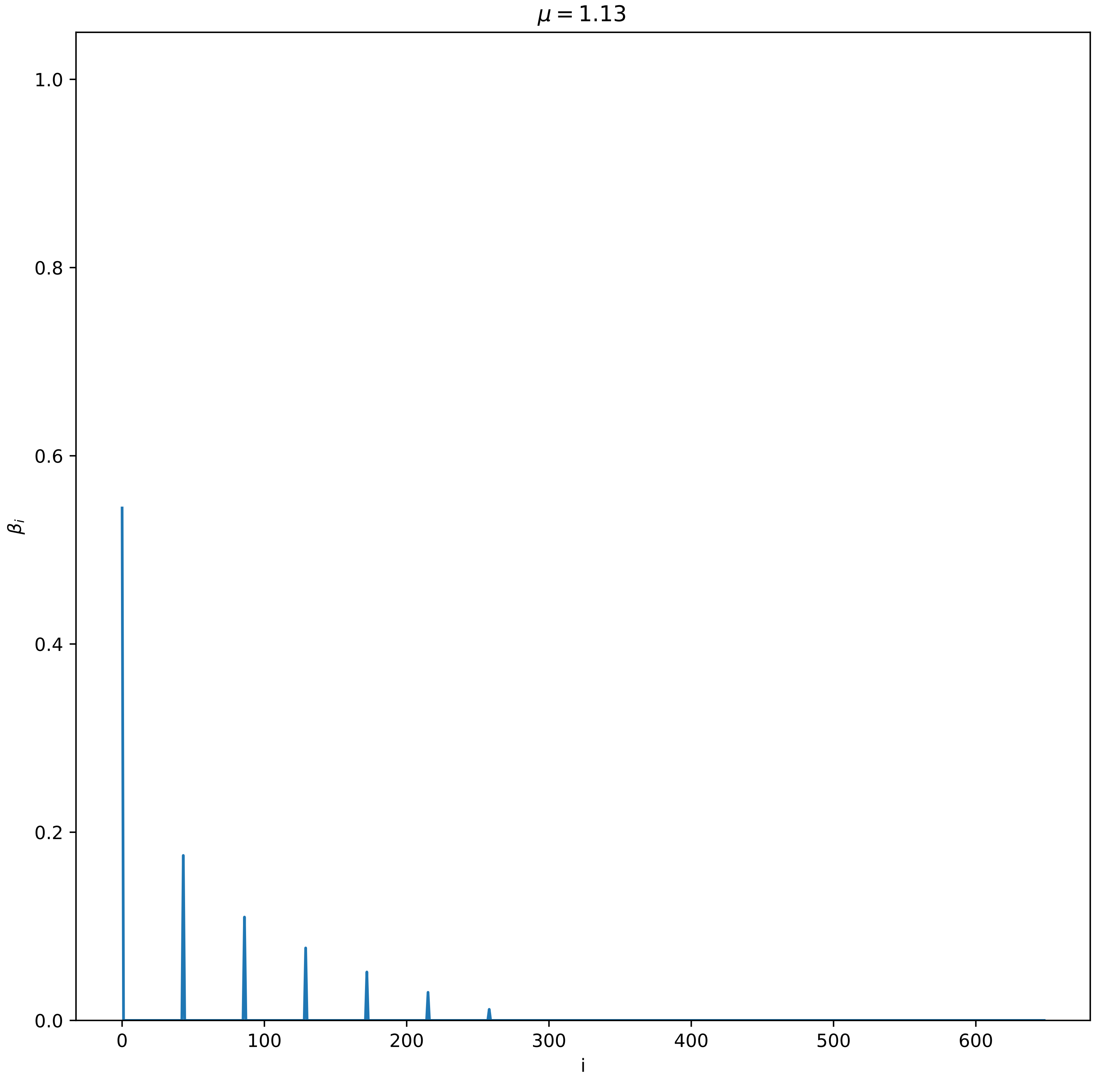


$\mu = 1.10$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

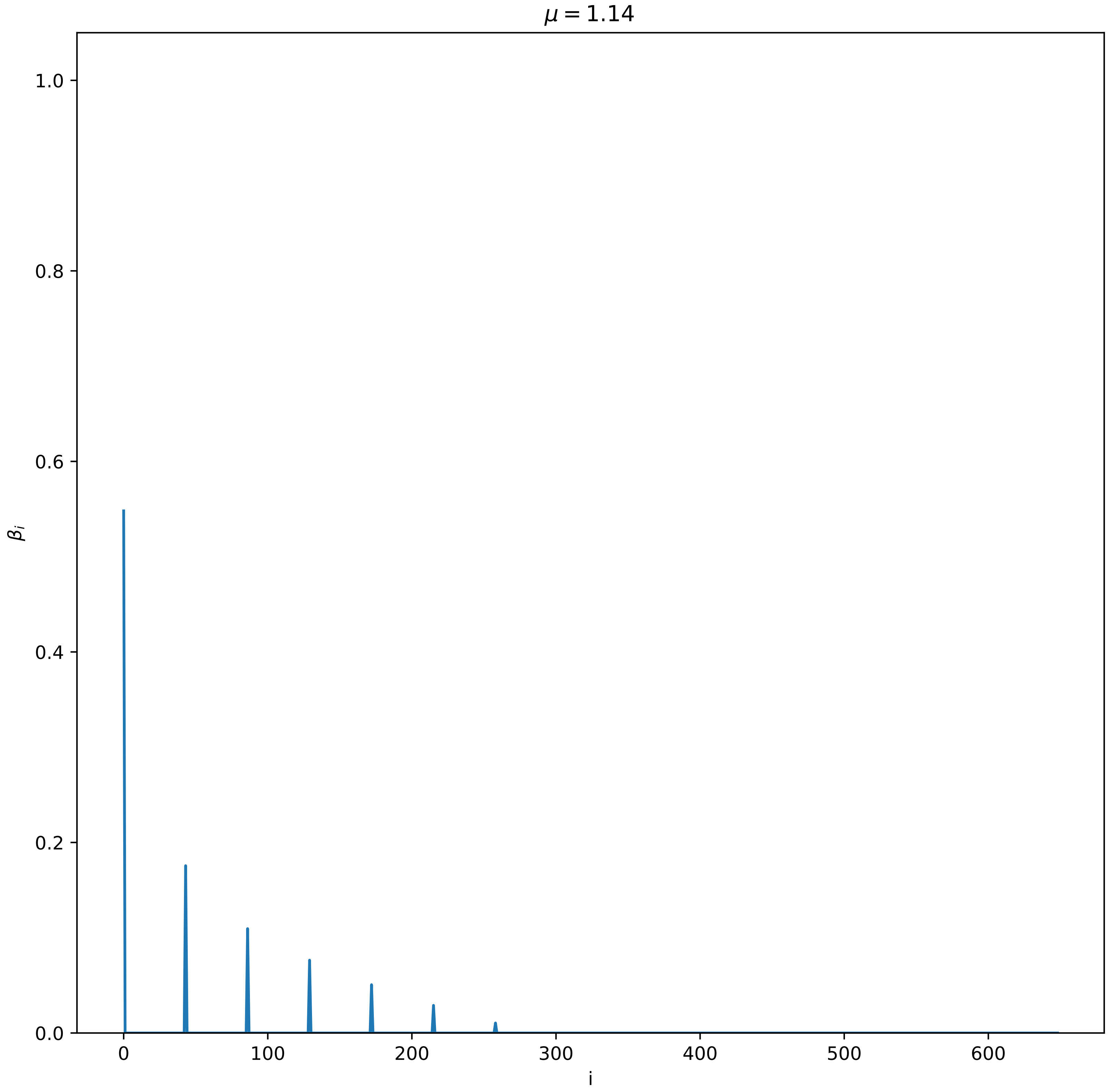

$\mu = 1.11$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



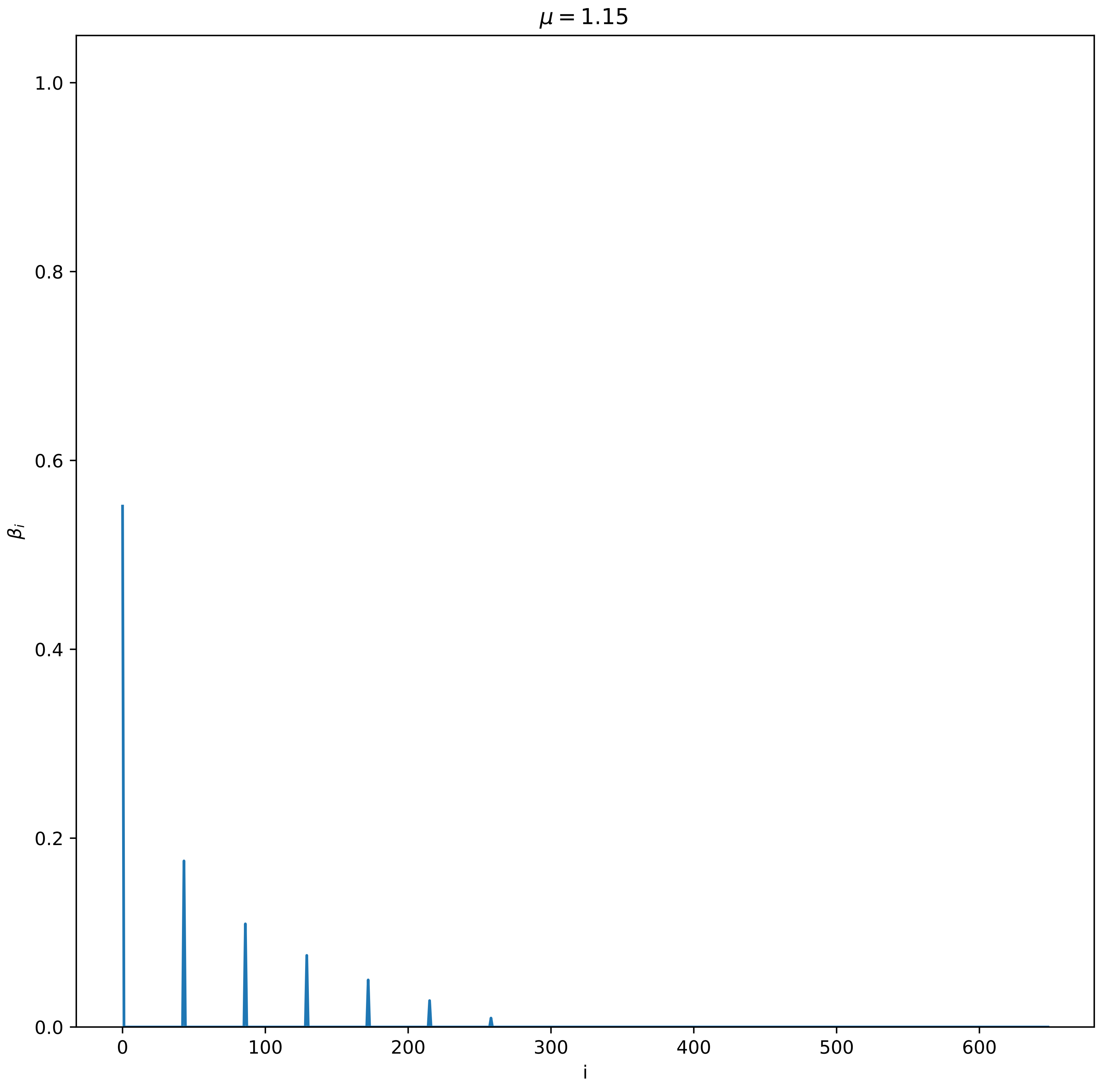
$\mu = 1.12$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



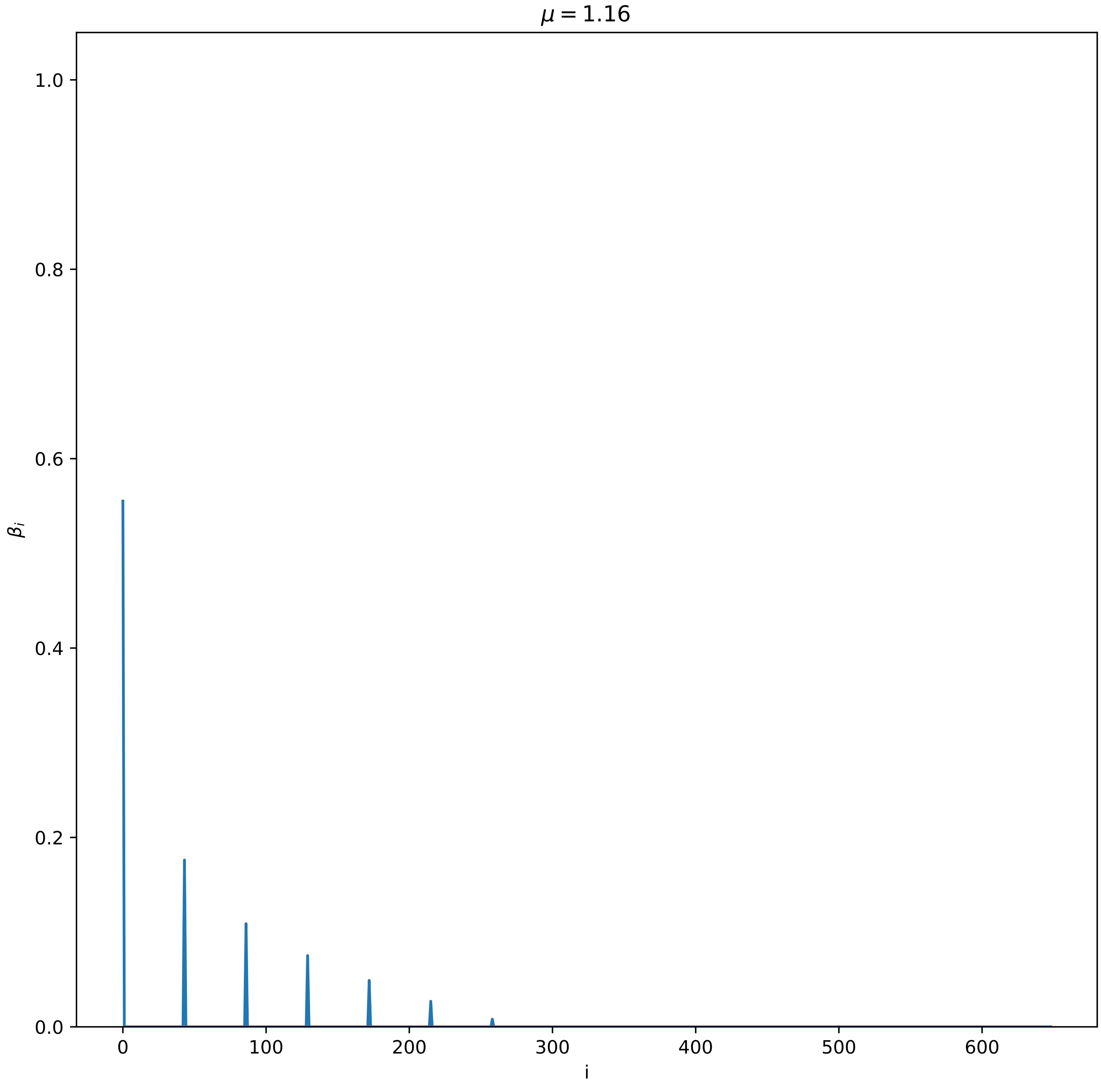
$\mu = 1.13$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



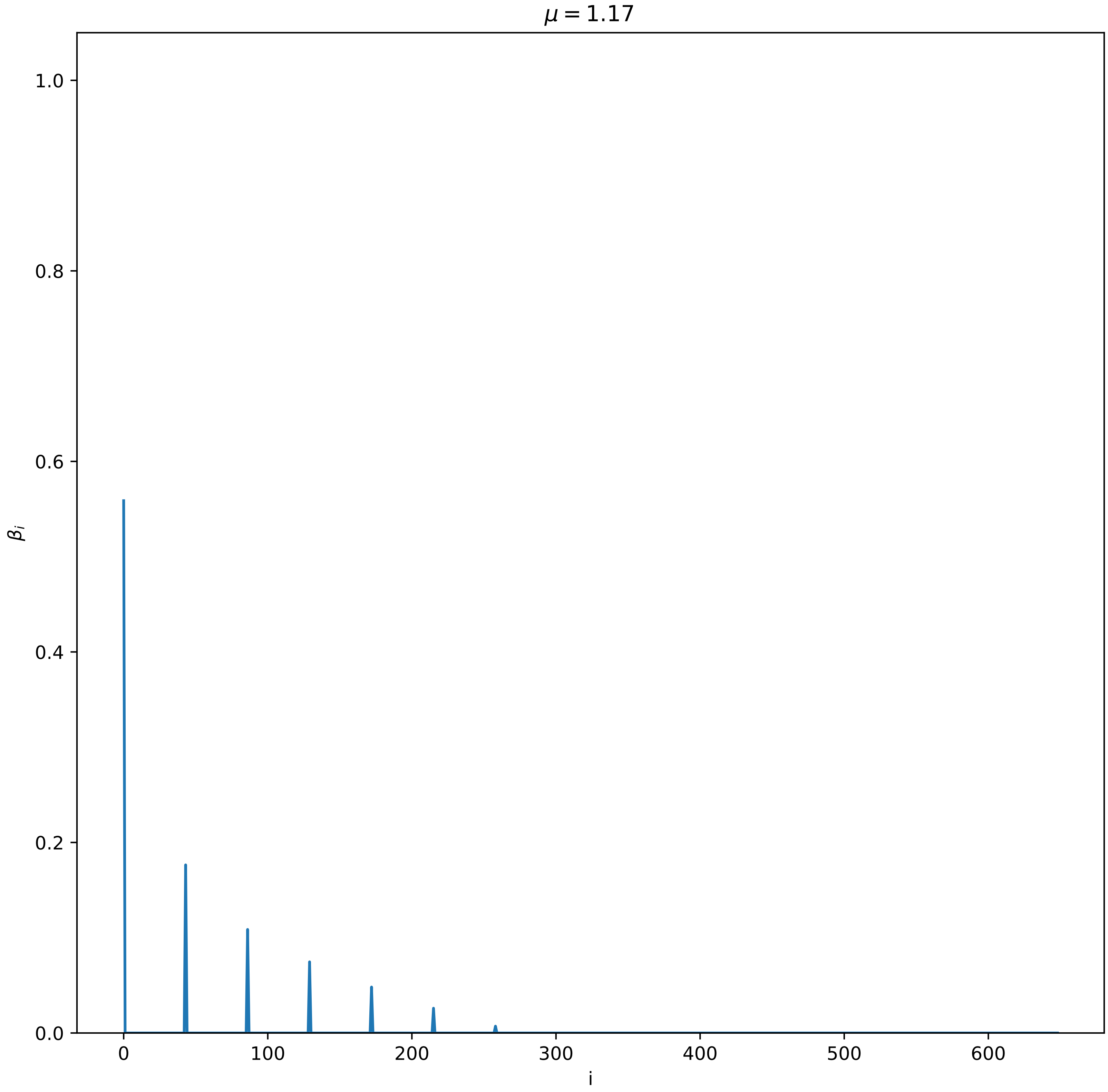
$\mu = 1.14$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



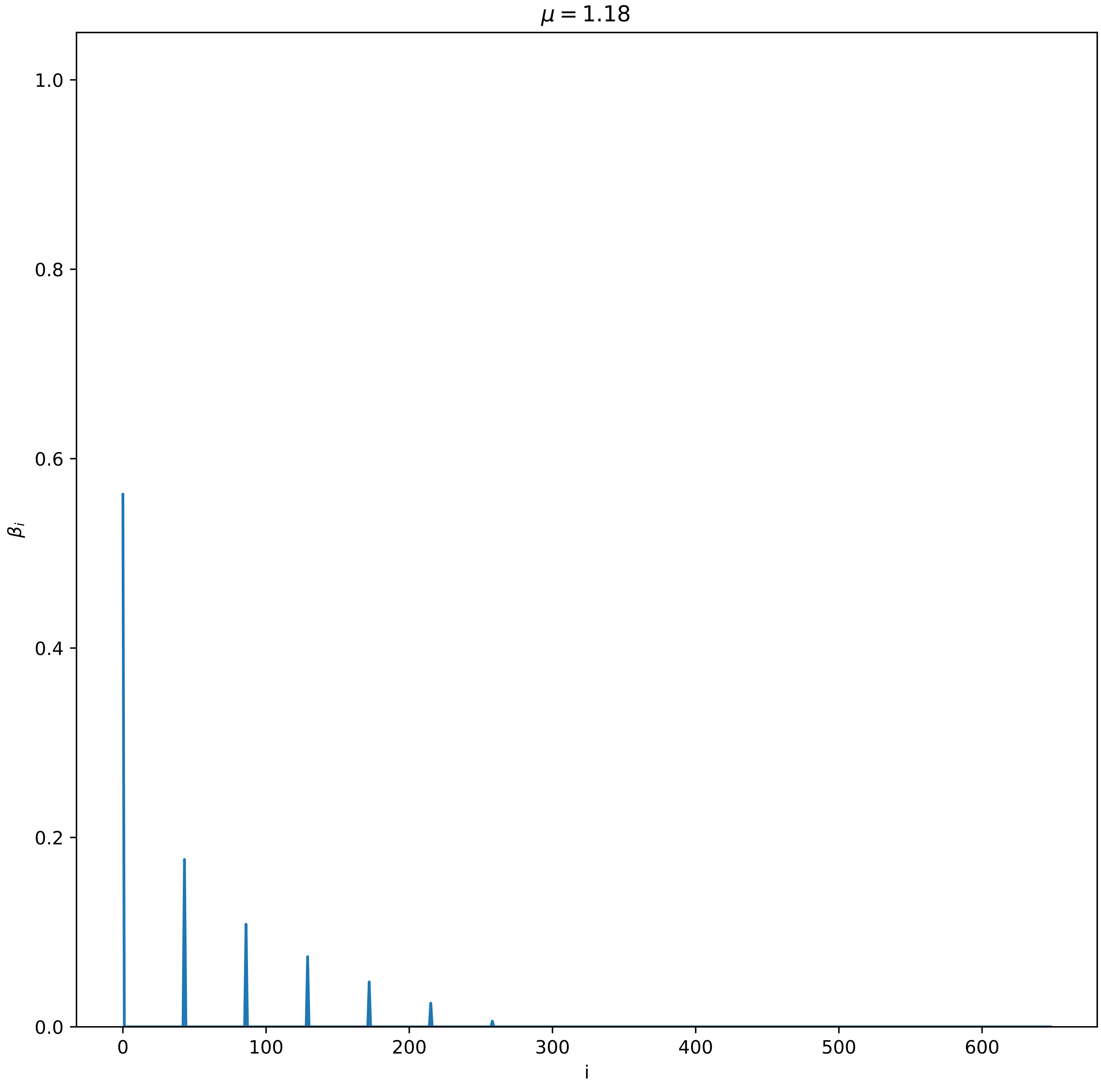
$\mu = 1.15$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



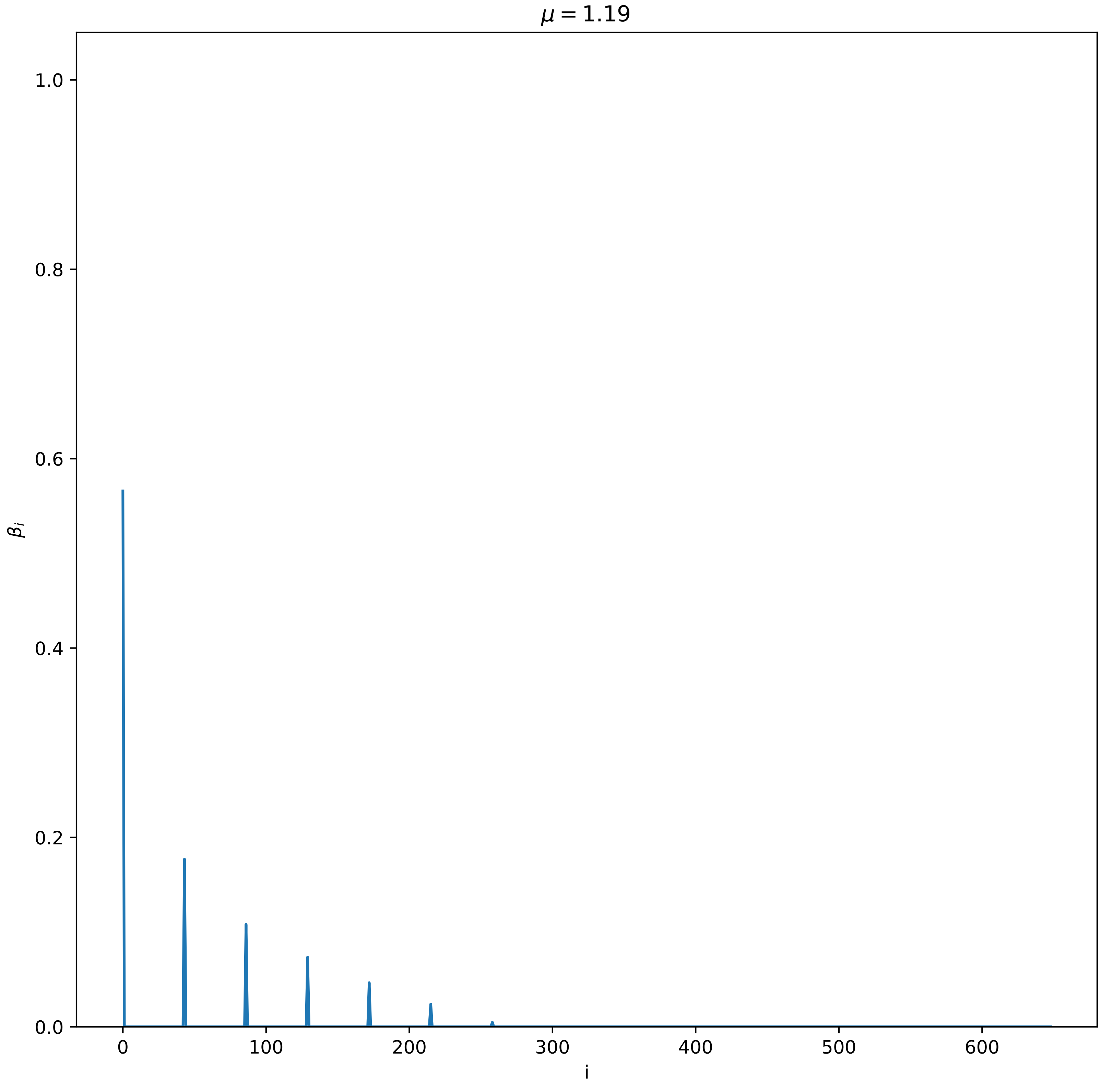
$\mu = 1.16$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



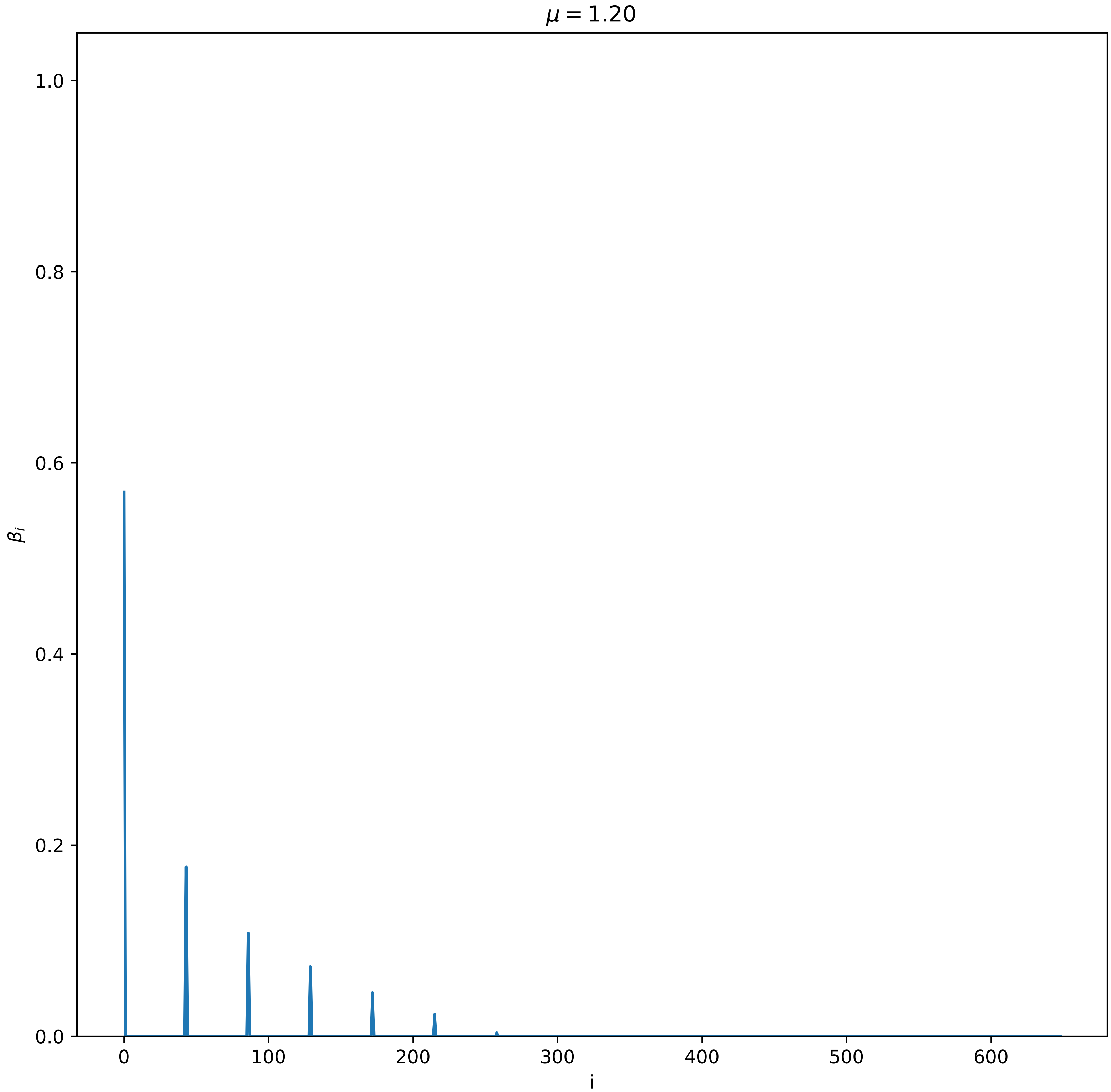
$\mu = 1.17$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



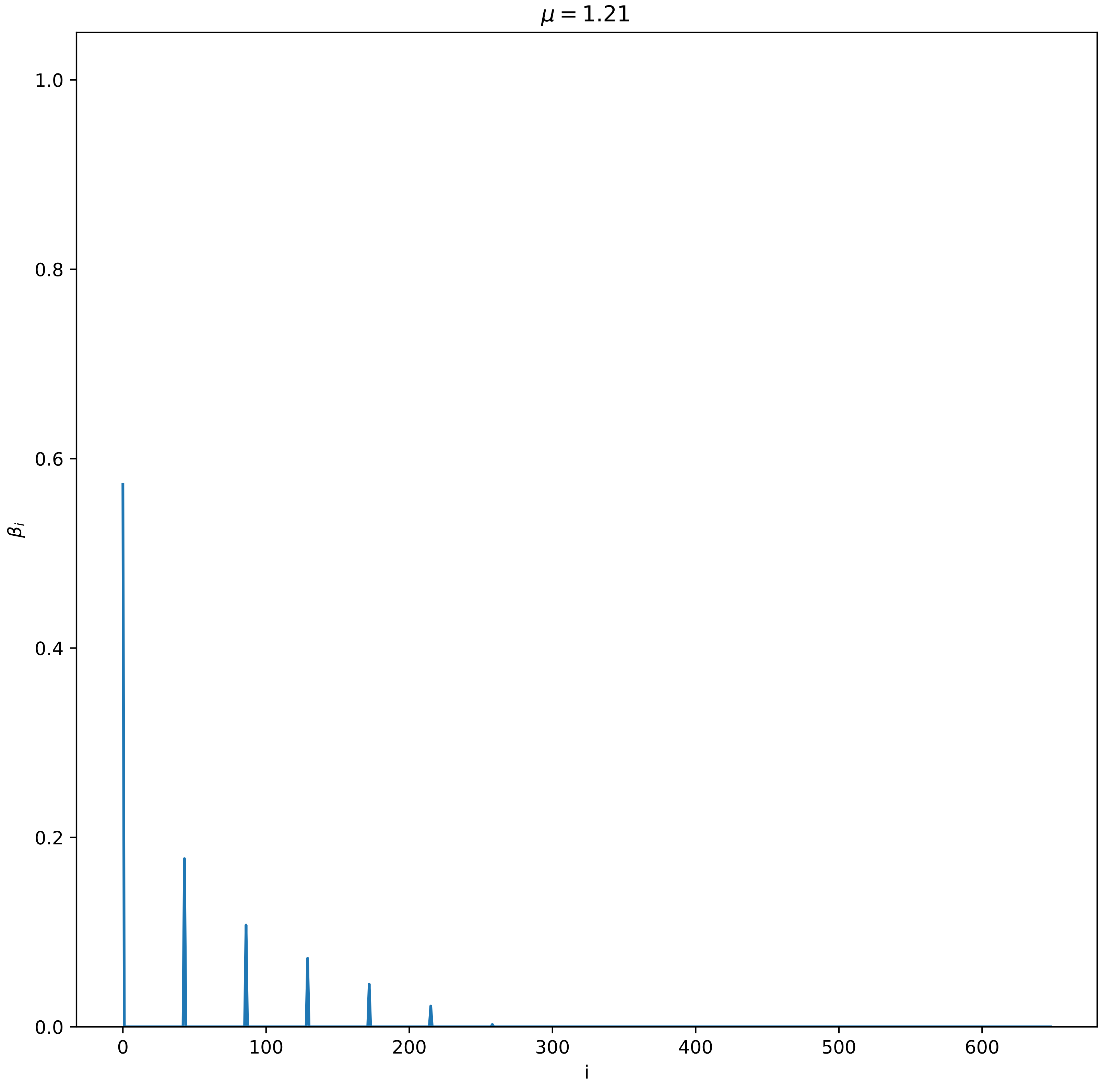
$\mu = 1.18$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

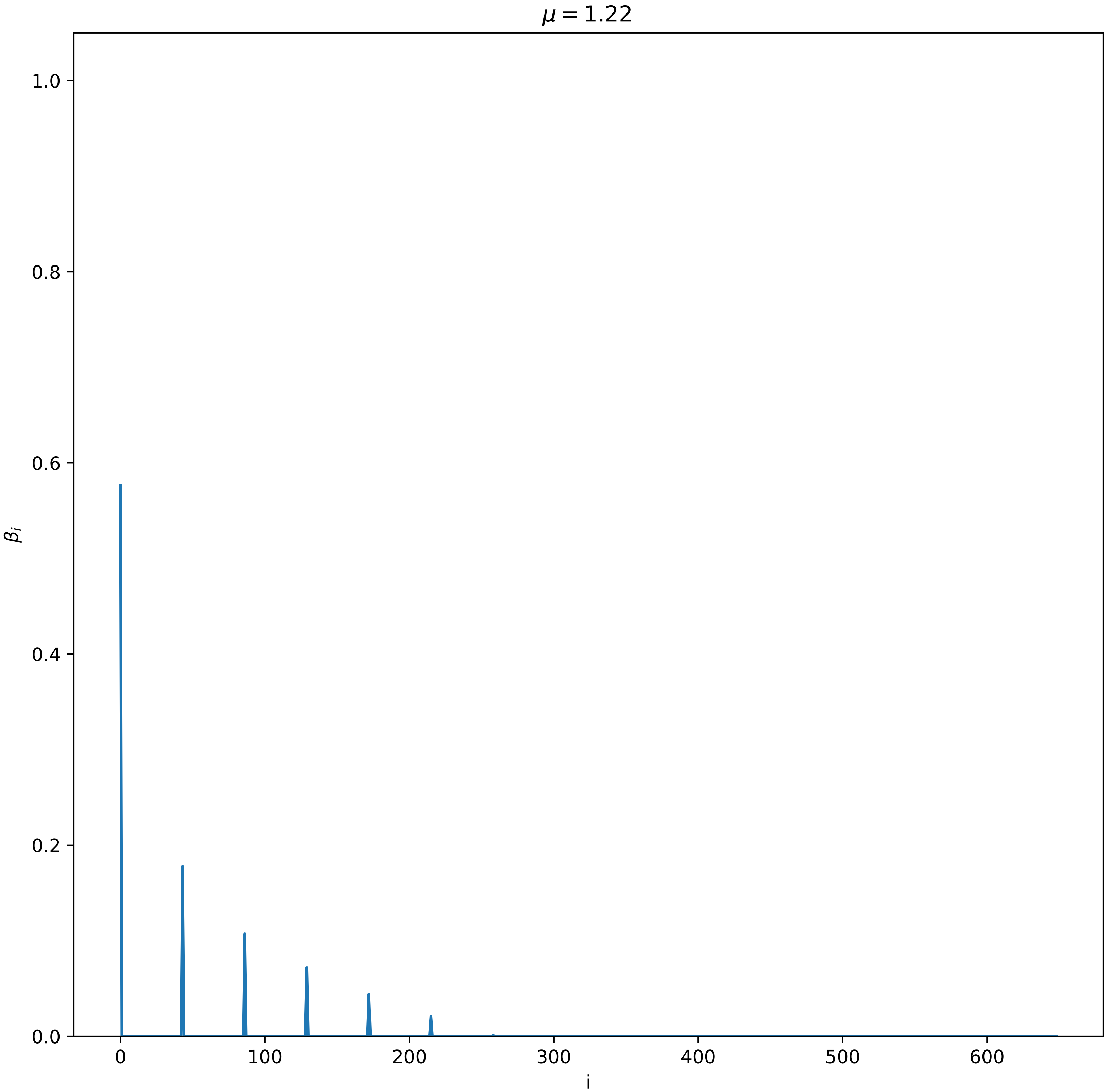


$\mu = 1.19$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

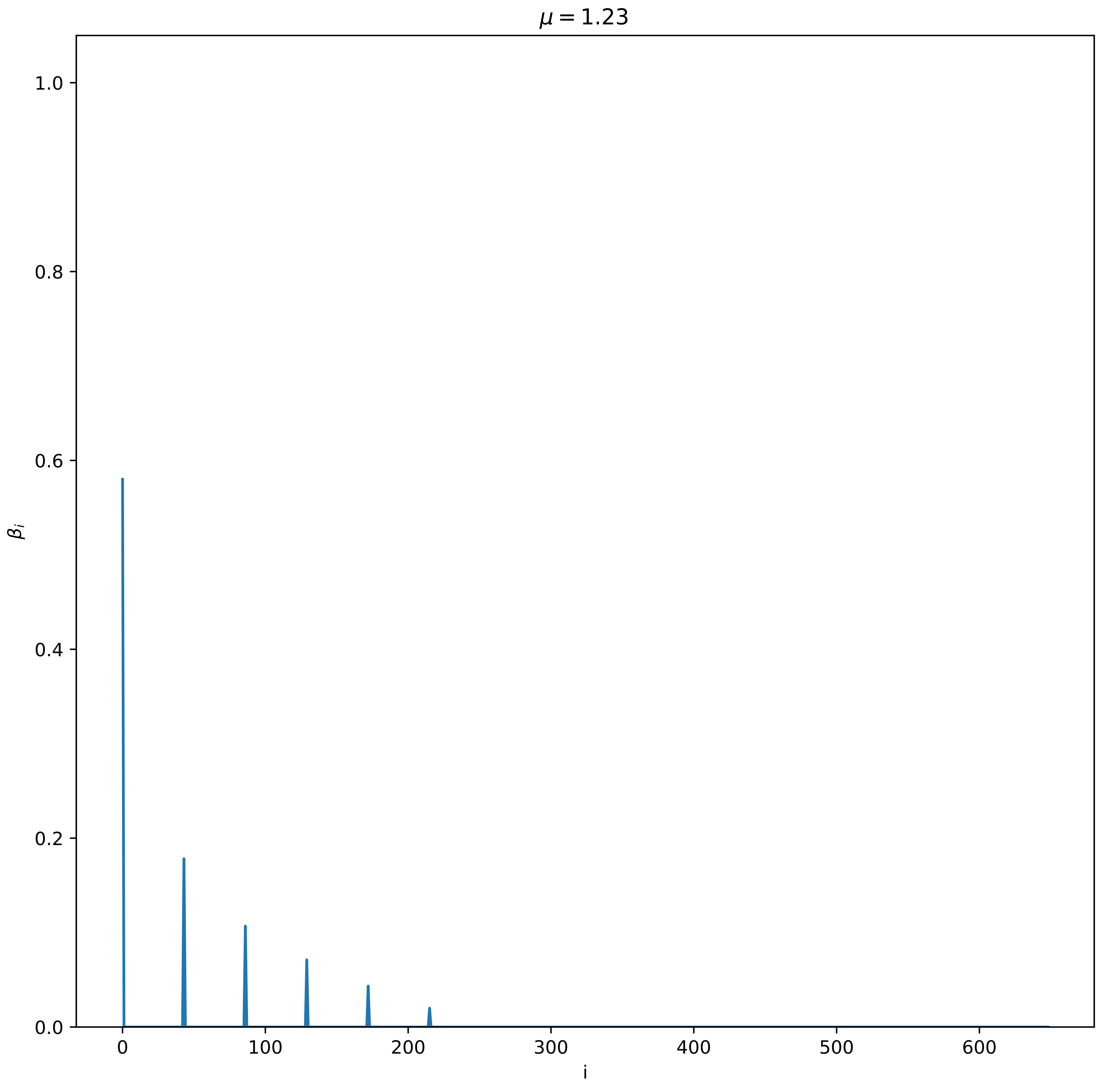


$\mu = 1.20$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

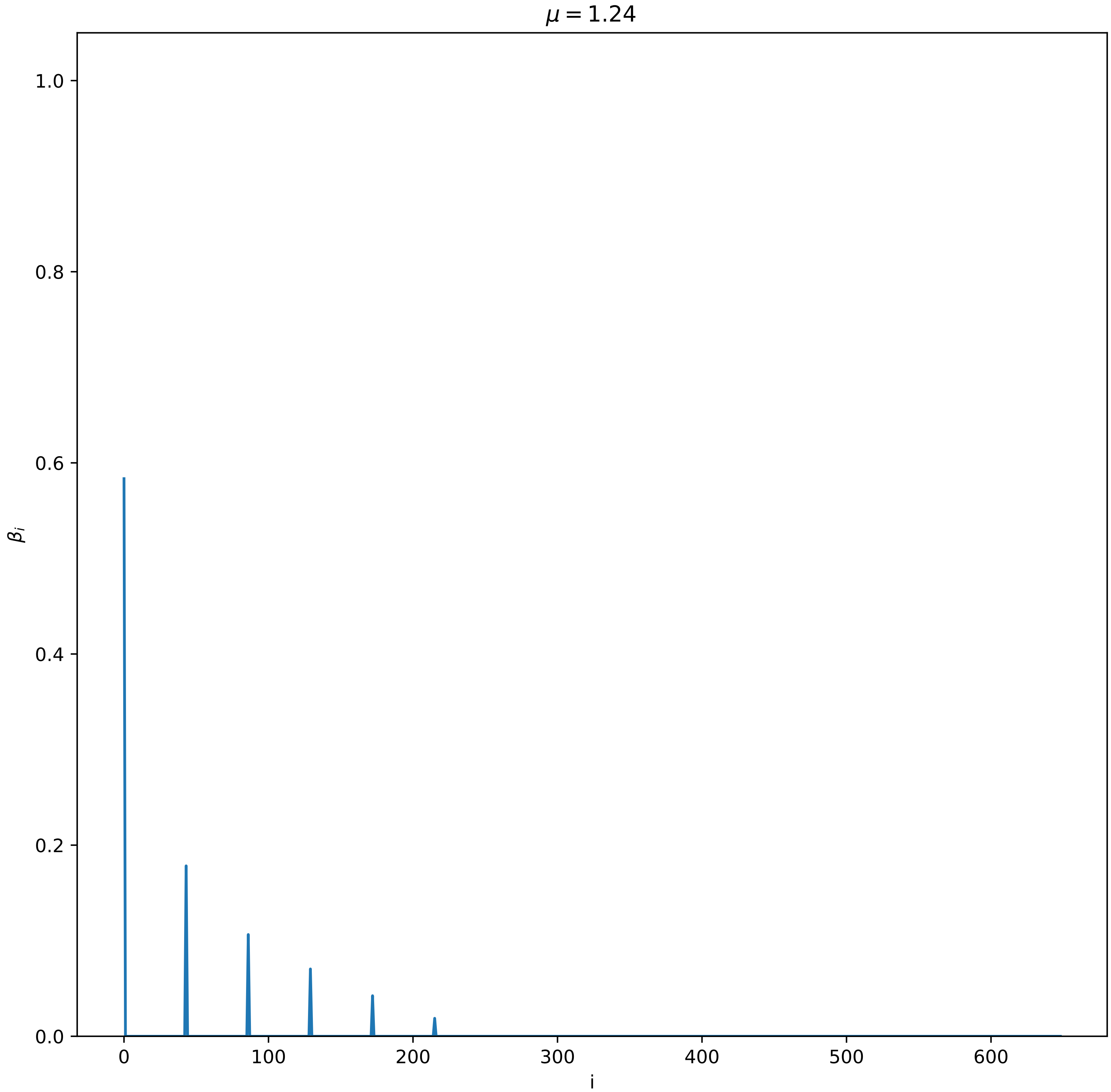

$\mu = 1.21$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



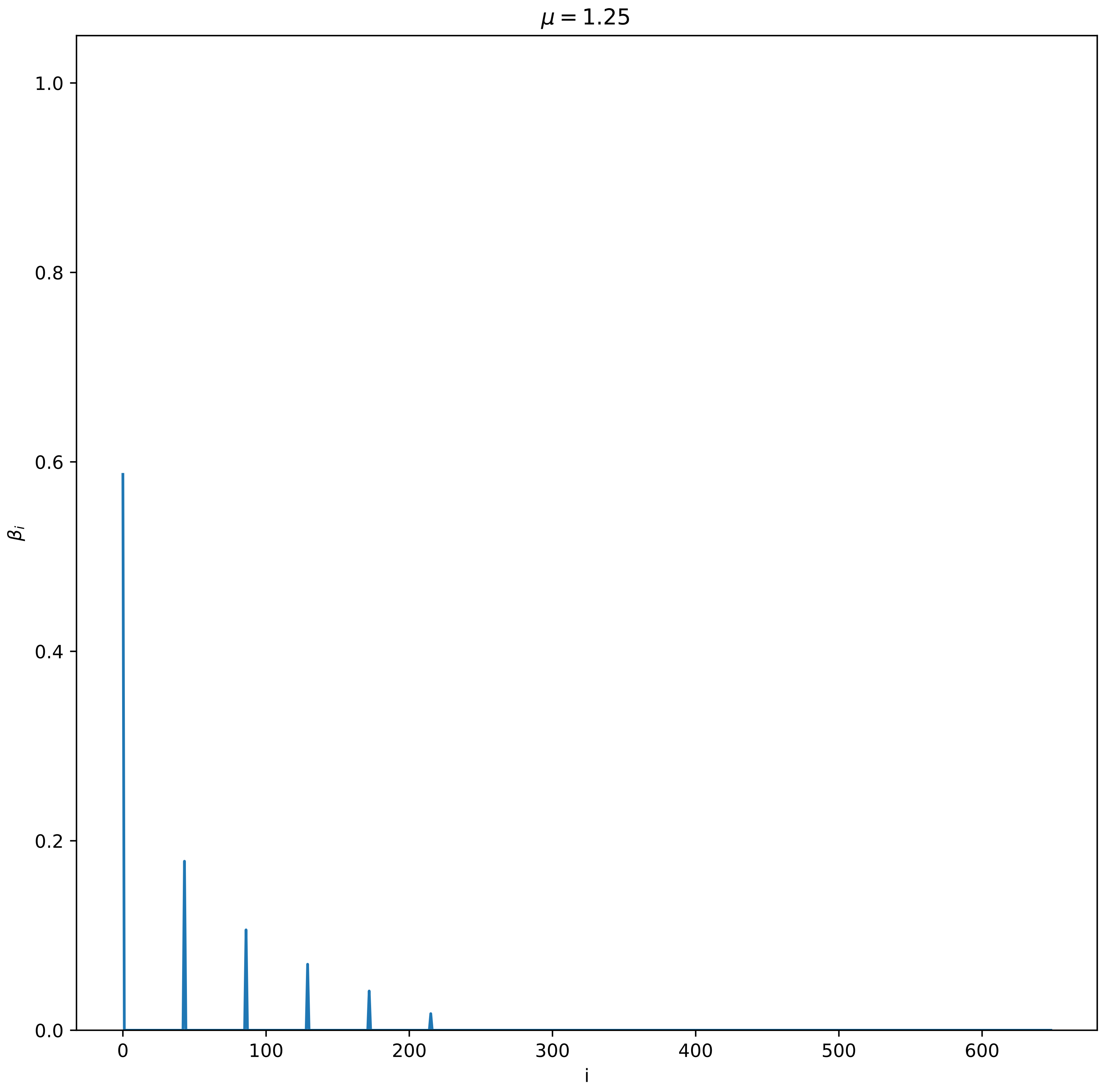
$\mu = 1.22$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



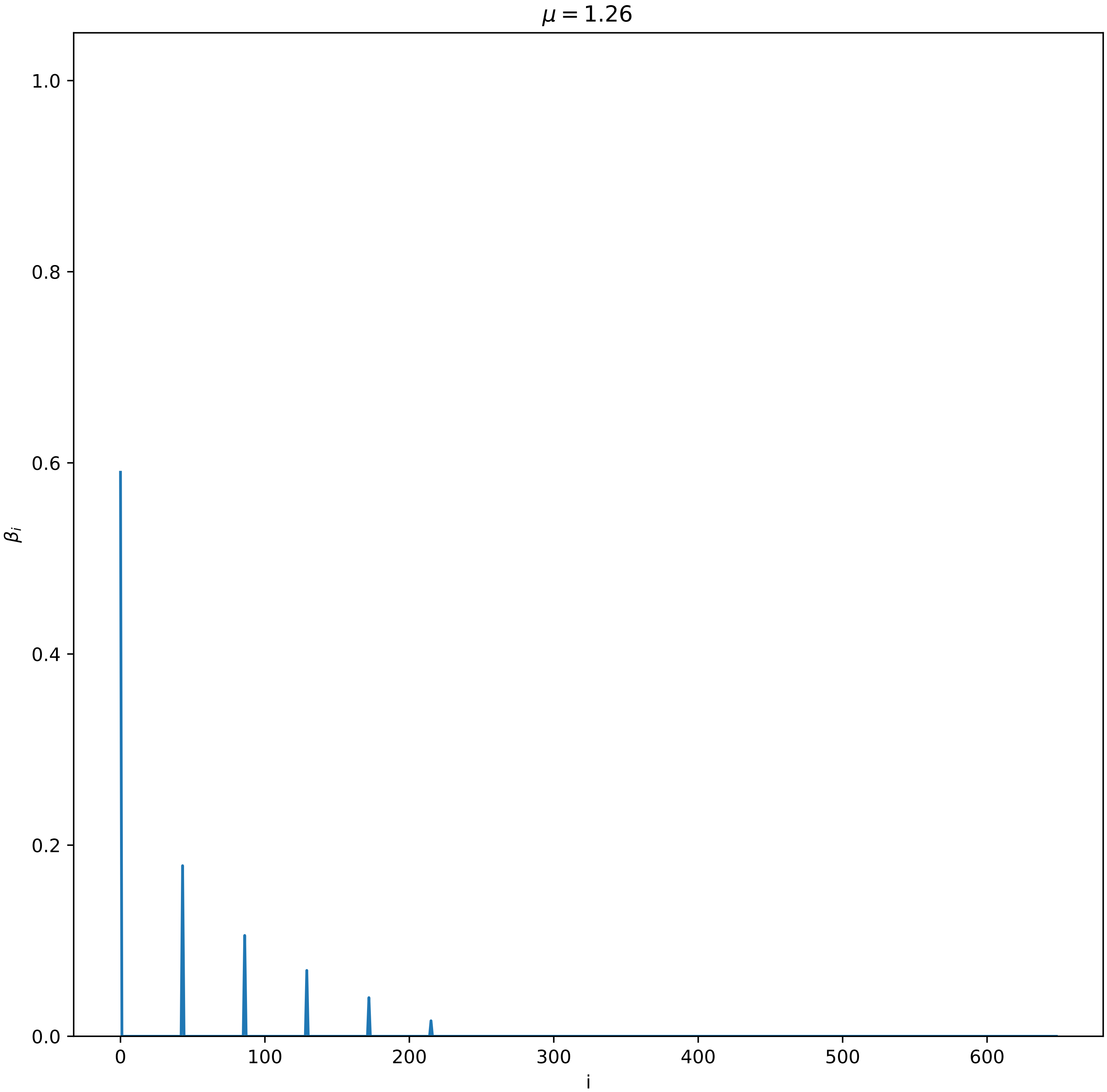
$\mu = 1.23$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



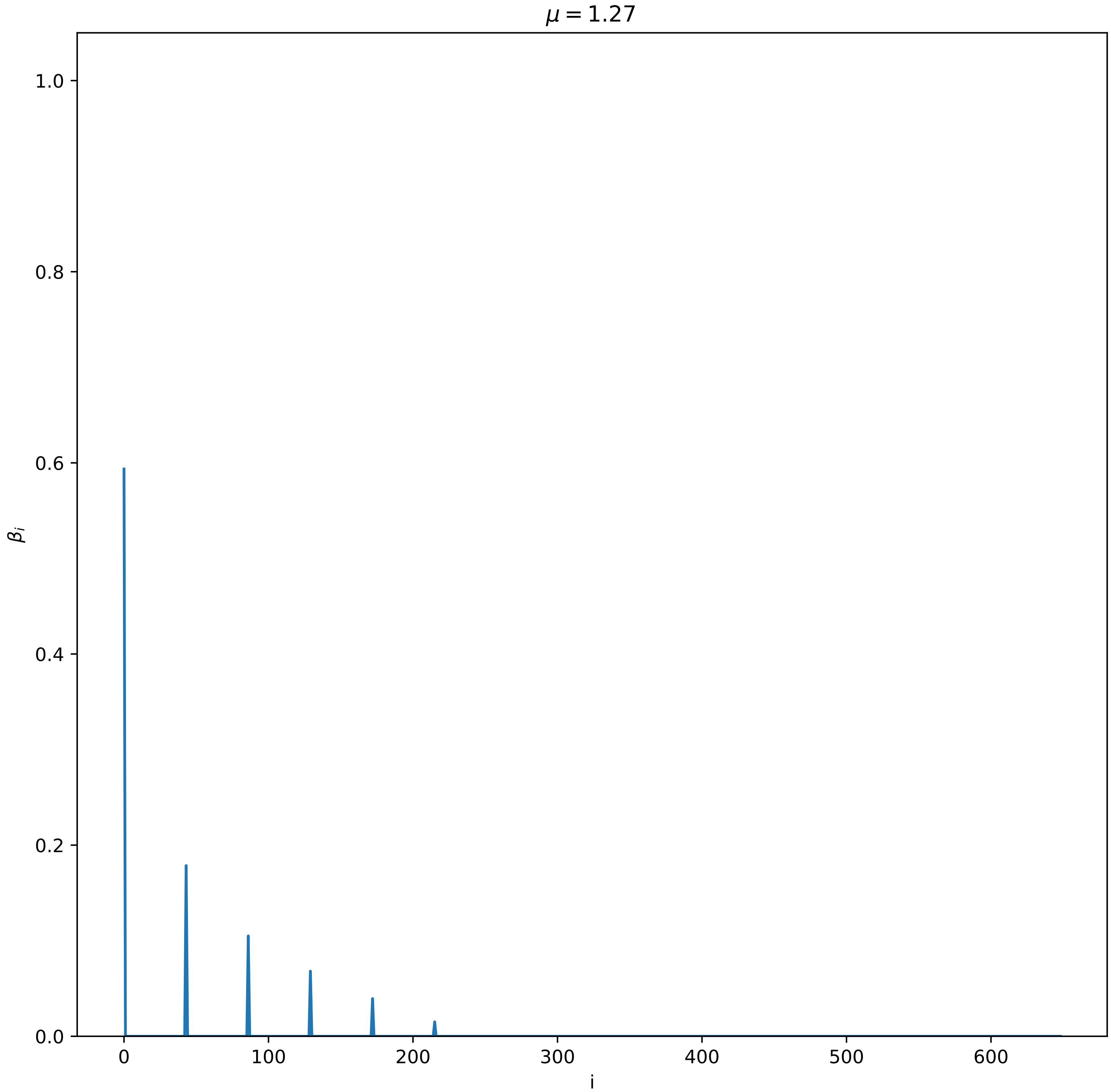
$\mu = 1.24$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



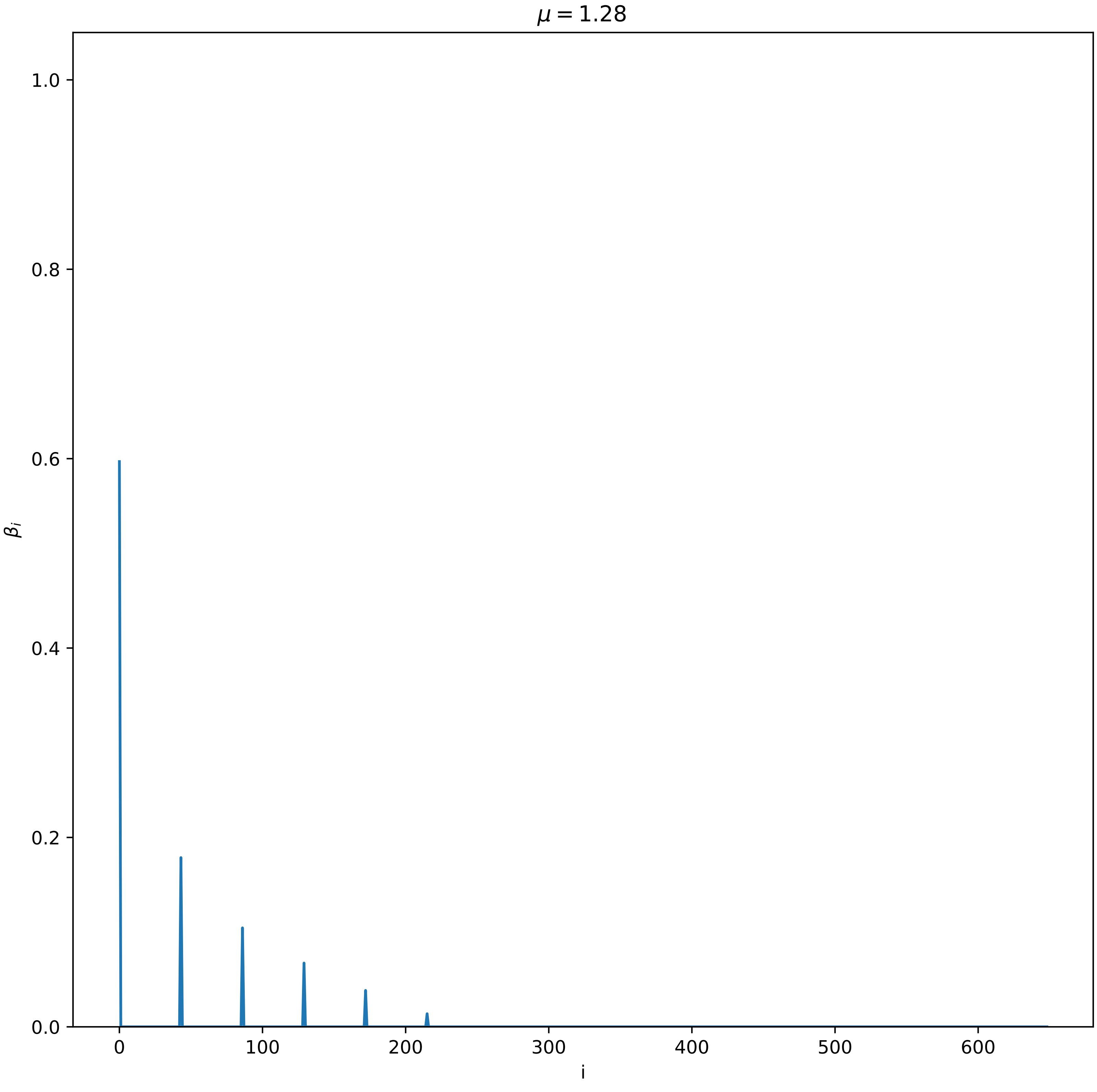
$\mu = 1.25$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



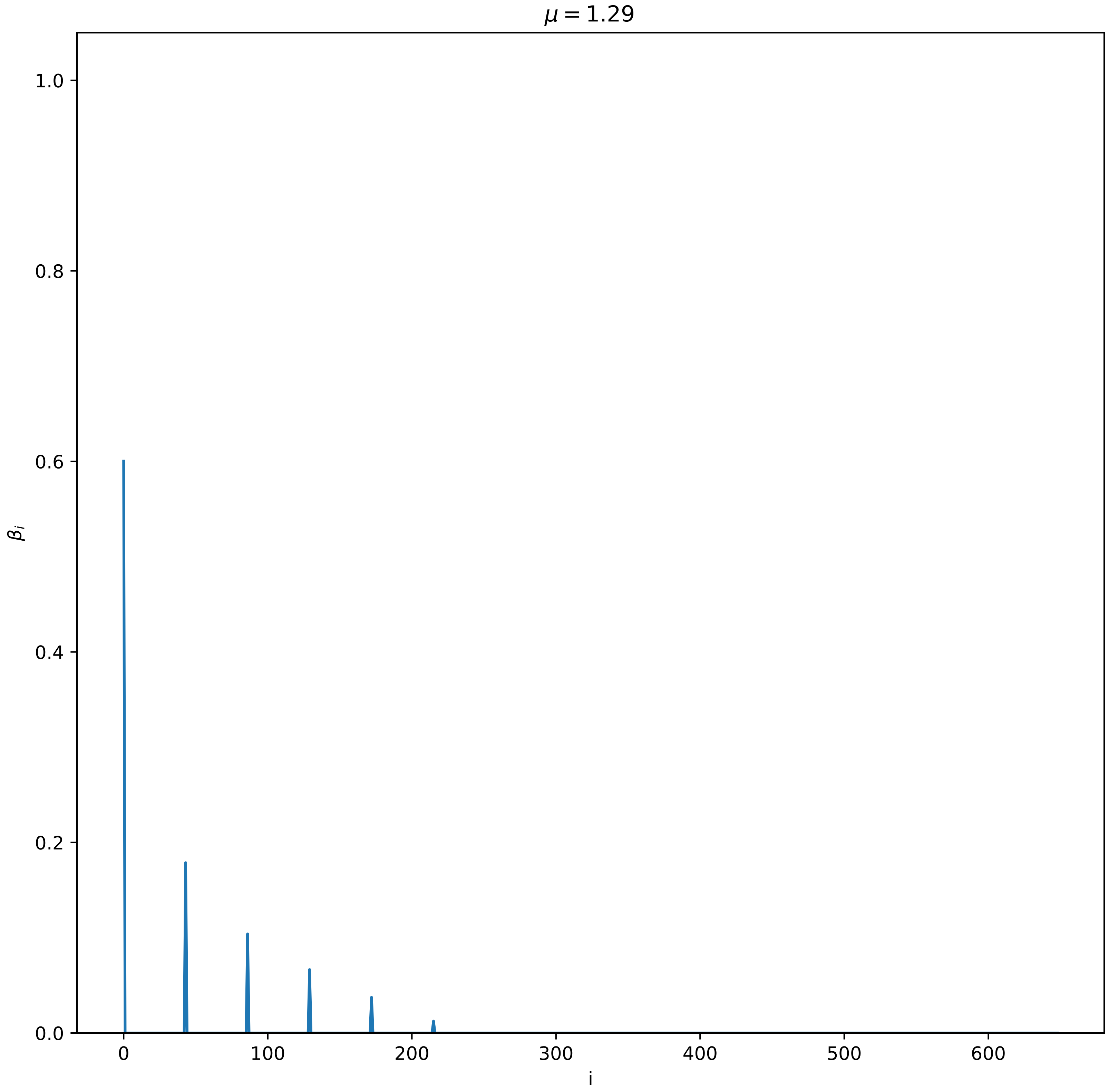
$\mu = 1.26$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



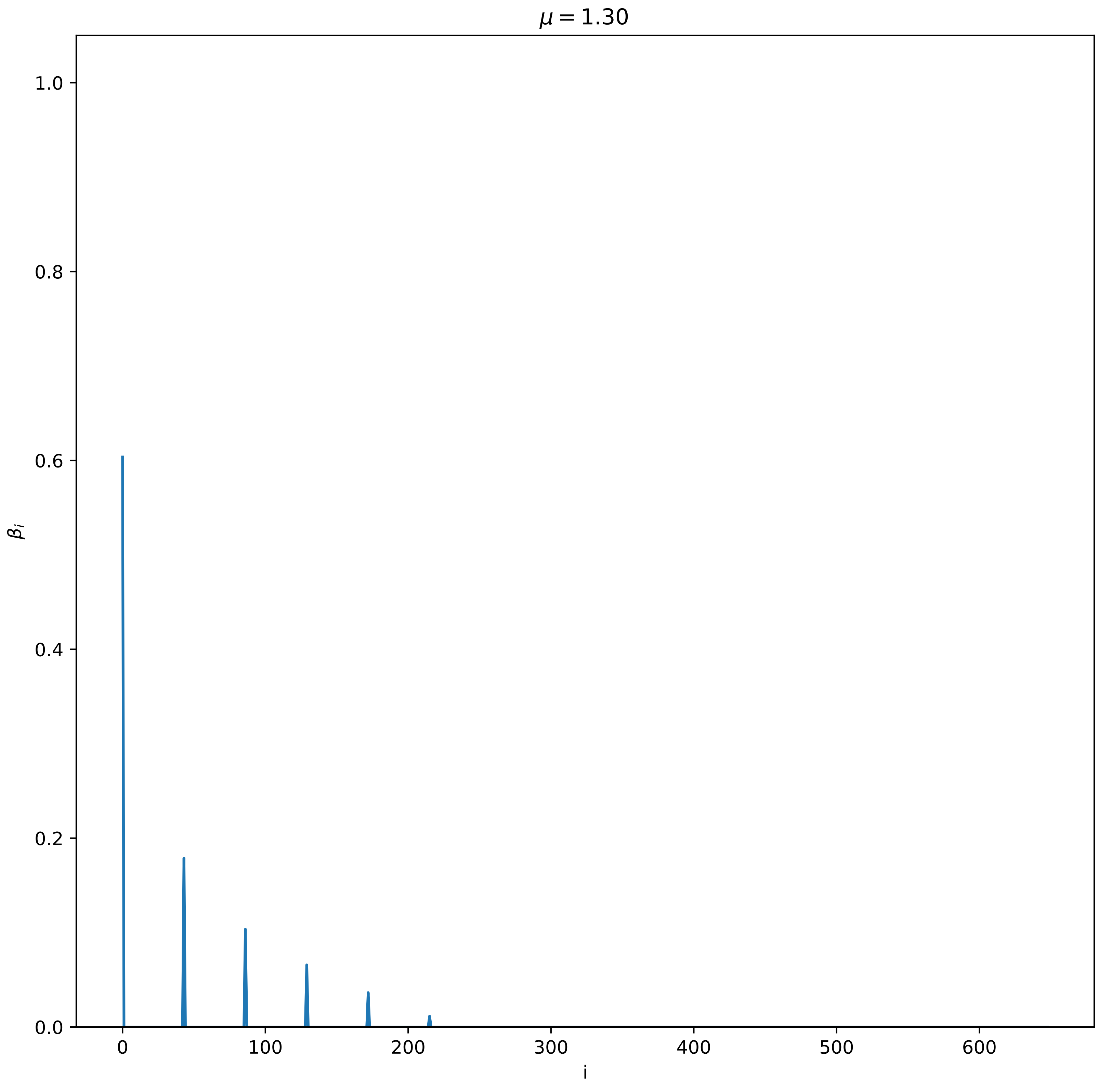
$\mu = 1.27$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



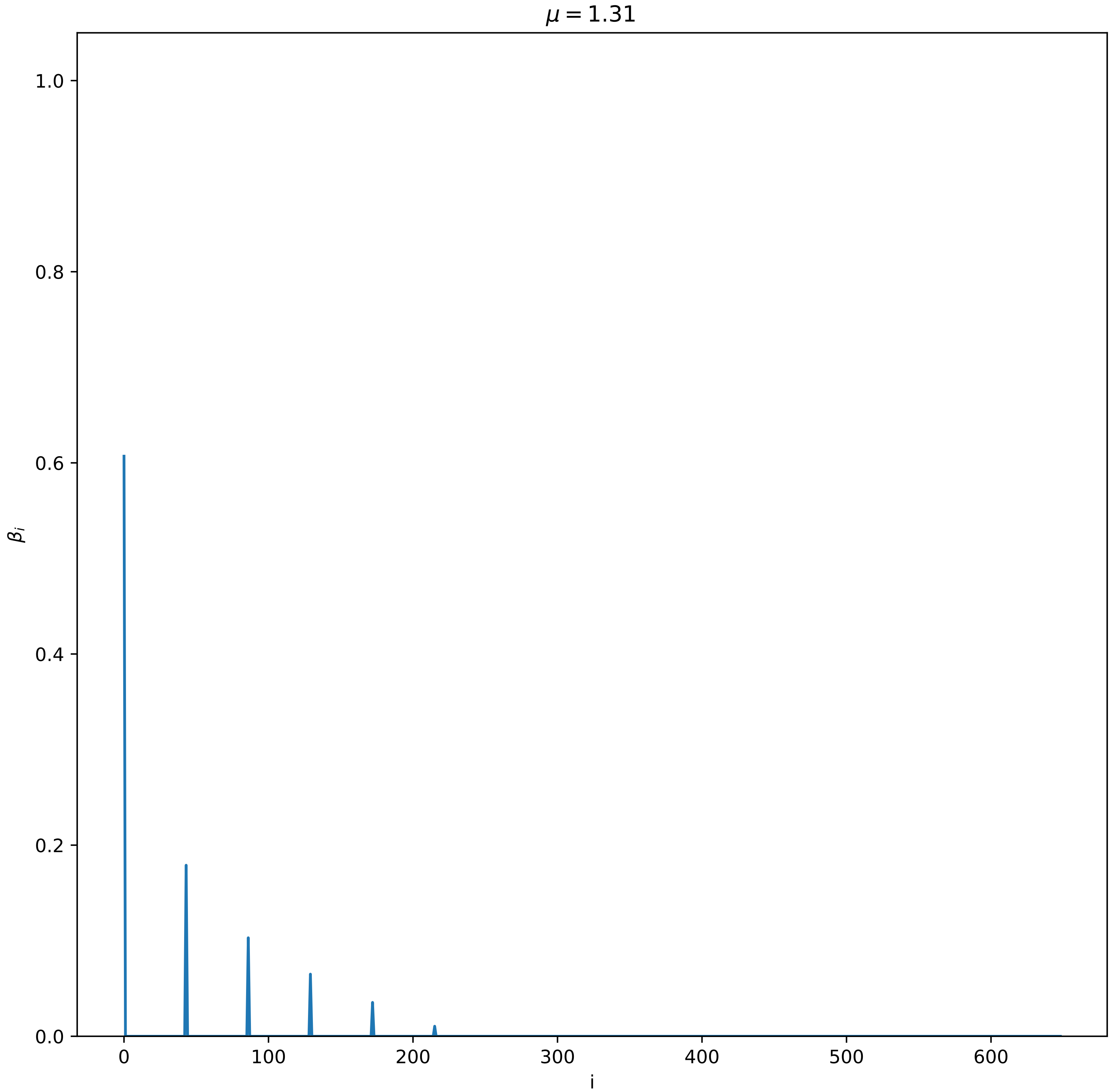
$\mu = 1.28$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



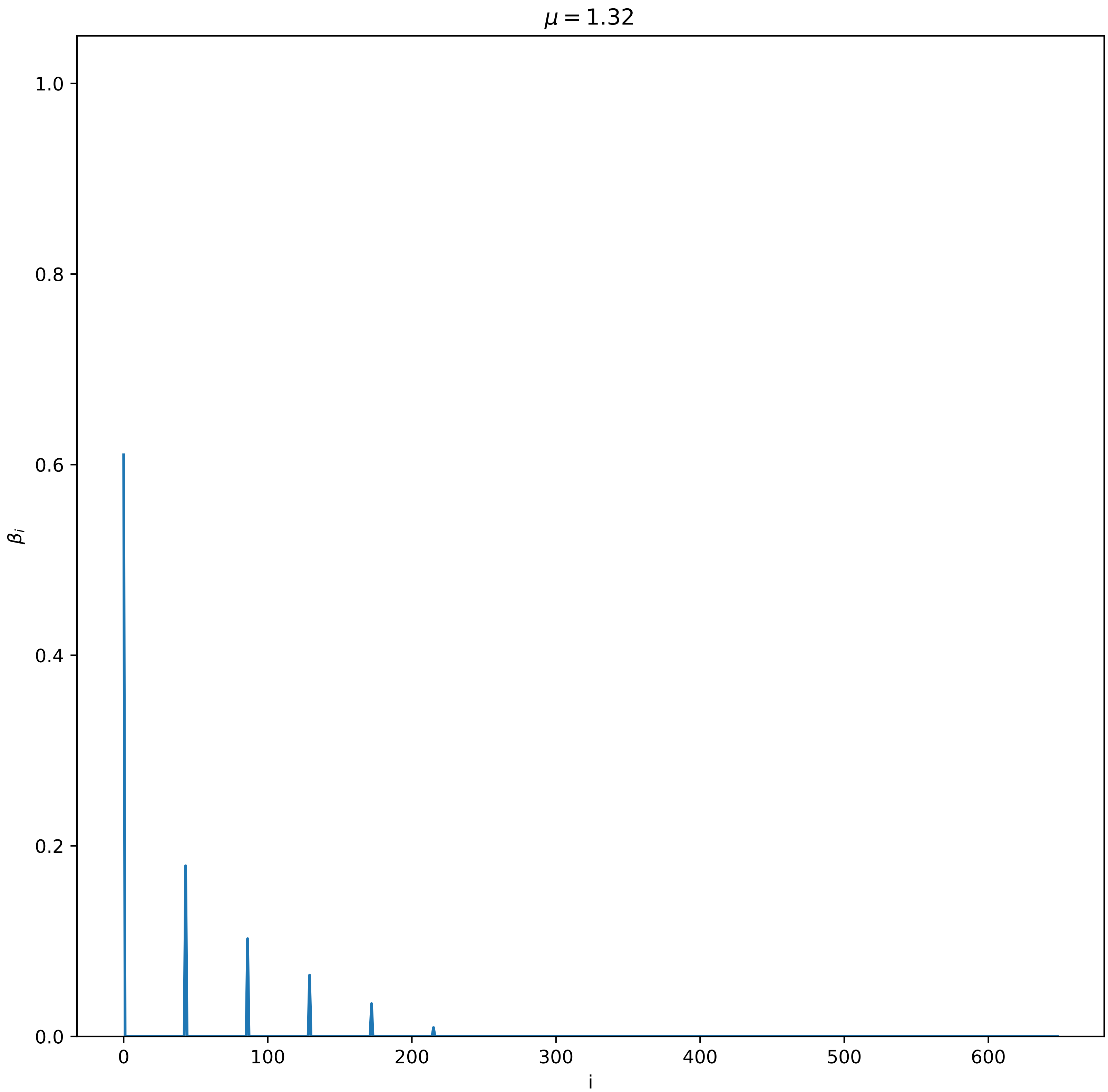
$\mu = 1.29$
β_i
i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600



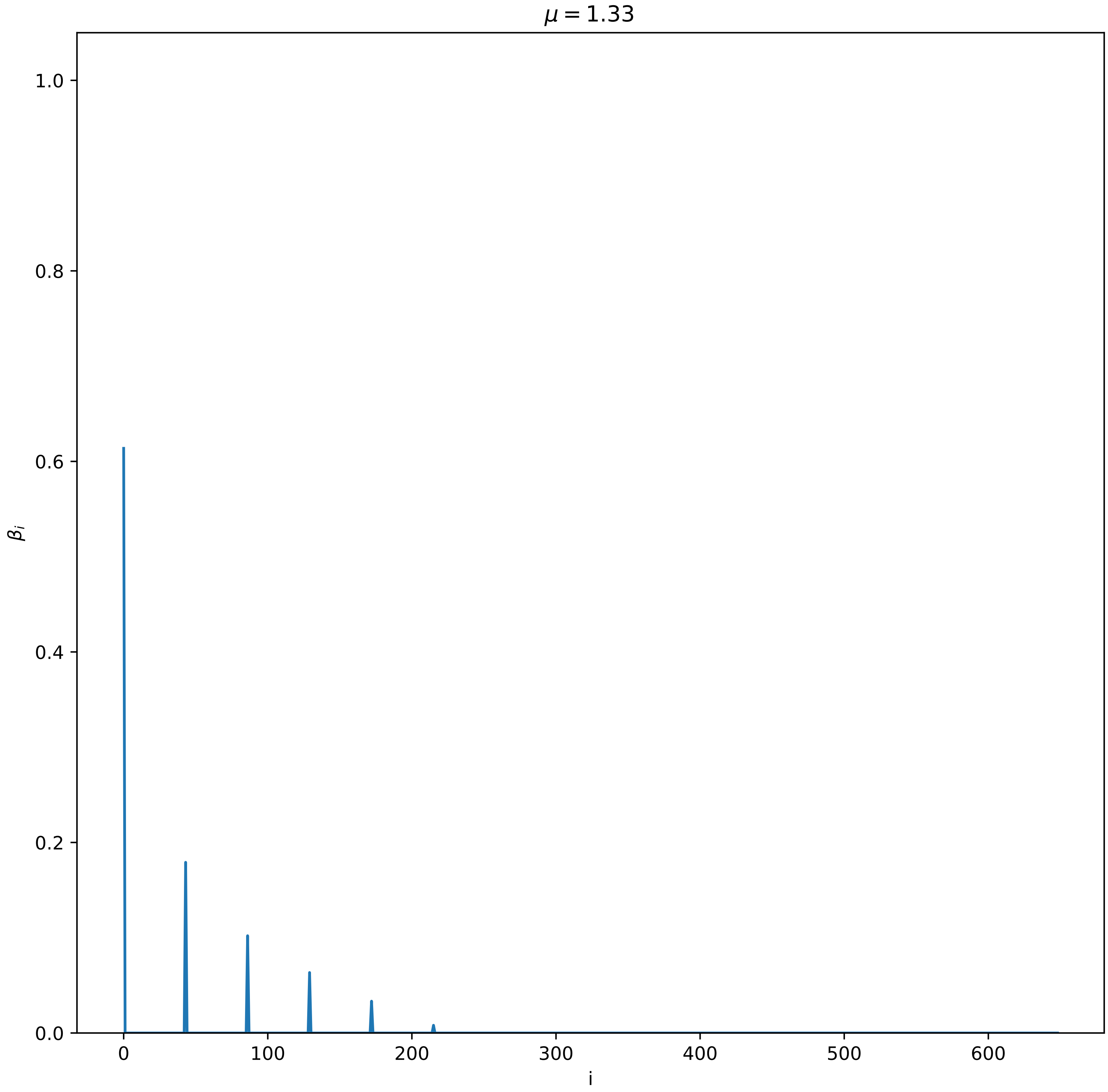
$\mu = 1.30$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



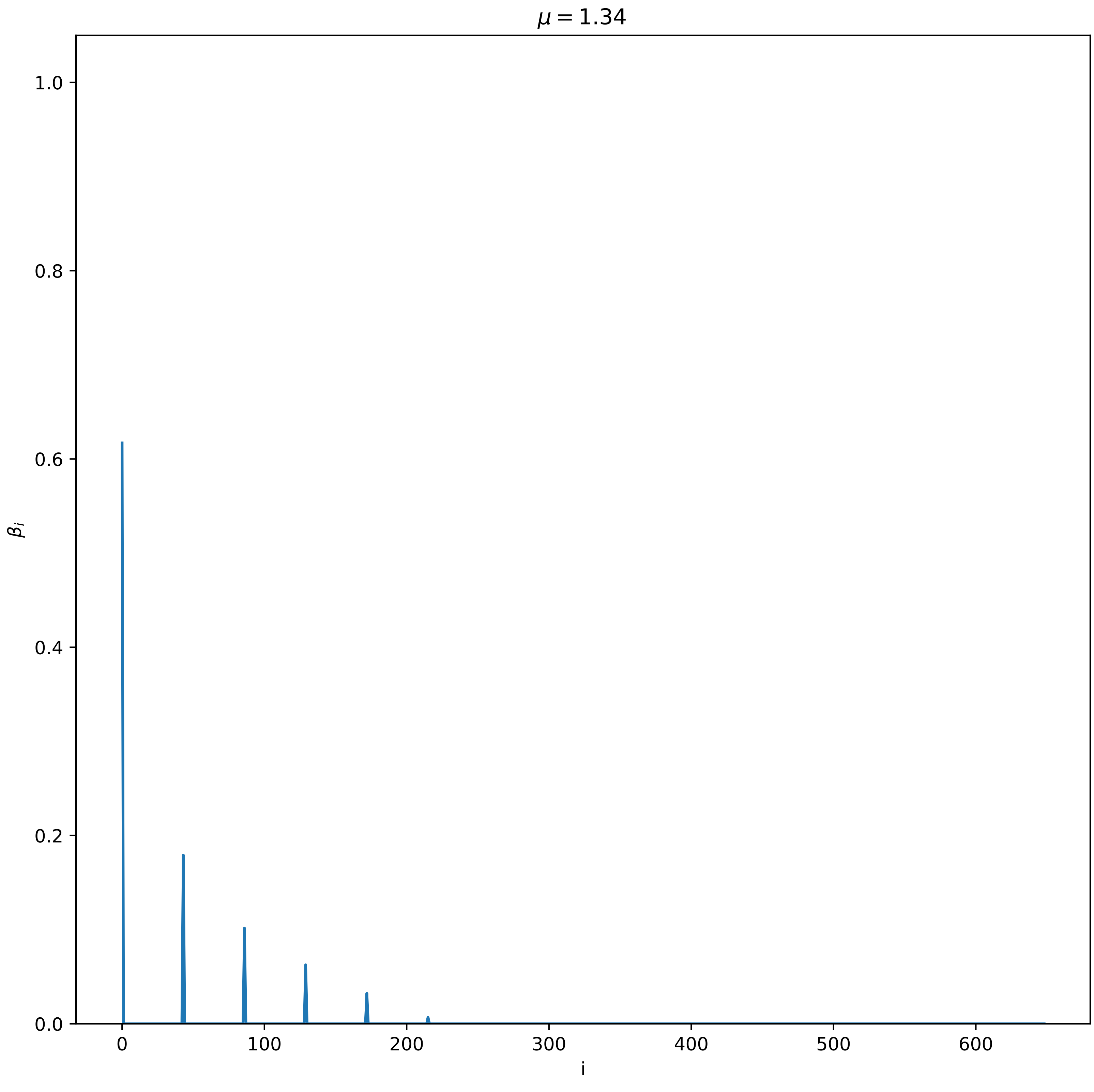
$\mu = 1.31$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

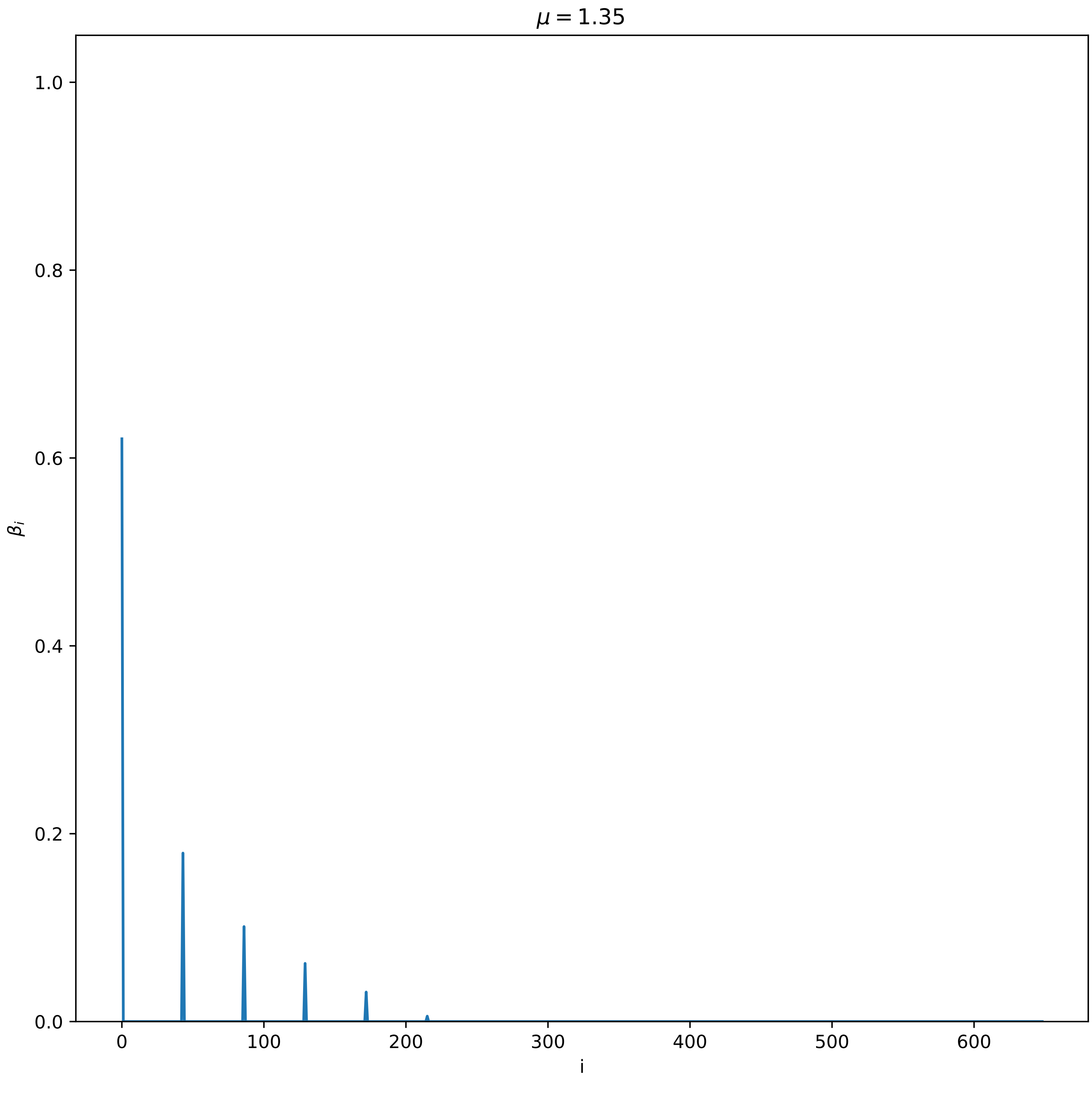


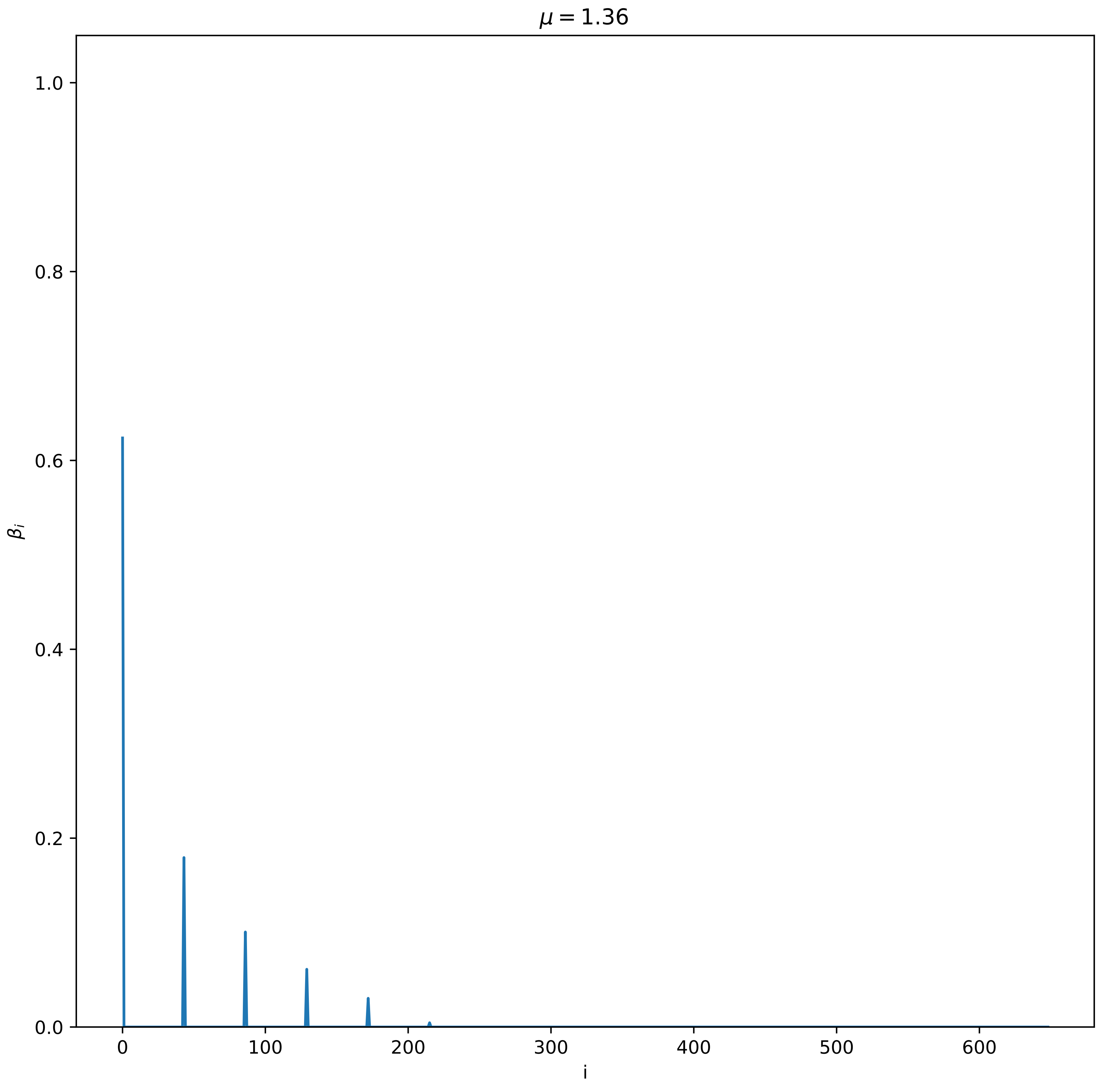
$\mu = 1.32$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

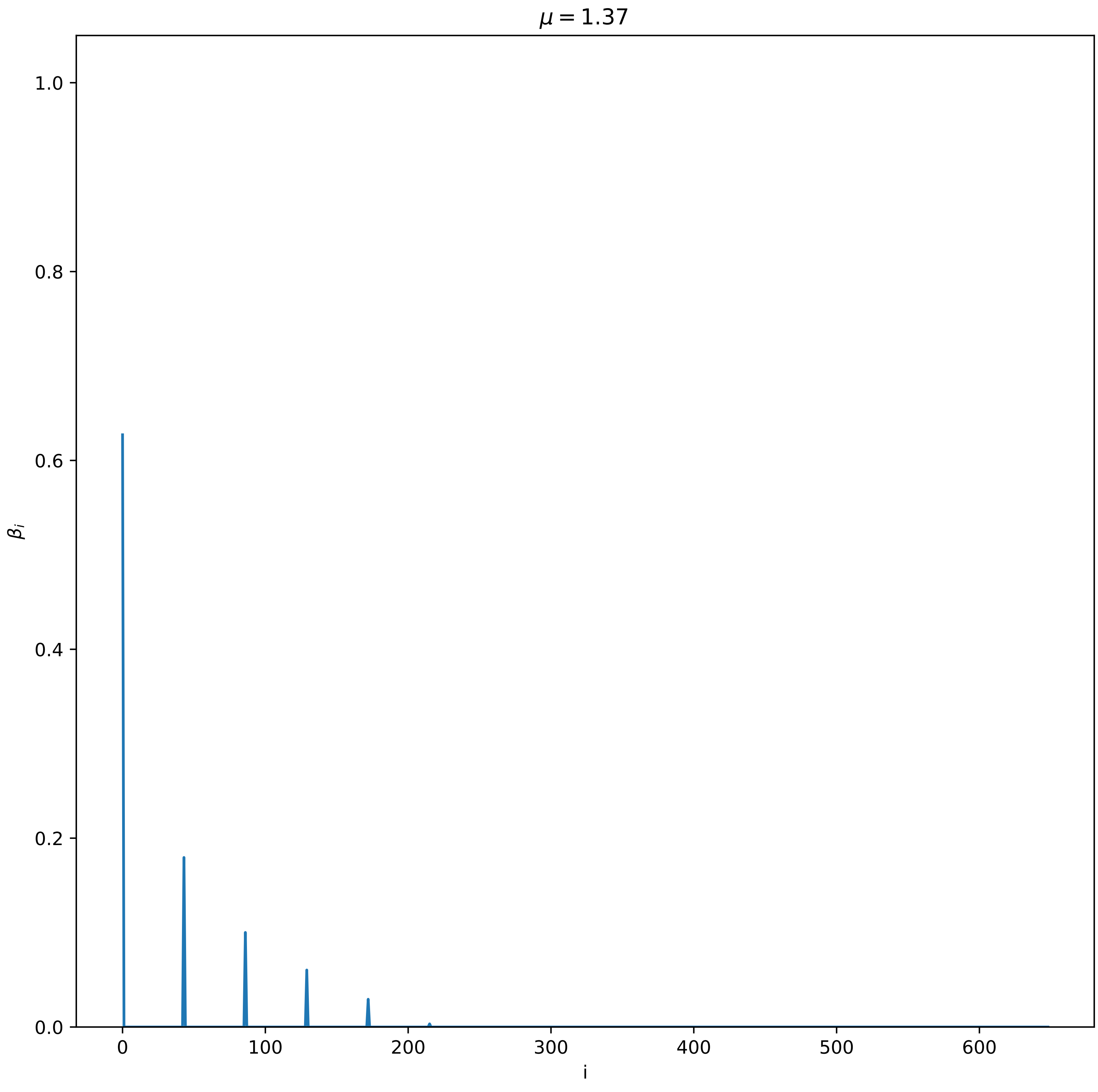


$\mu = 1.33$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

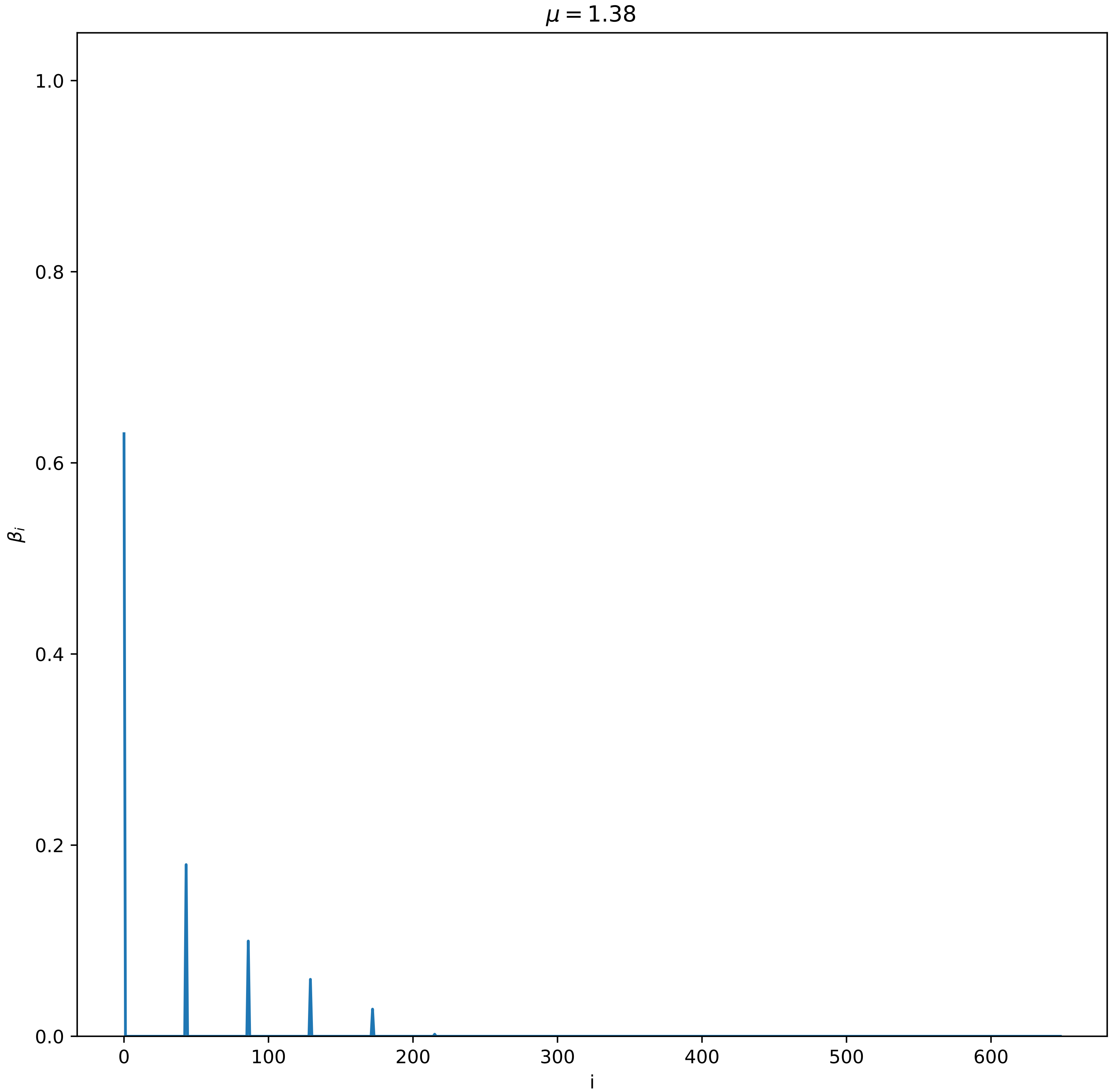

$\mu = 1.34$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.35$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

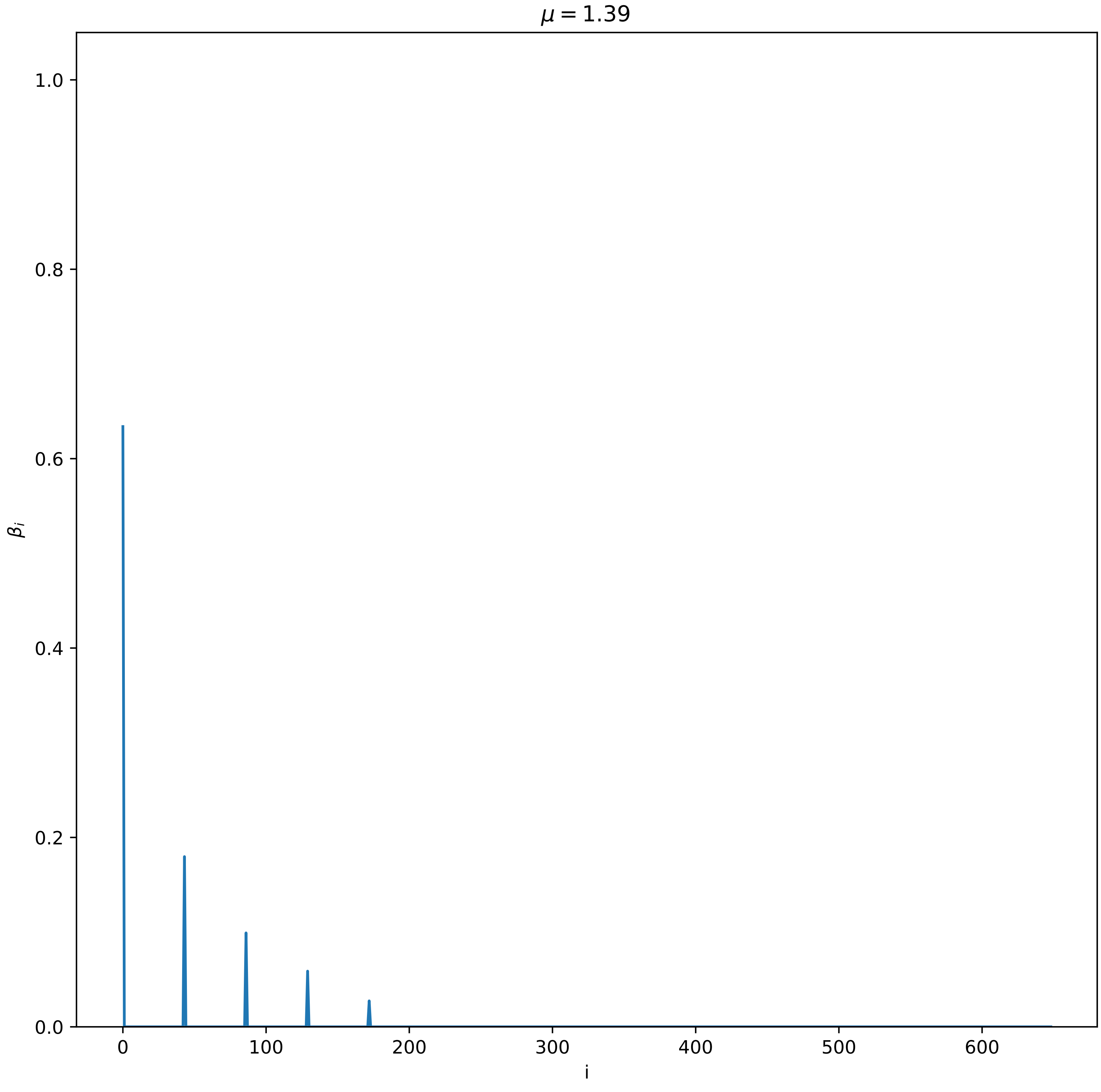

$\mu = 1.36$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



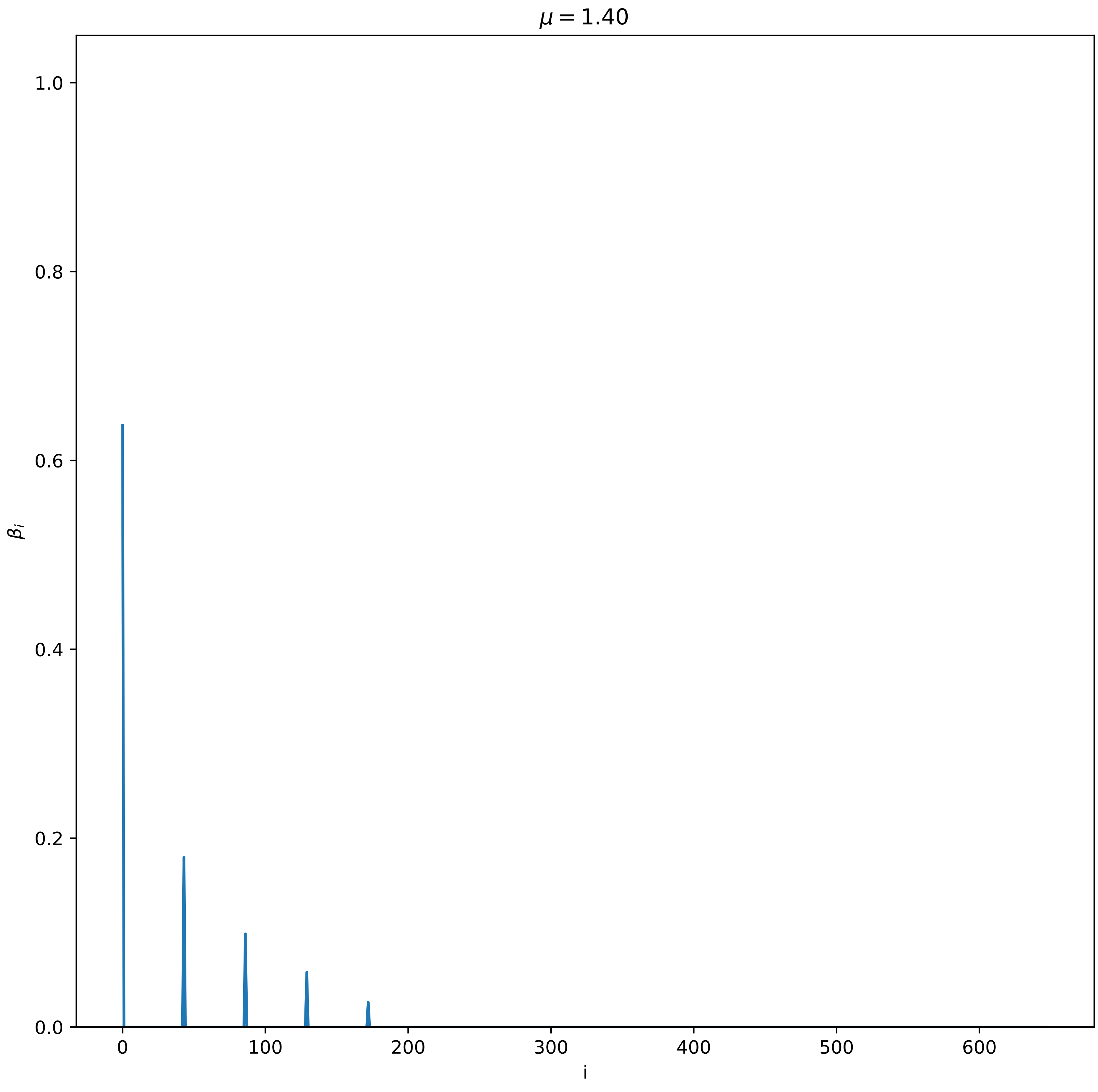
$\mu = 1.37$
β_i
i
0.0
0.2
0.4
0.6
0.8
1.0
0
100
200
300
400
500
600



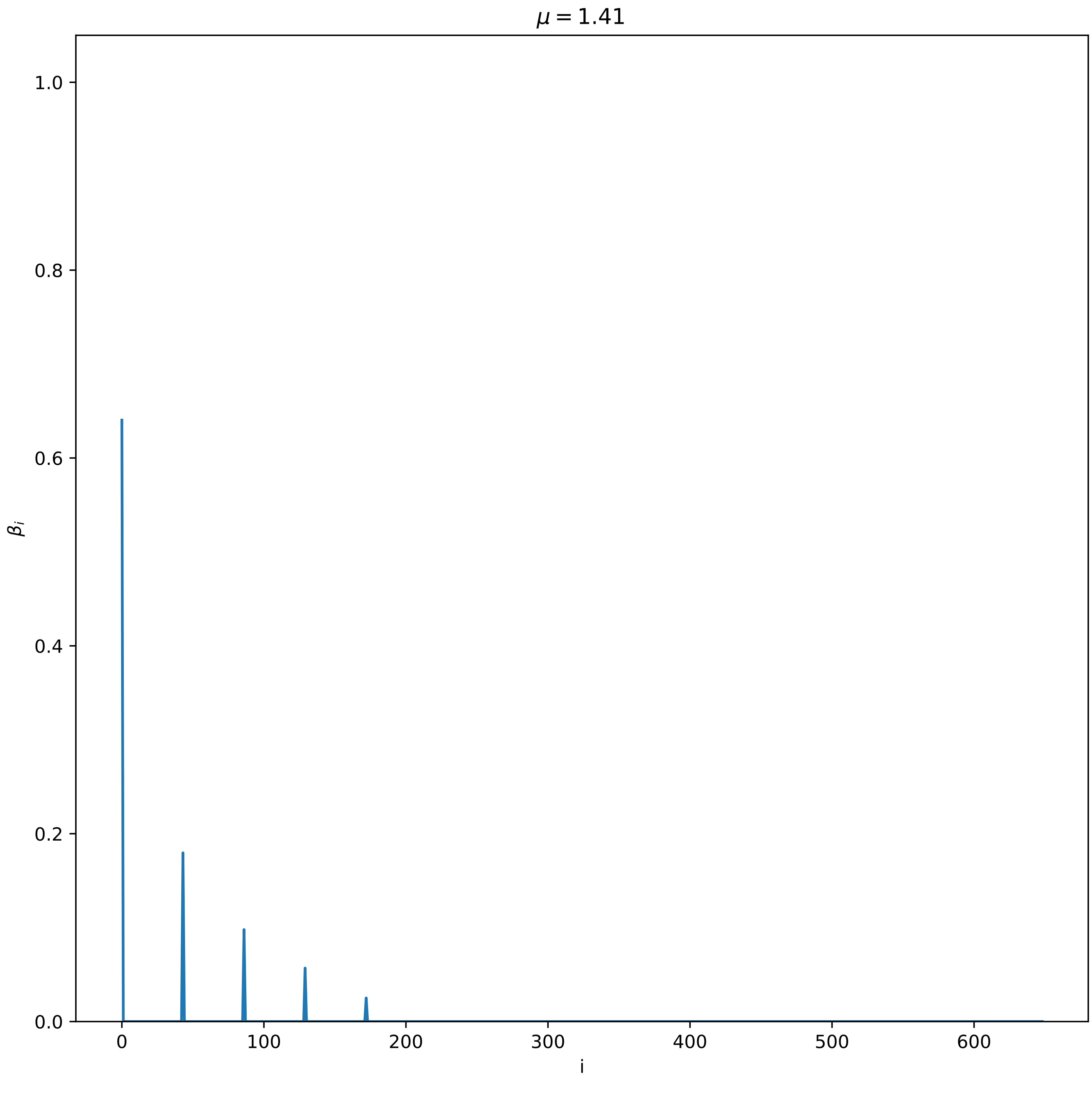
$\mu = 1.38$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



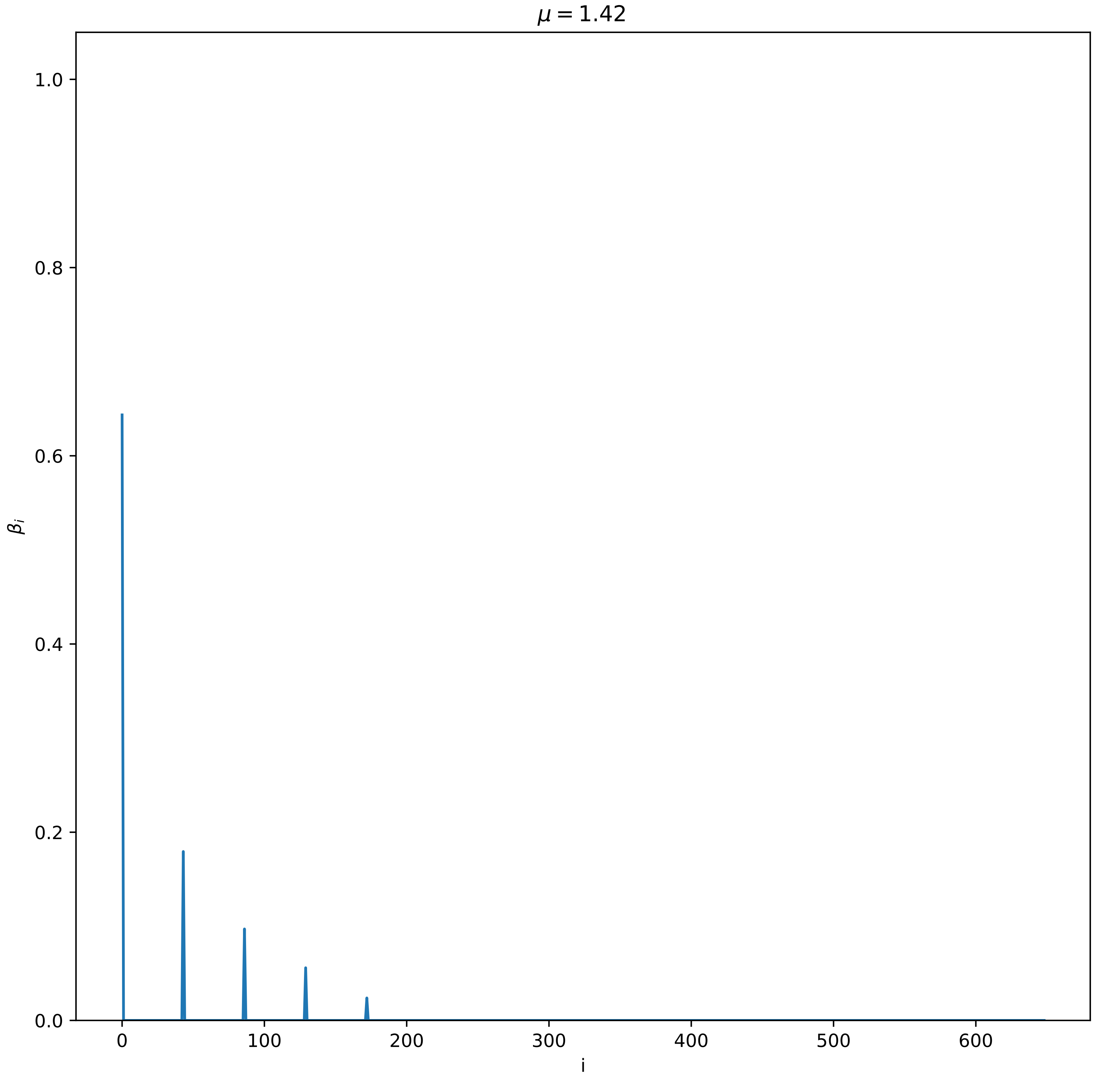
$\mu = 1.39$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



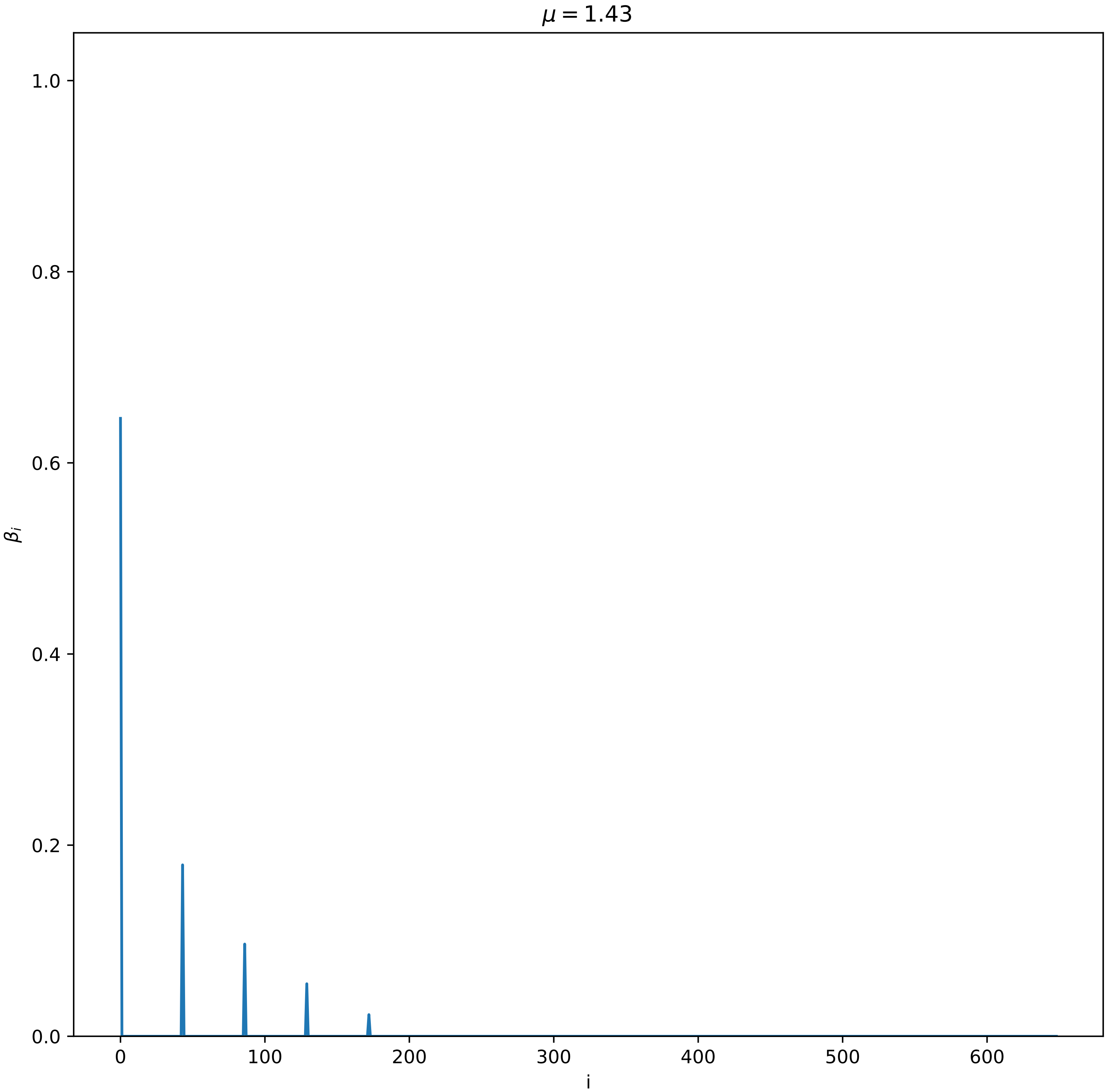
$\mu = 1.40$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



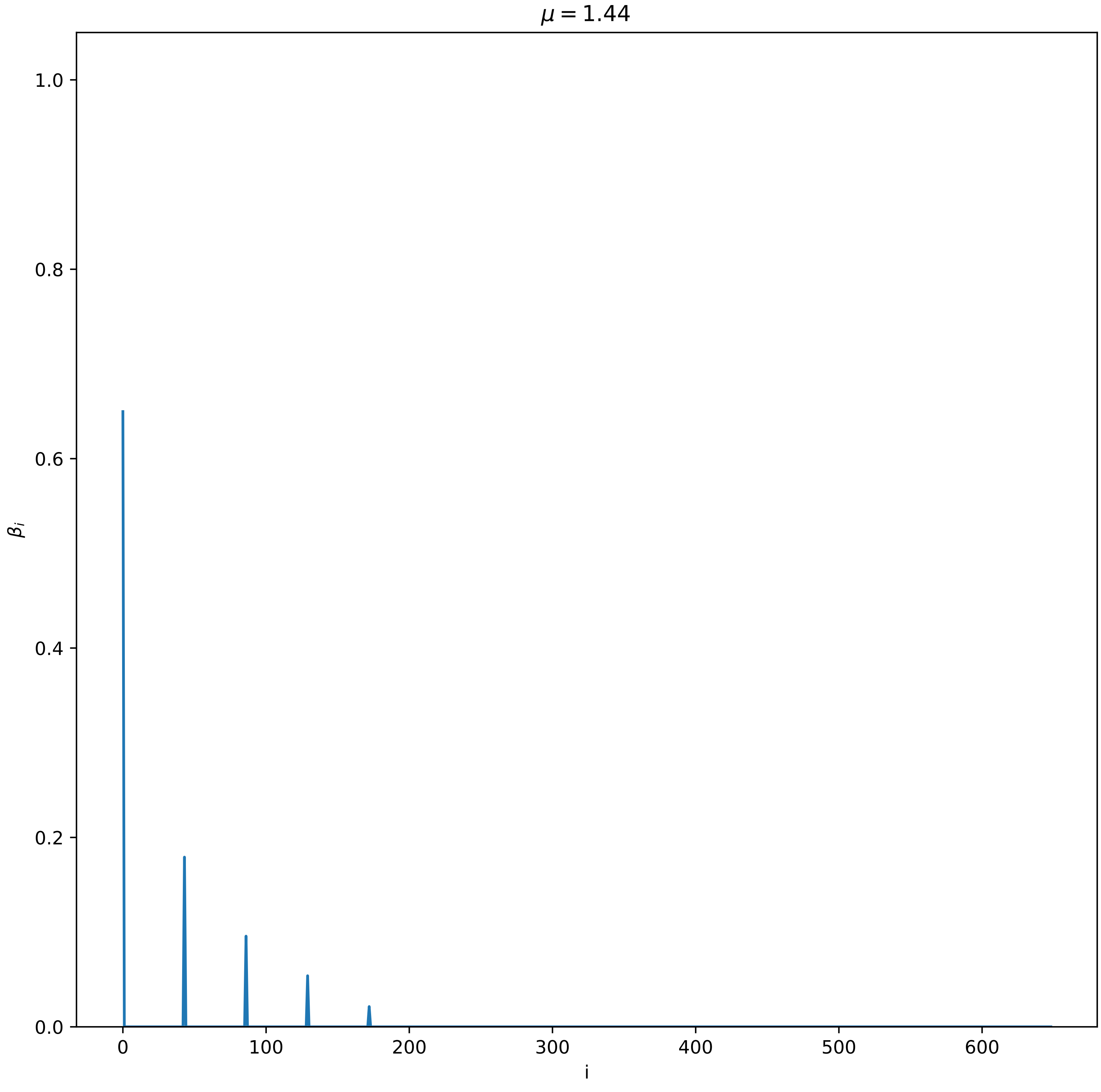
$\mu = 1.41$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



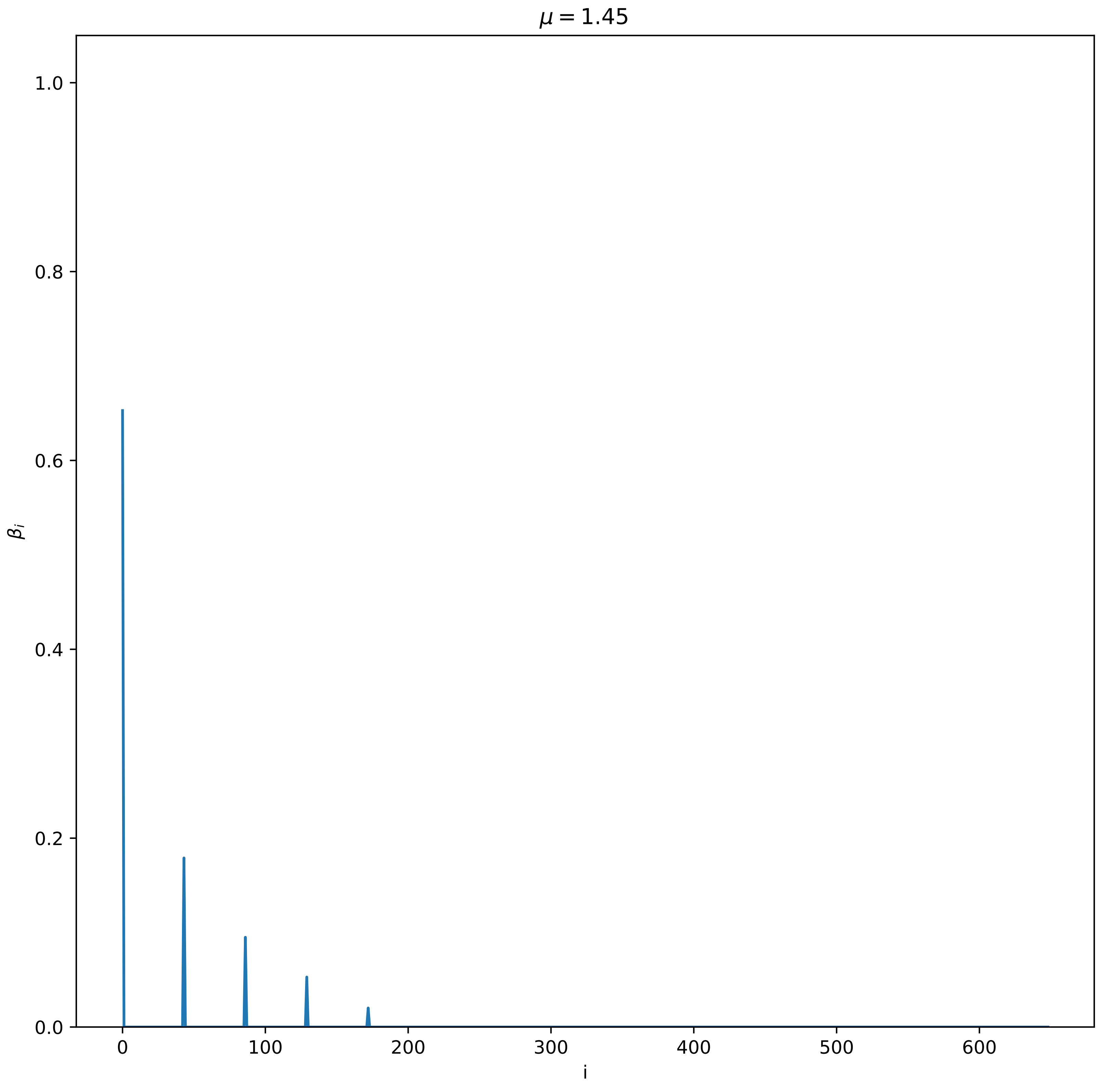
$\mu = 1.42$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



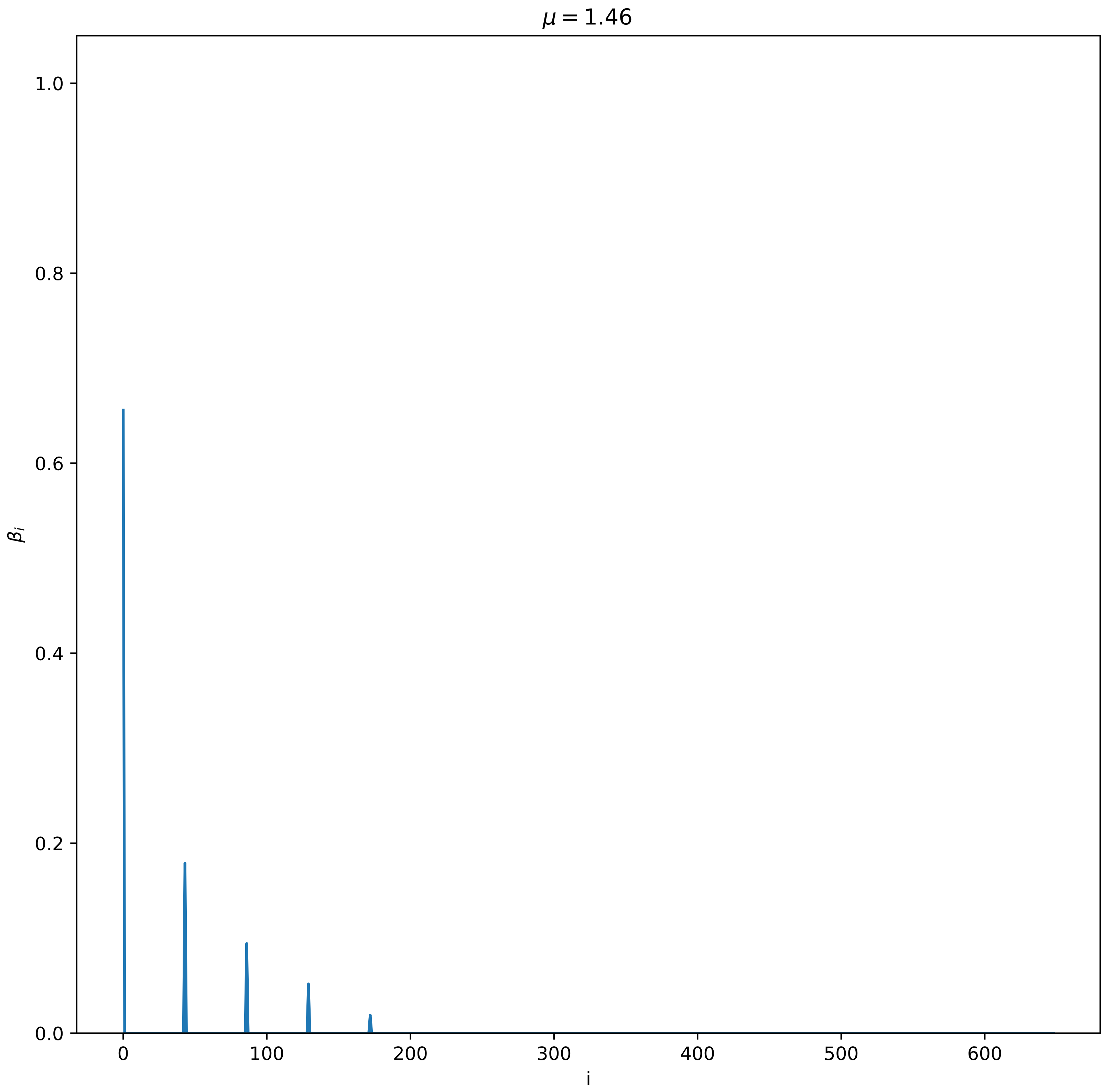
$\mu = 1.43$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



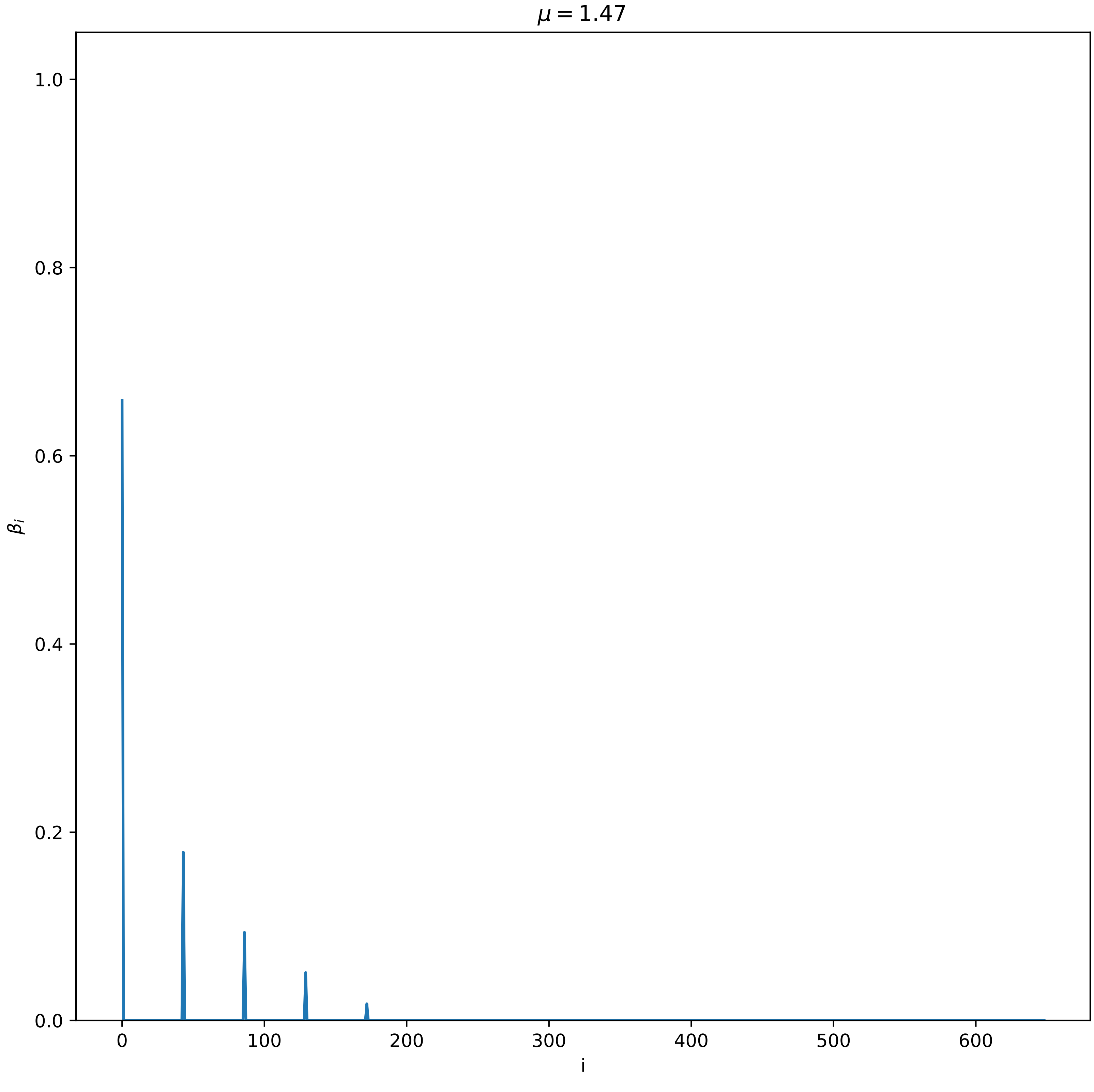
$\mu = 1.44$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



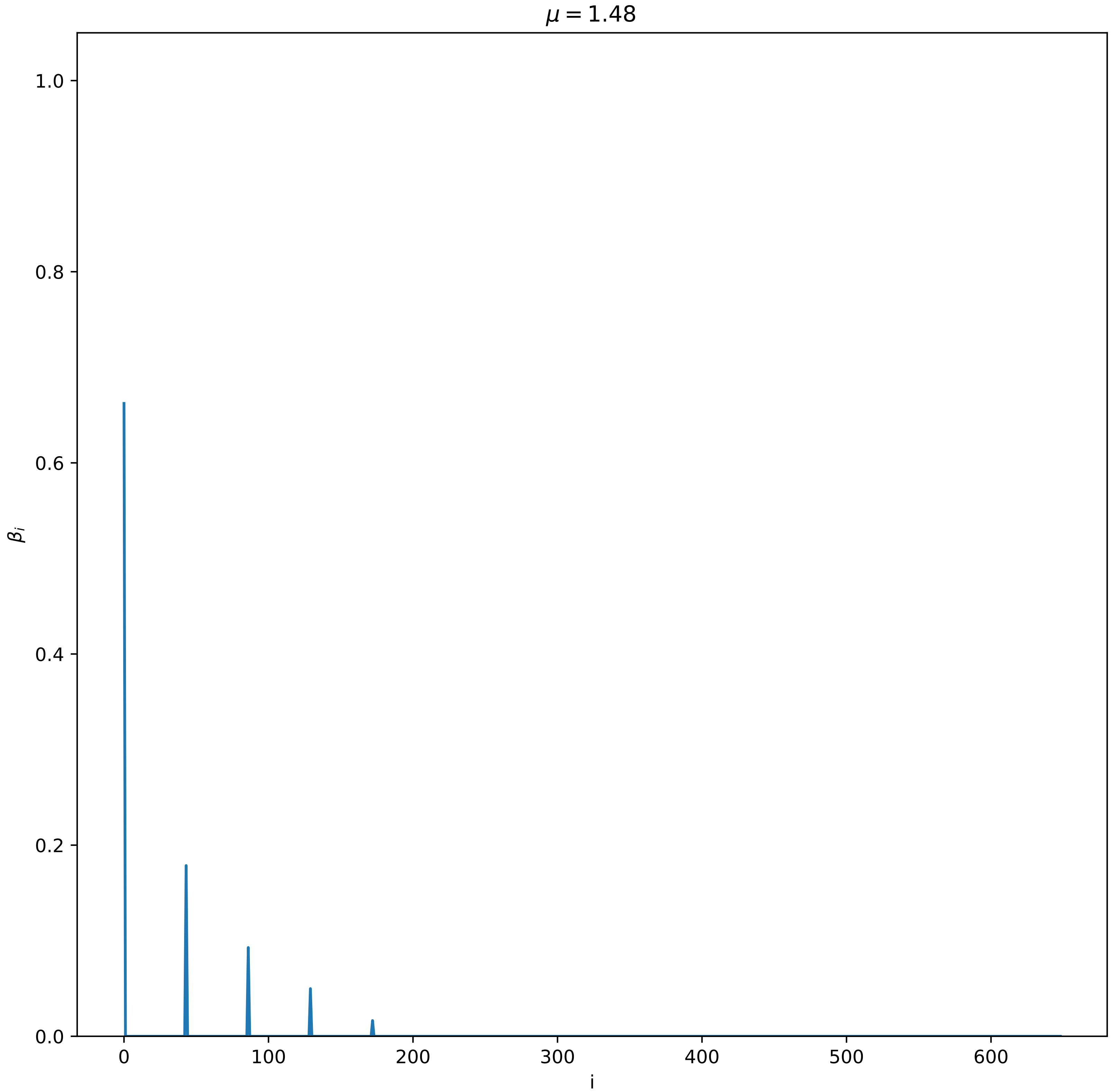
$\mu = 1.45$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



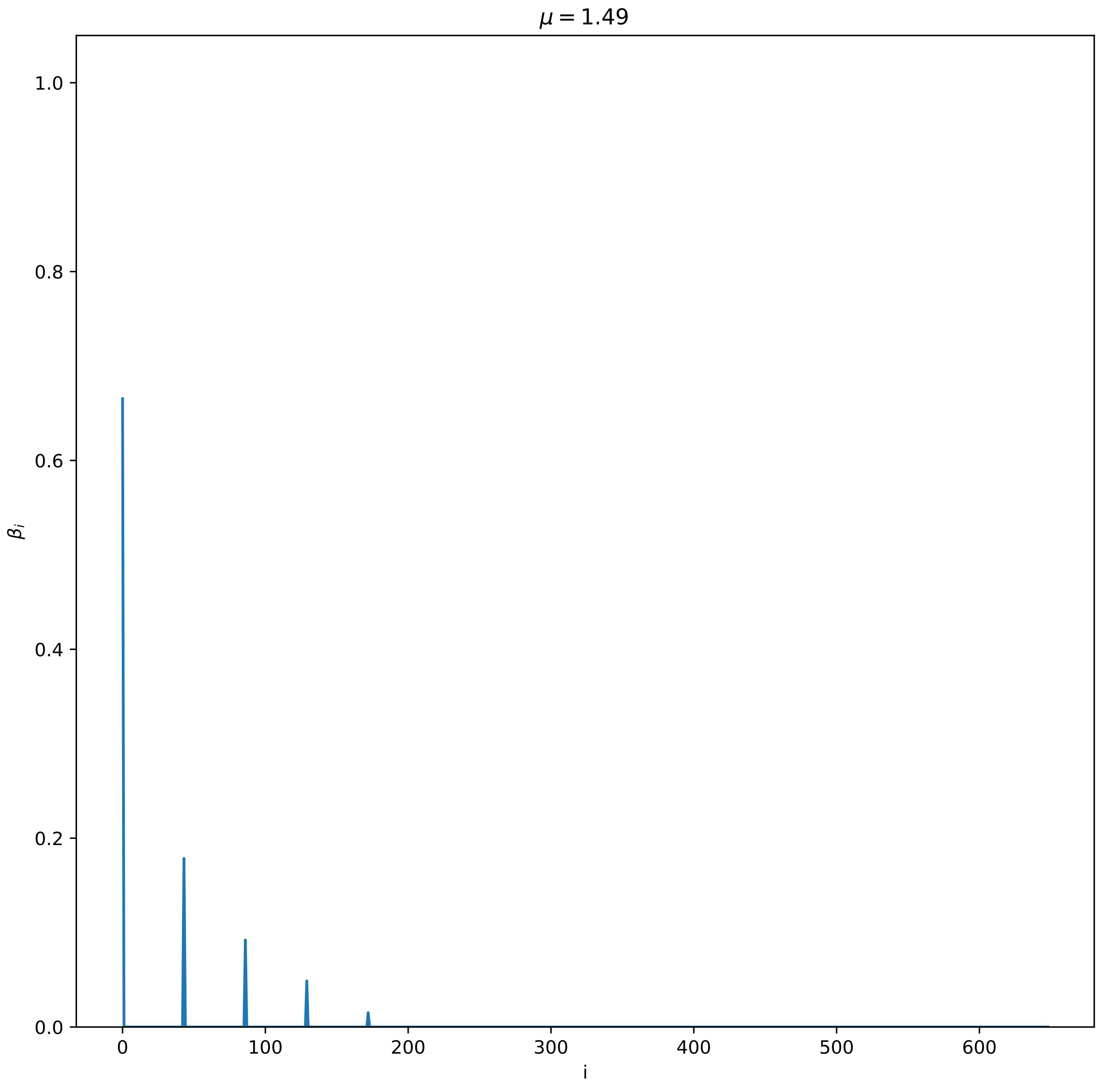
$\mu = 1.46$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

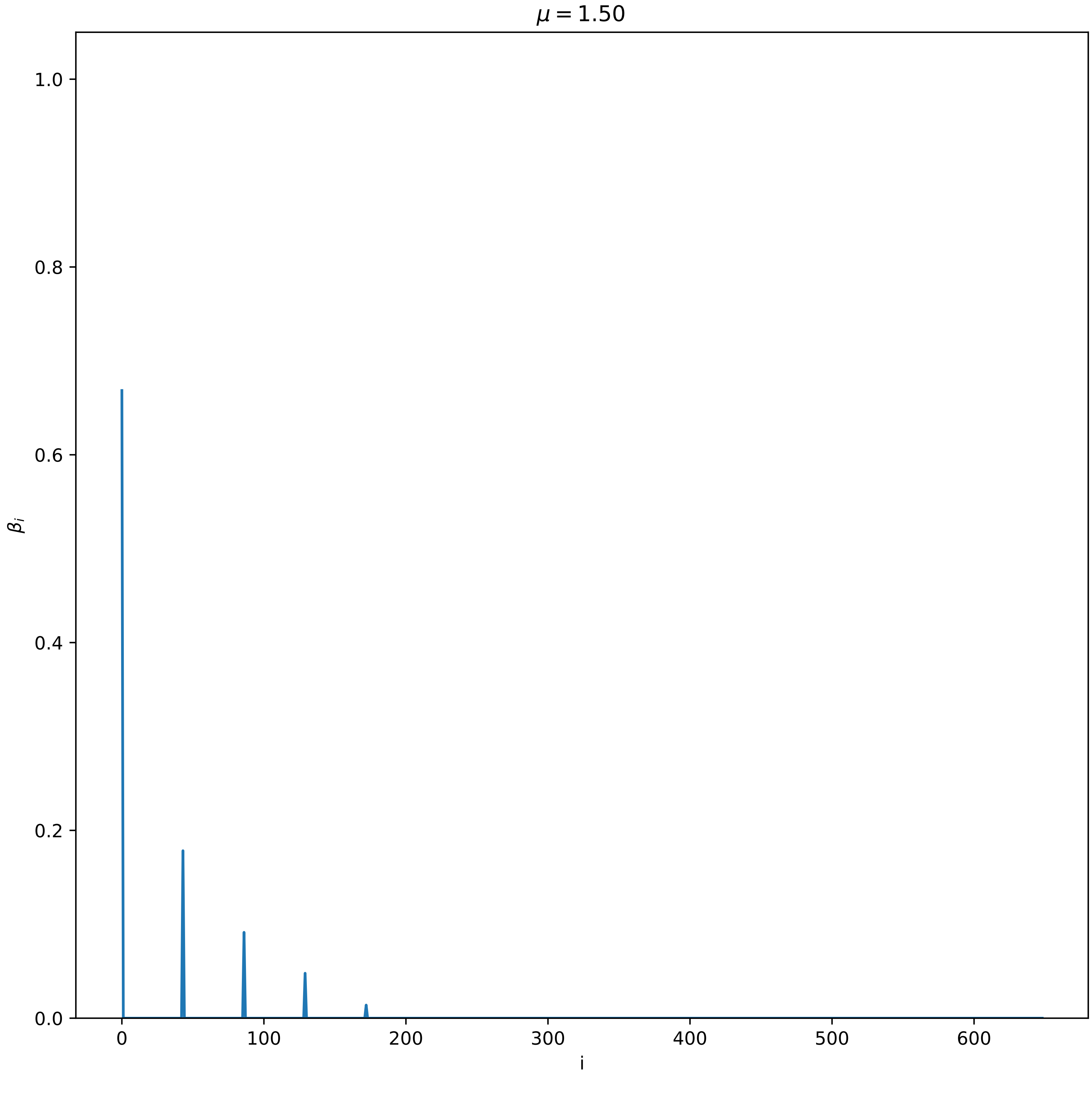


$\mu = 1.47$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

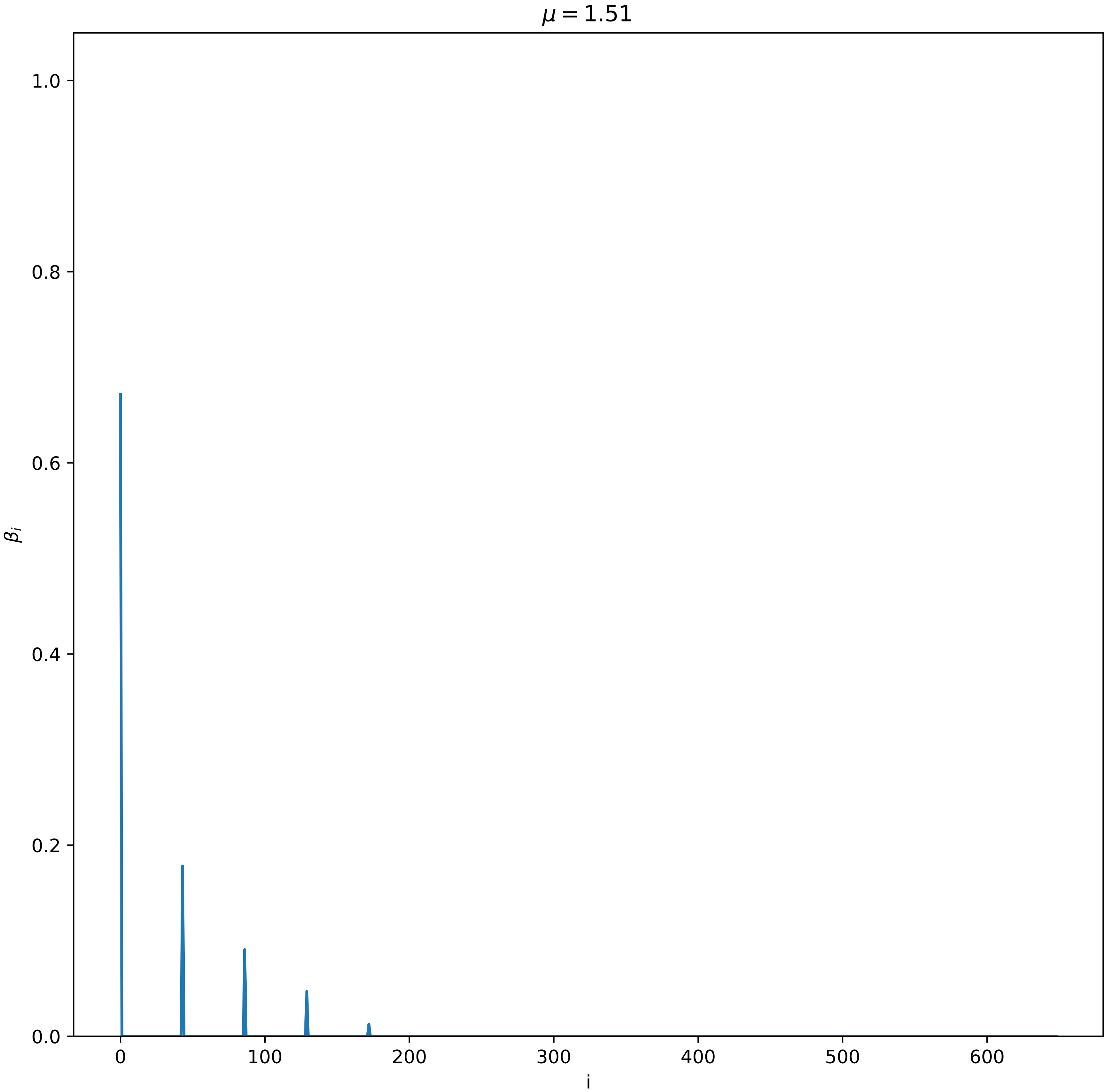


$\mu = 1.48$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

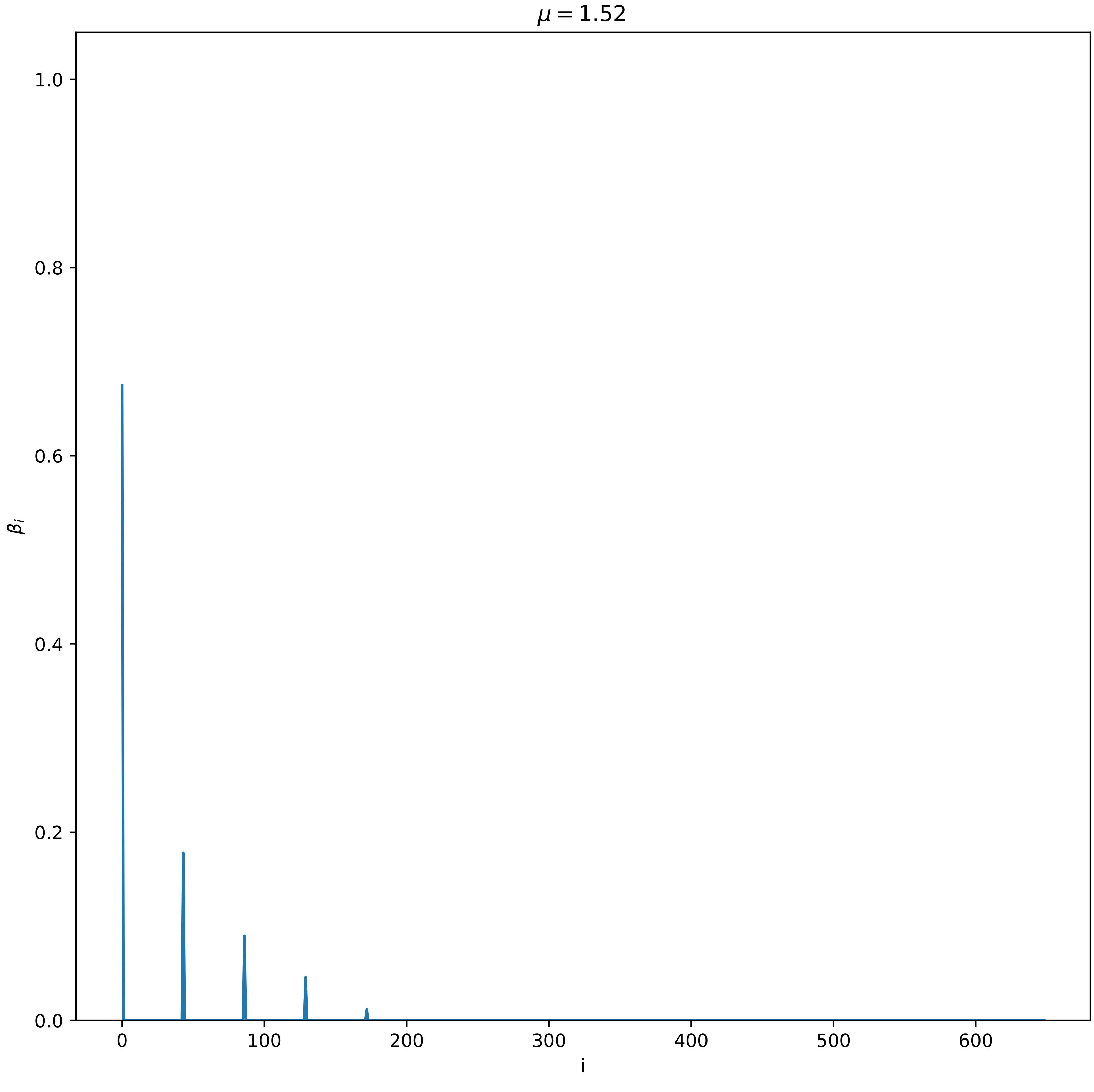

$\mu = 1.49$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



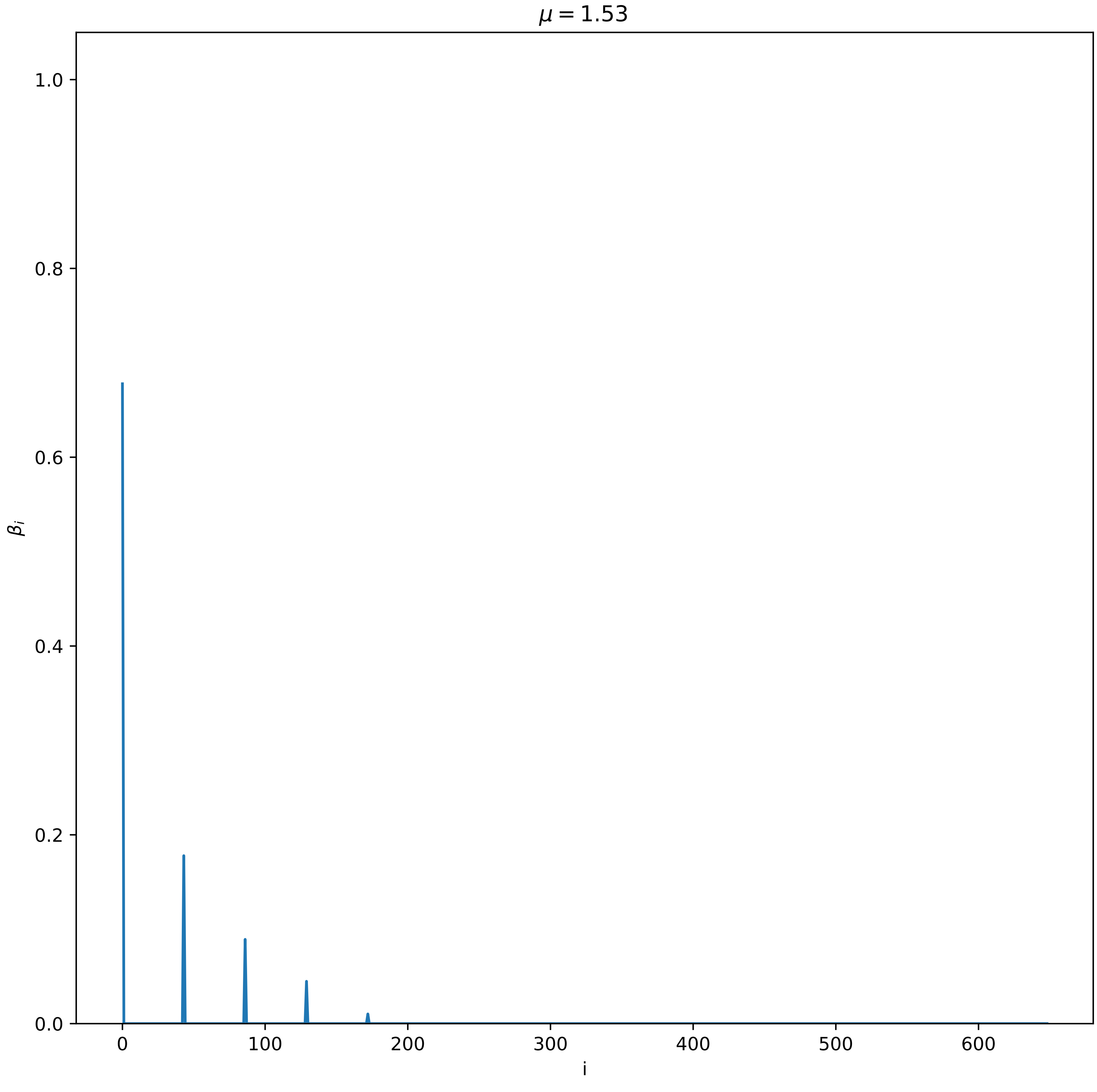
$\mu = 1.50$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

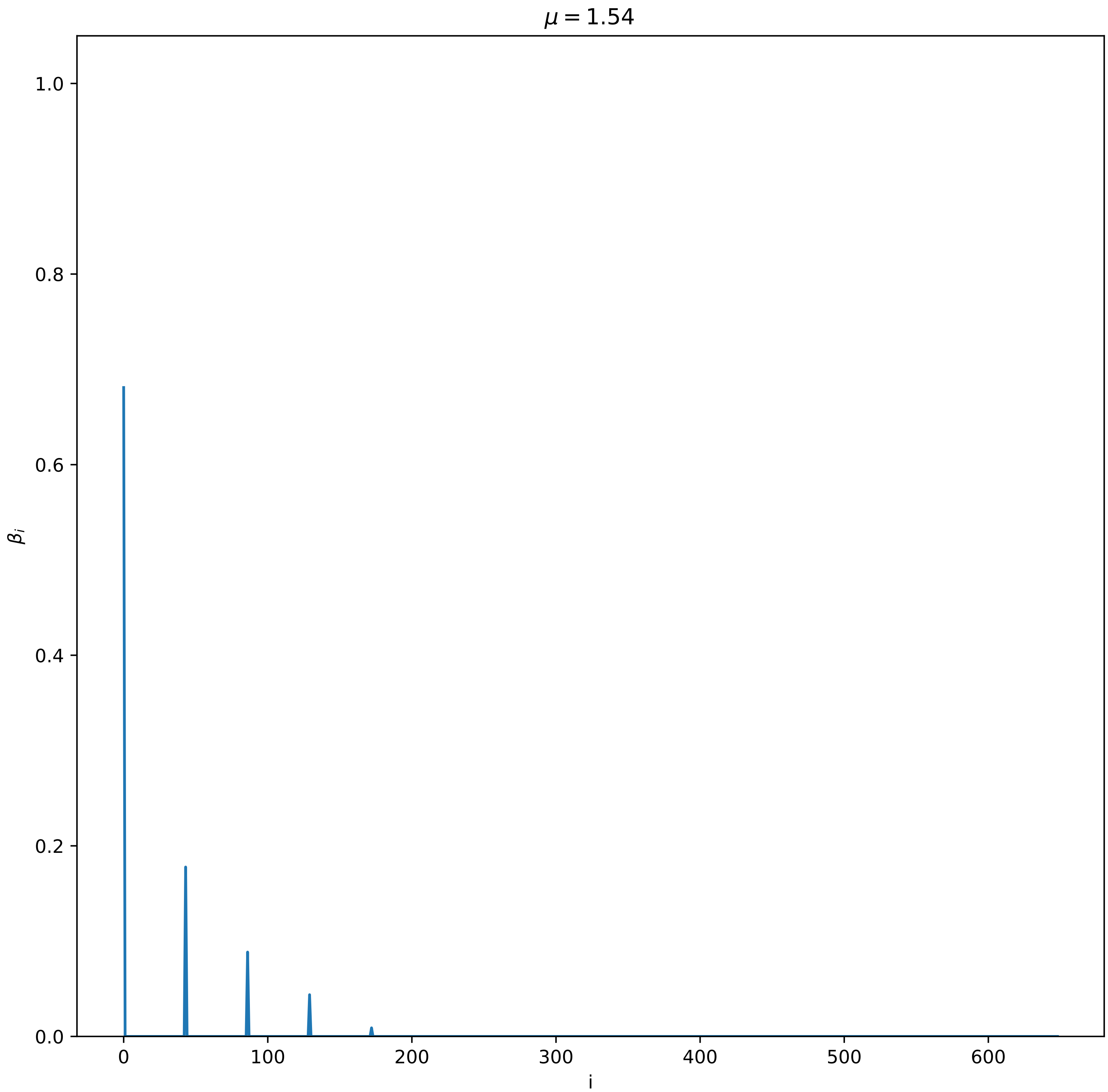


$\mu = 1.51$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

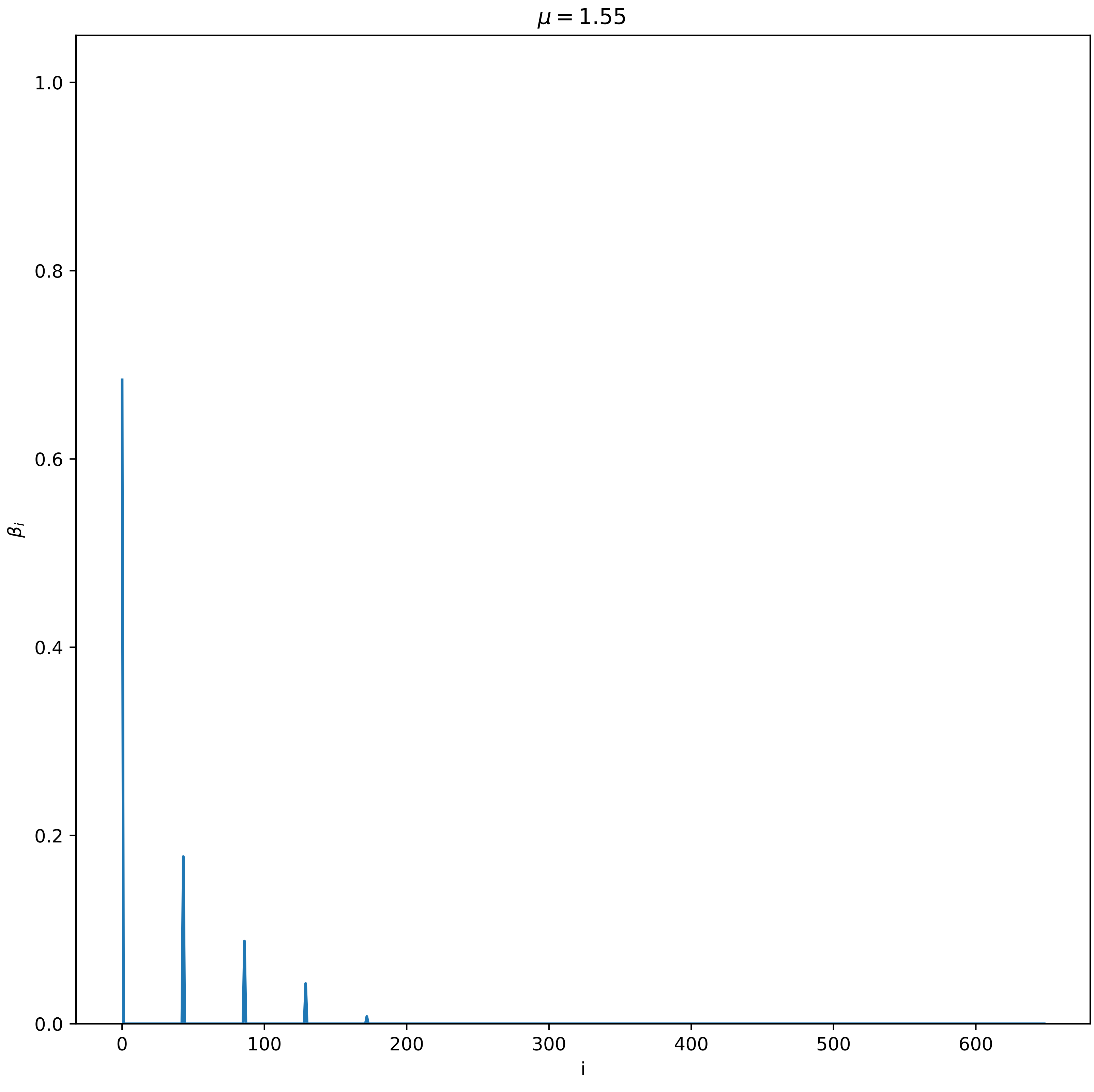


$\mu = 1.52$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

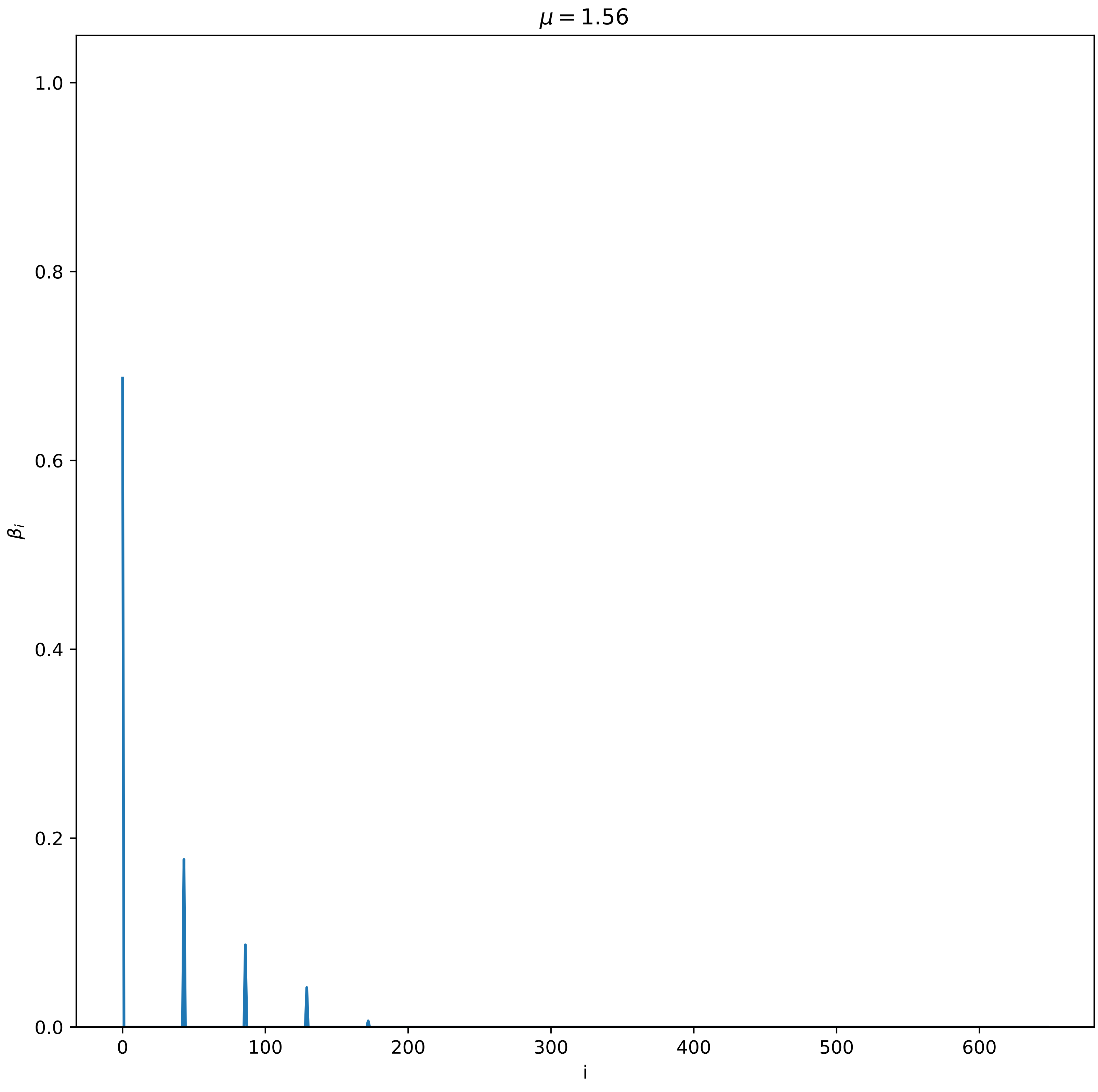

$\mu = 1.53$
β_i
i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600



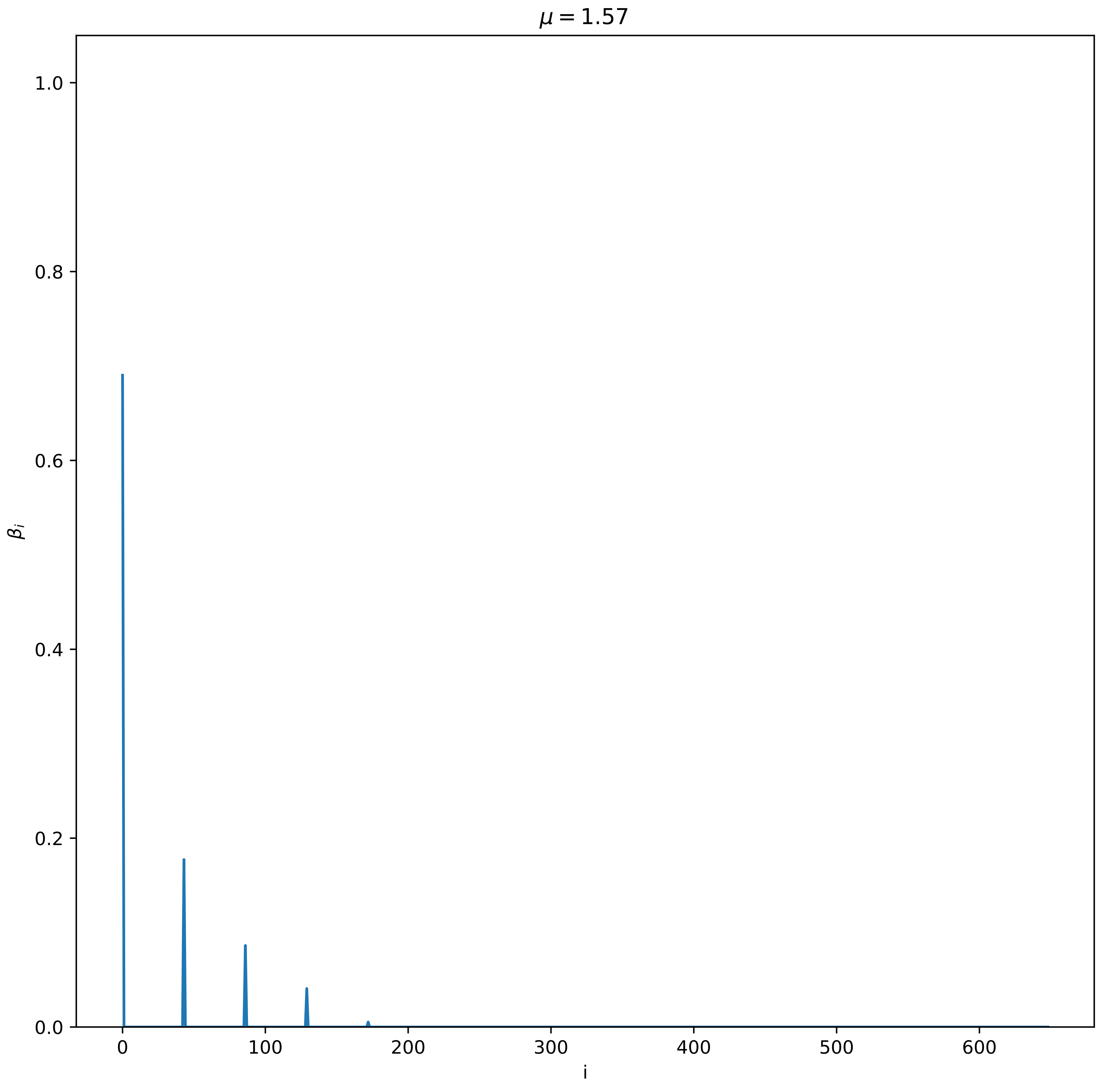
$\mu = 1.54$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

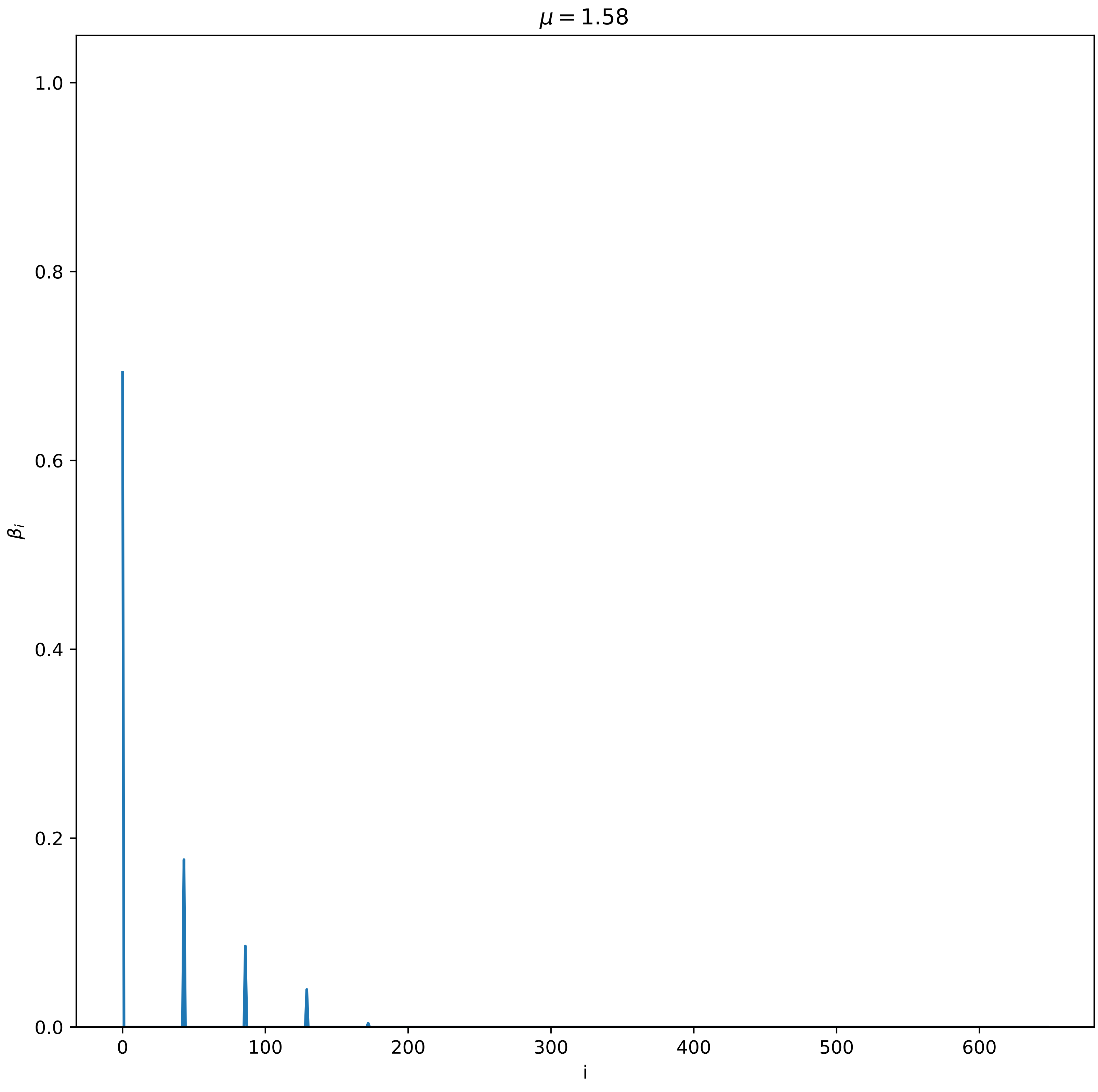


$\mu = 1.55$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

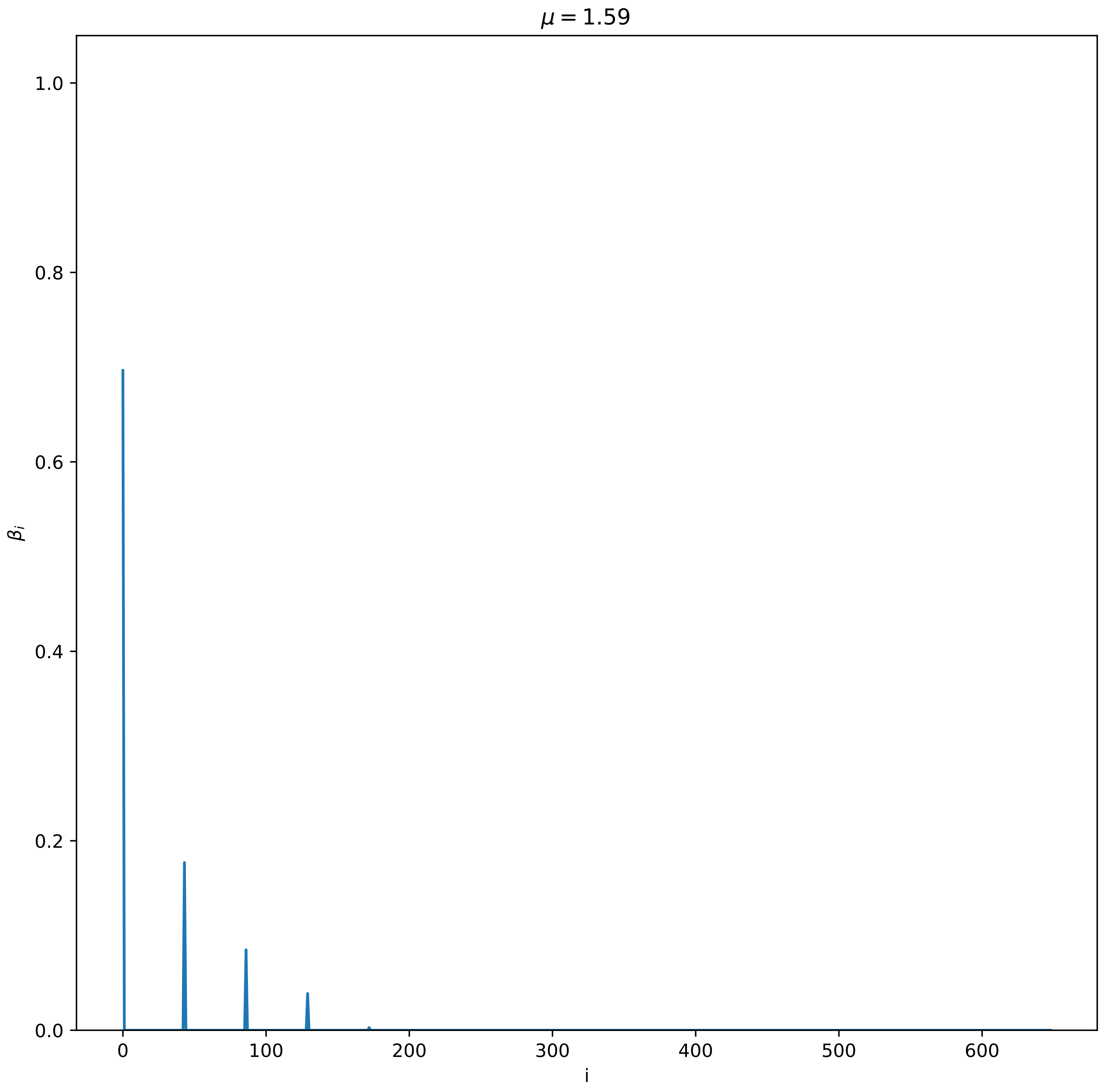


$\mu = 1.56$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

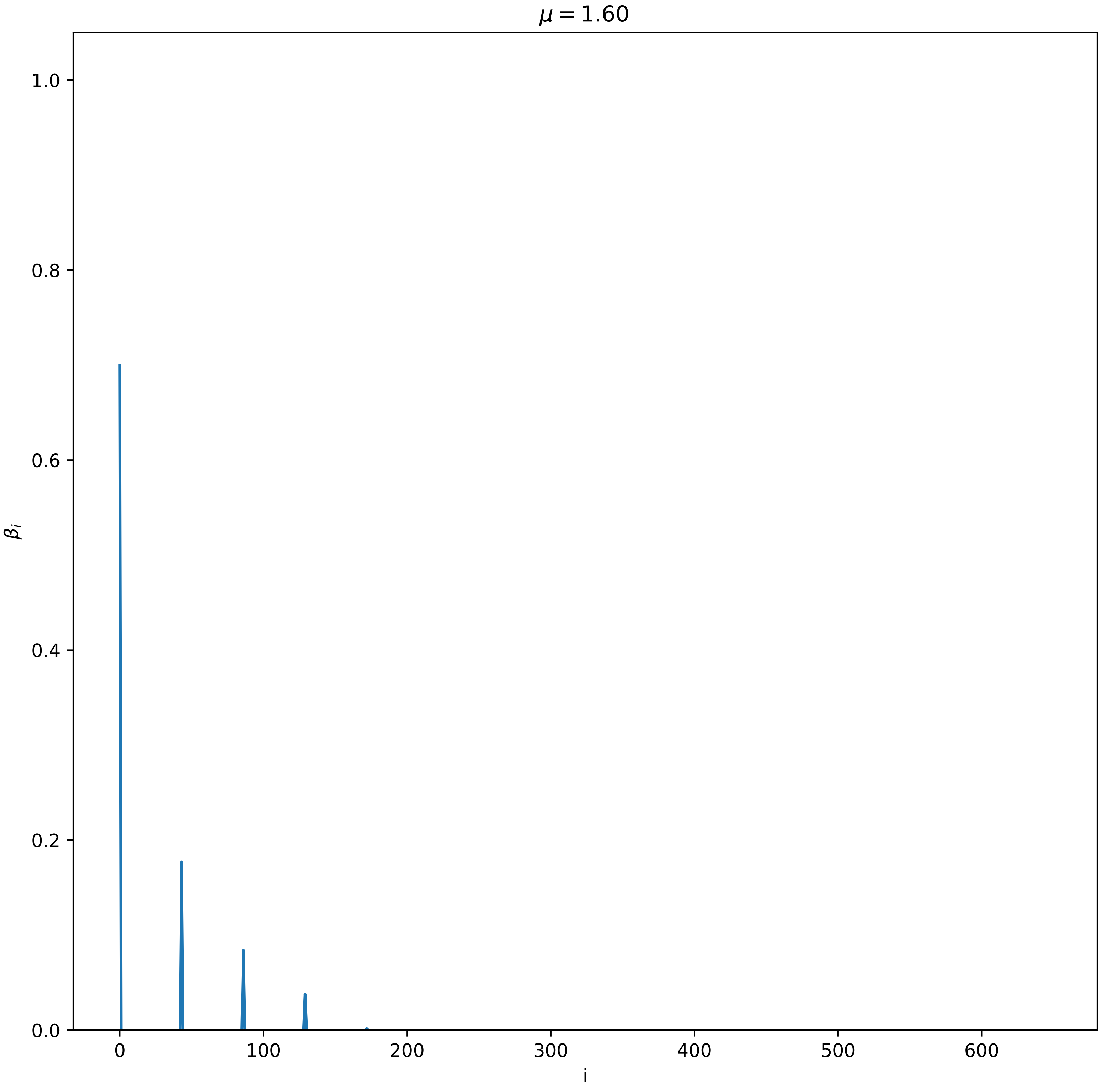

$\mu = 1.57$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



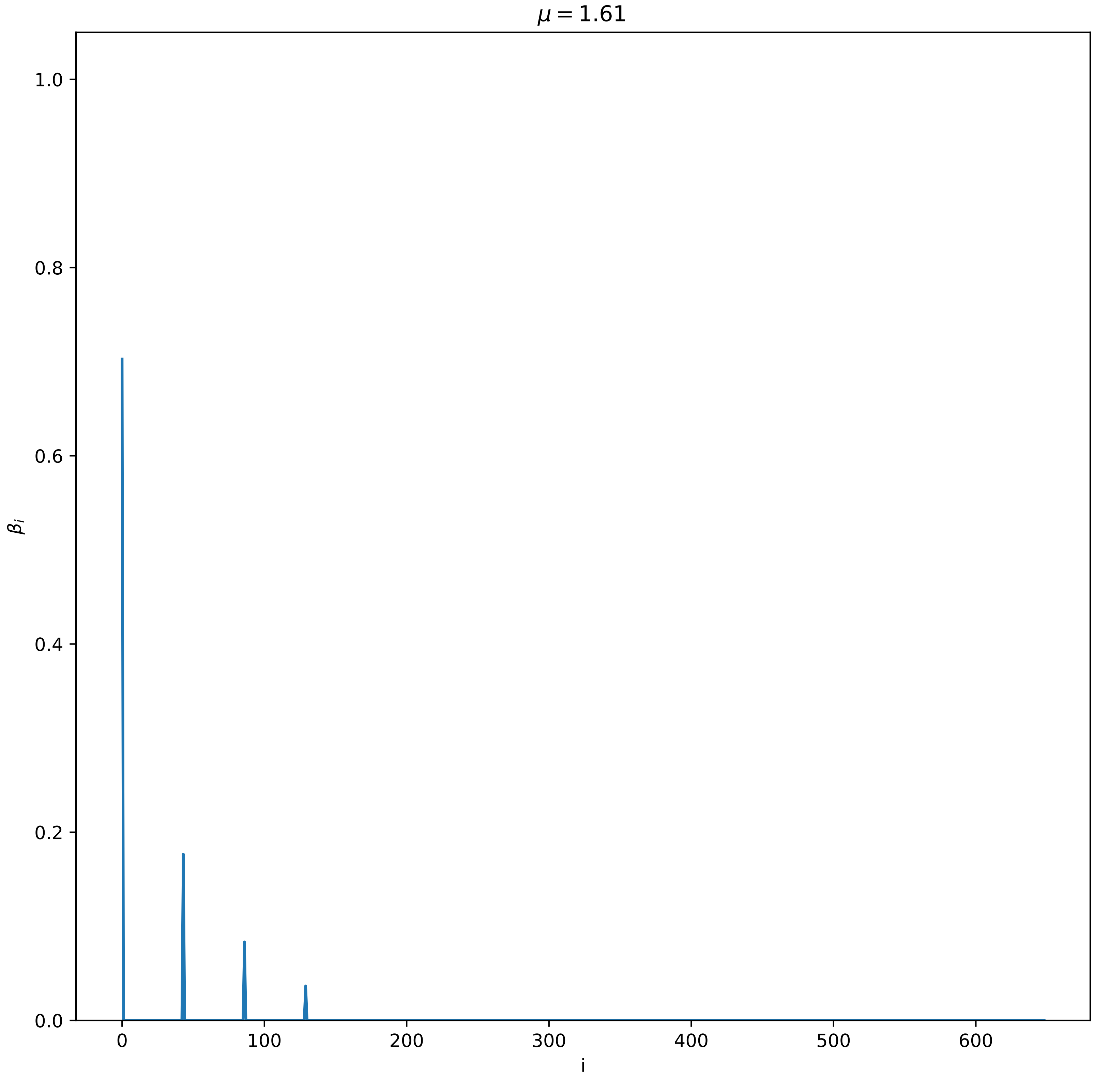
$\mu = 1.58$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



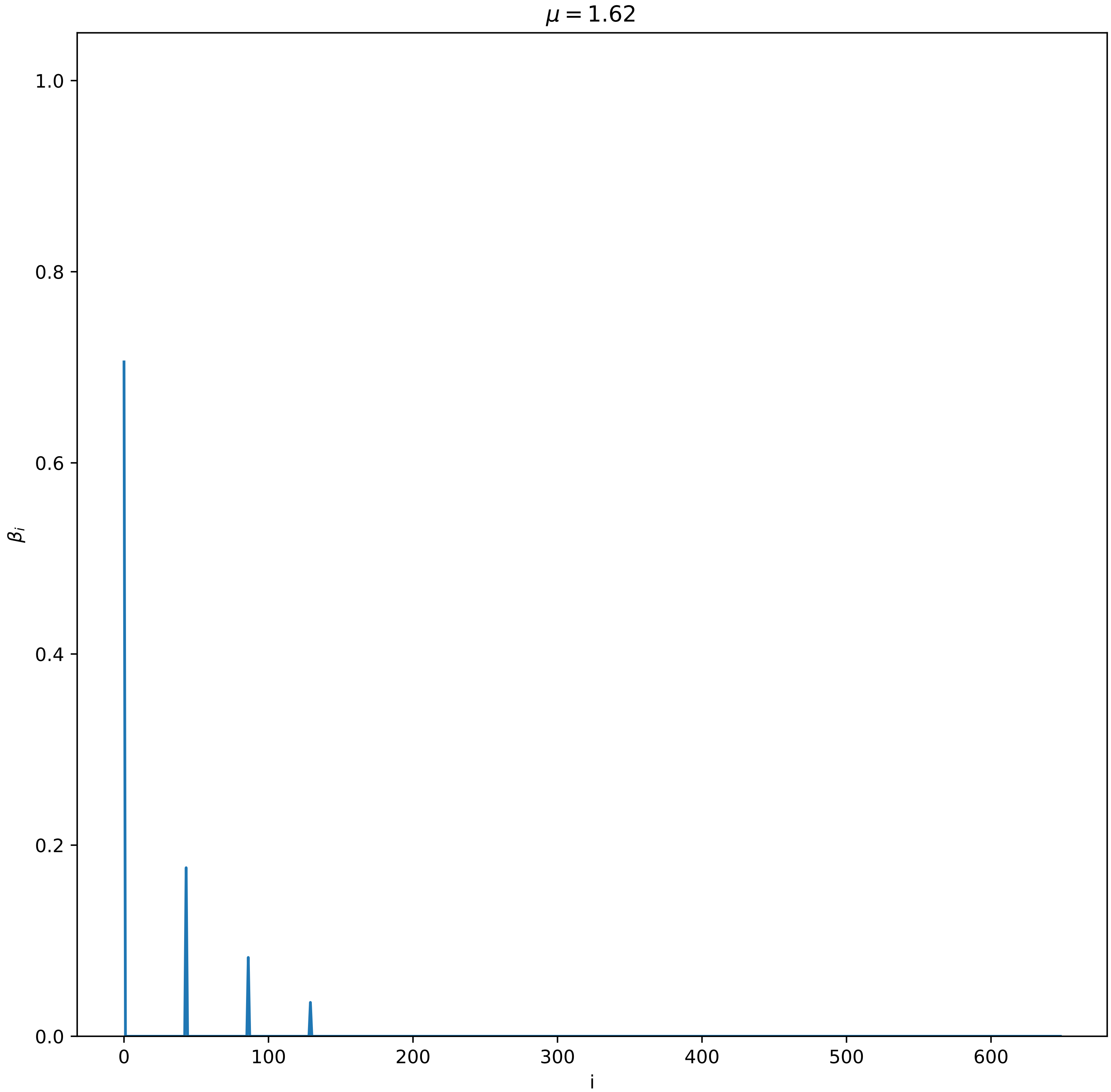
$\mu = 1.59$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



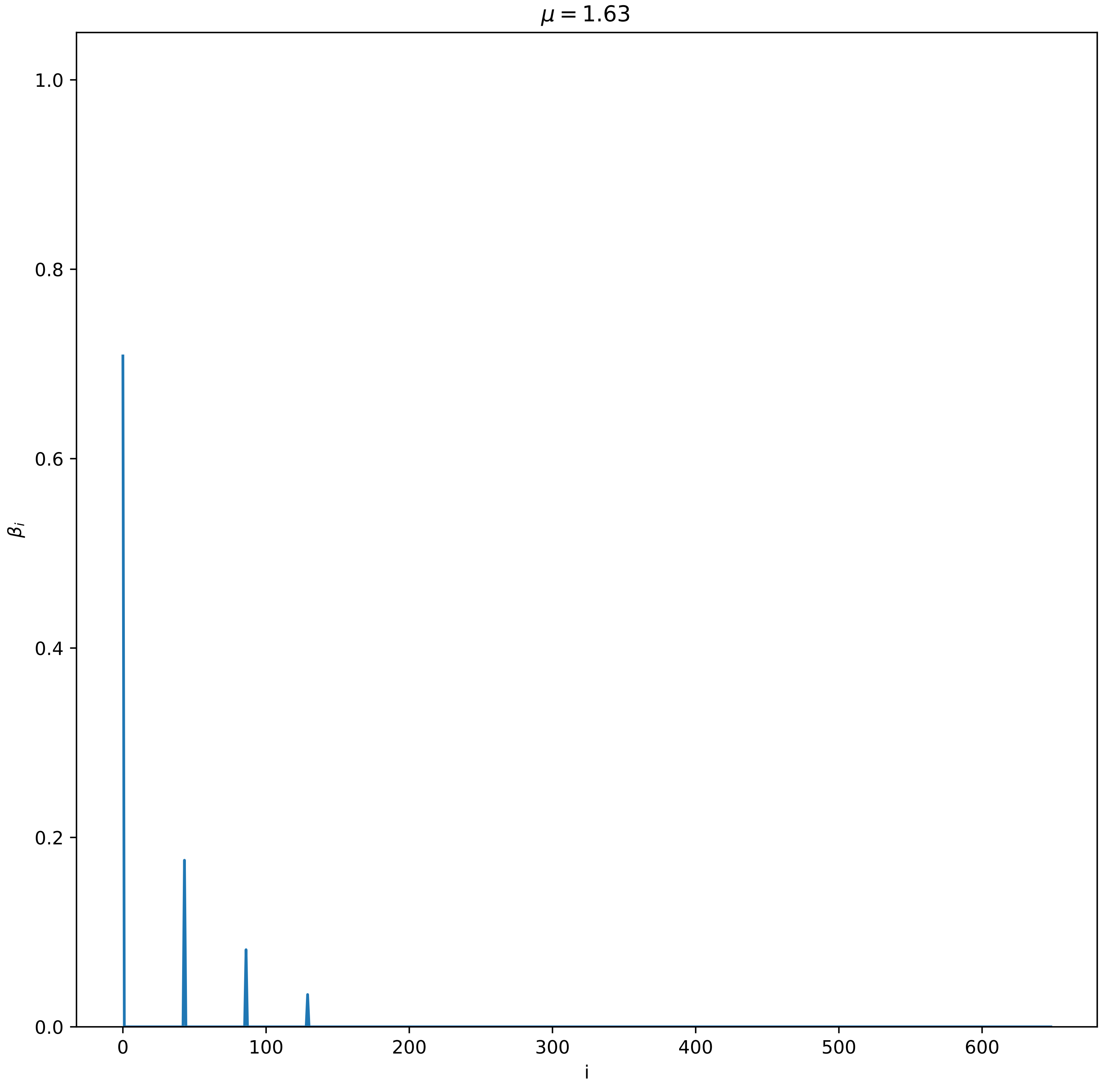
$\mu = 1.60$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

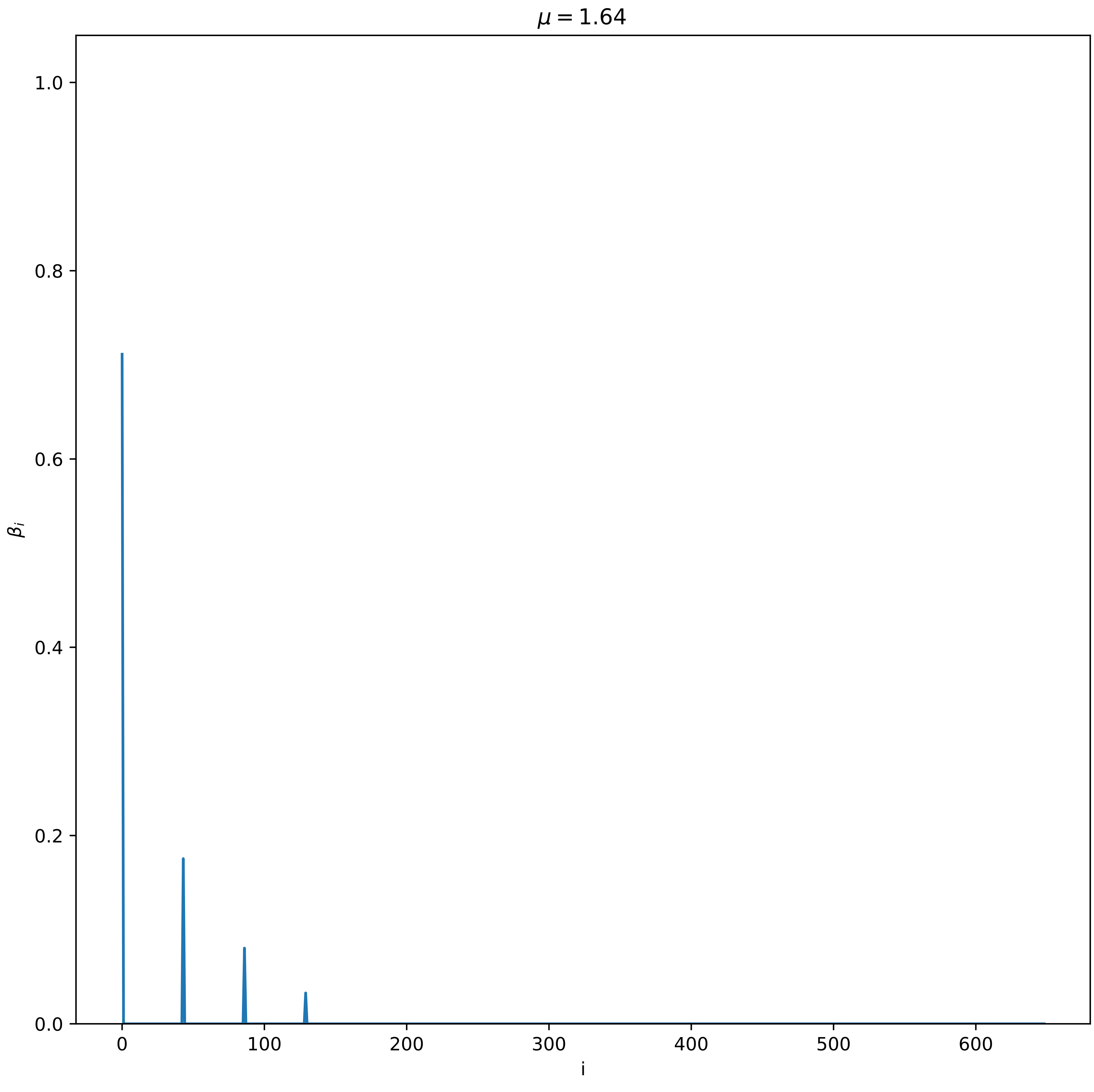


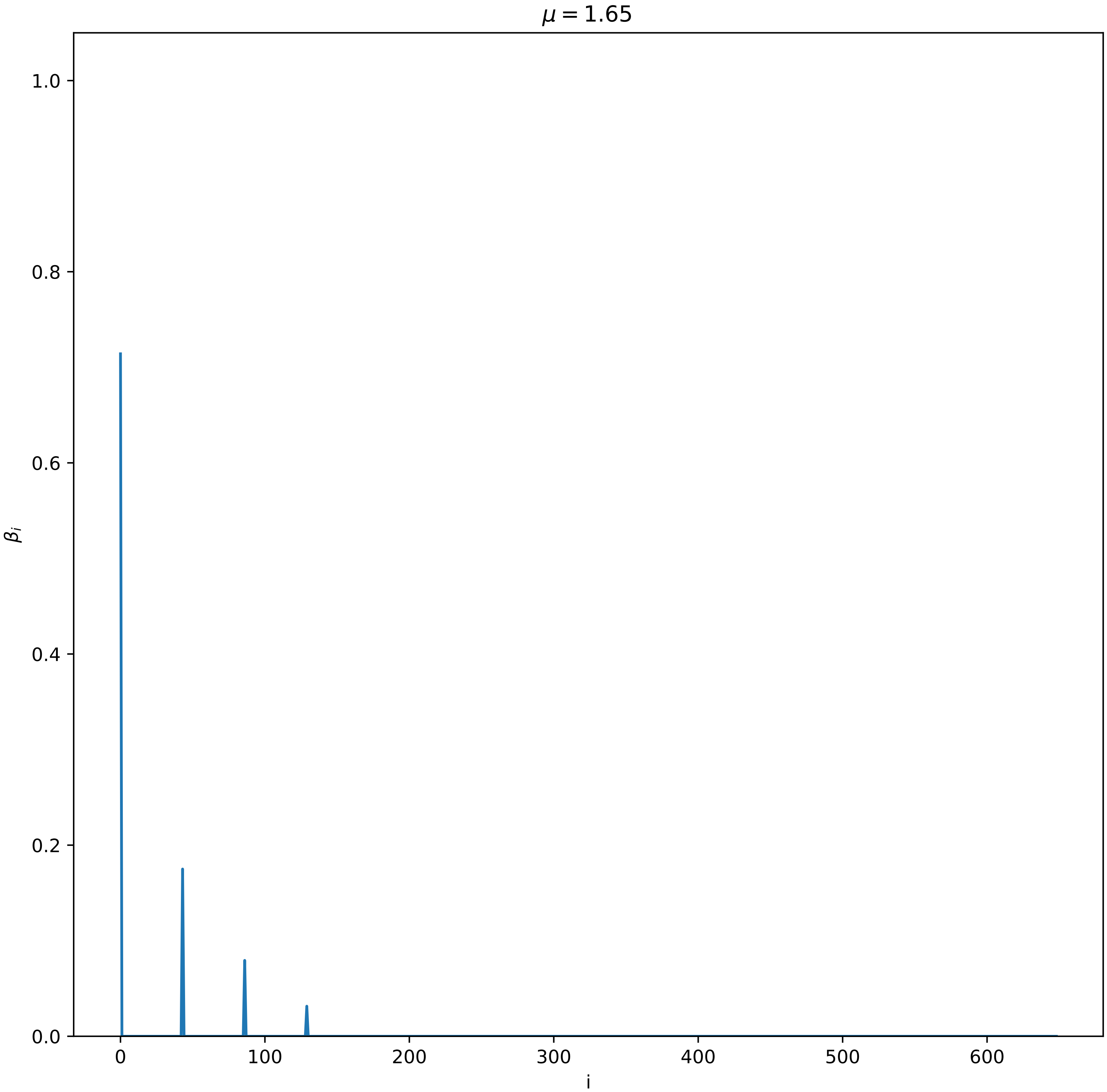
$\mu = 1.61$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



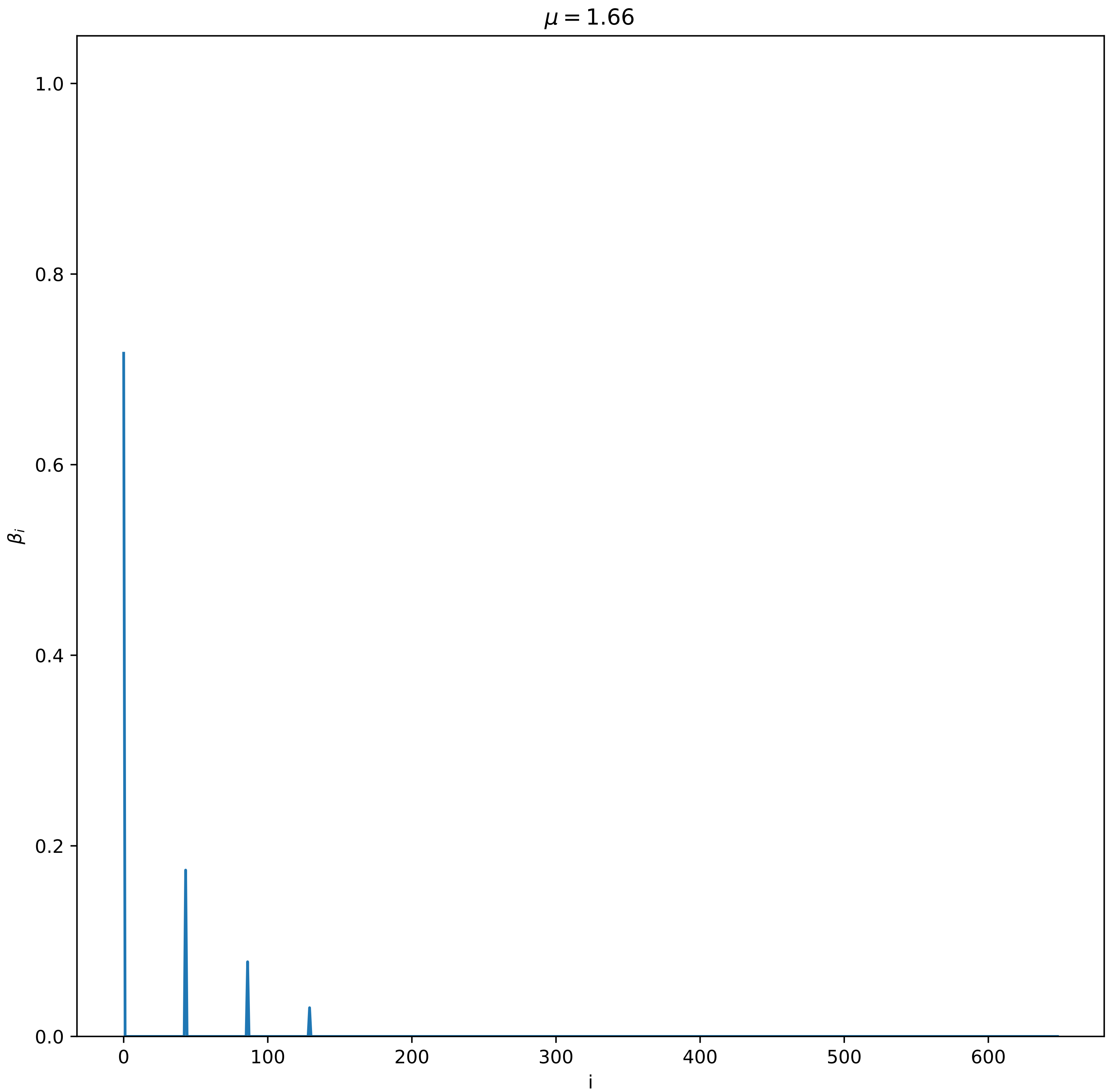
$\mu = 1.62$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.63$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

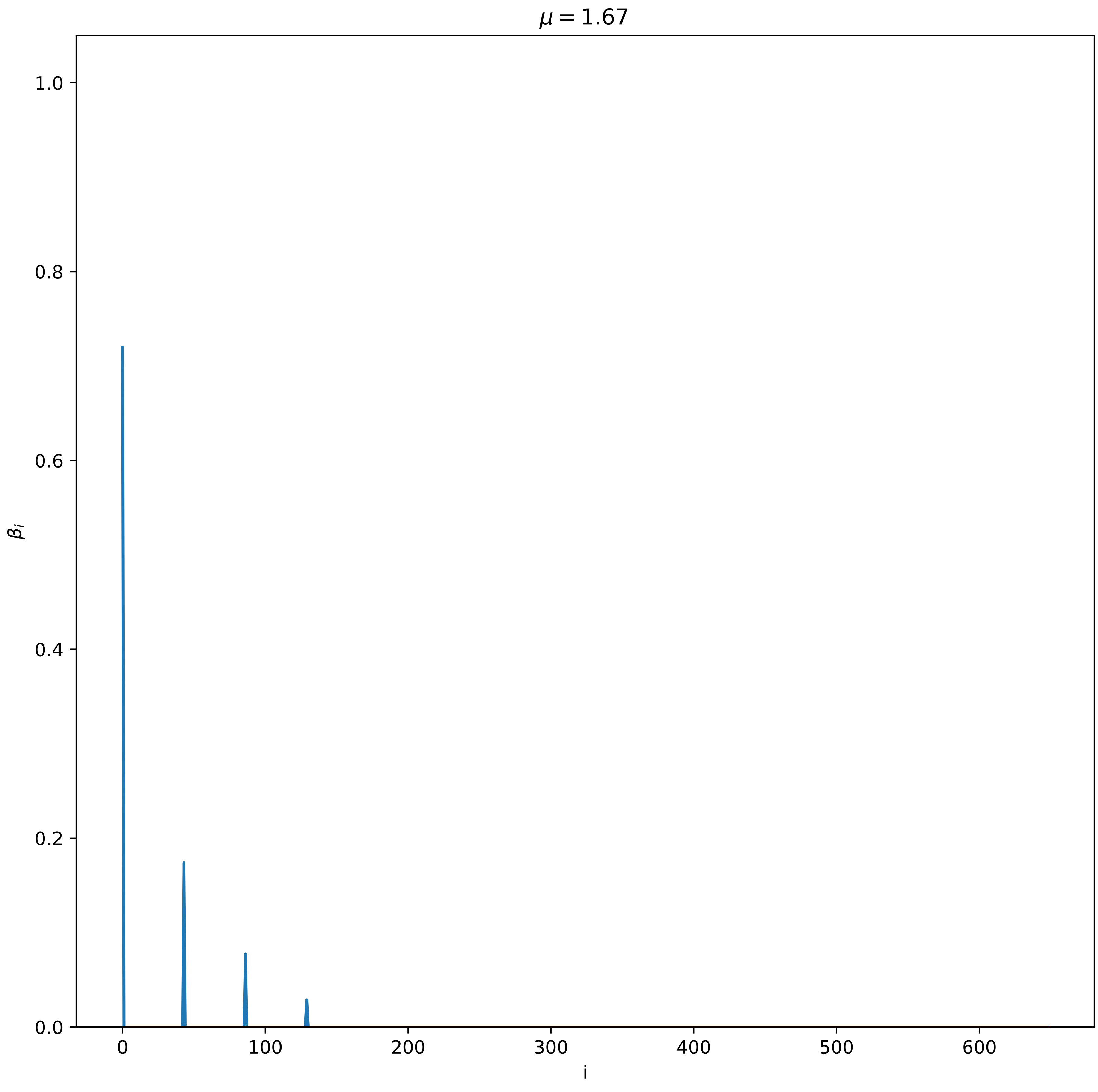

$\mu = 1.64$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

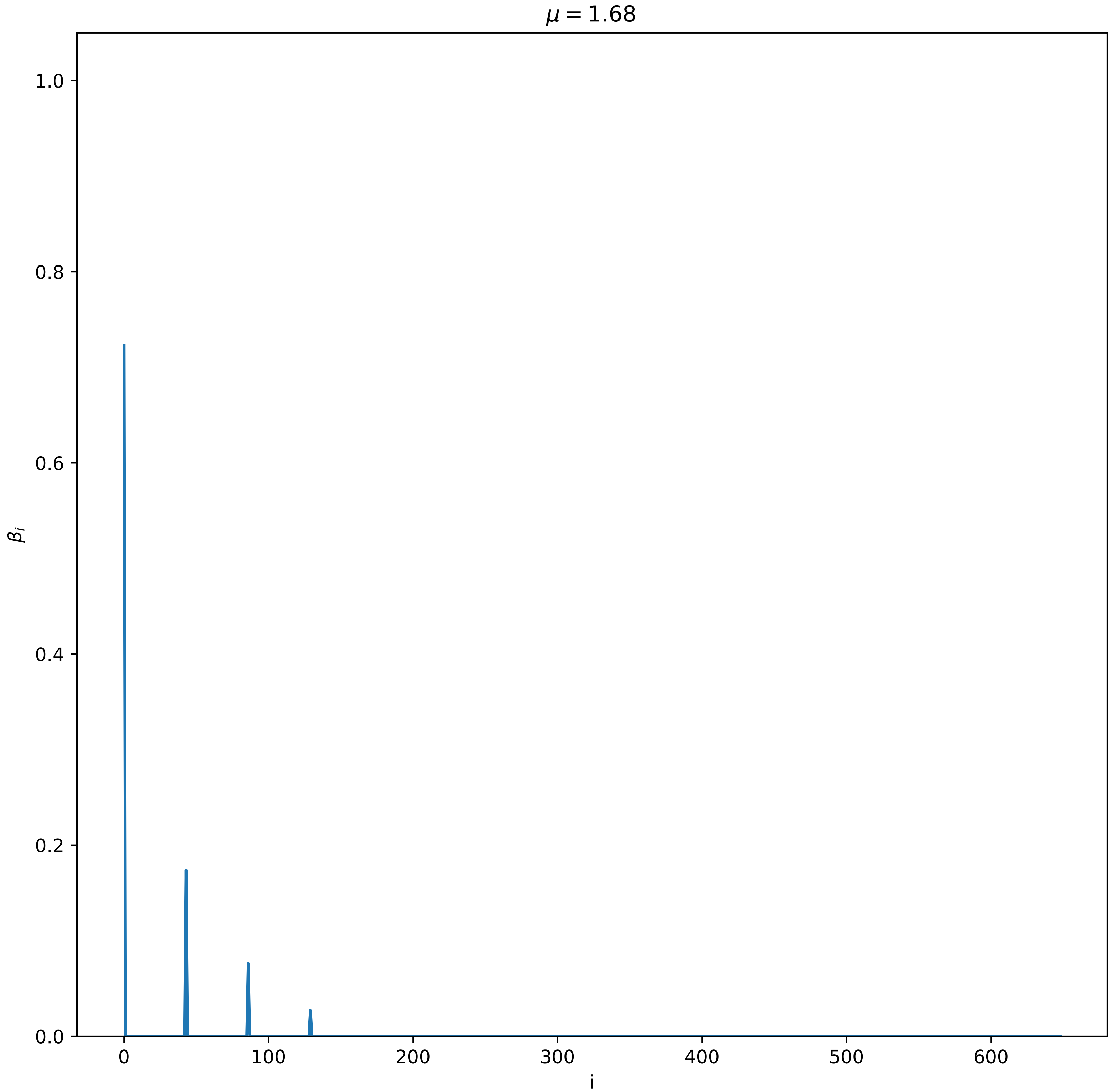


$\mu = 1.65$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

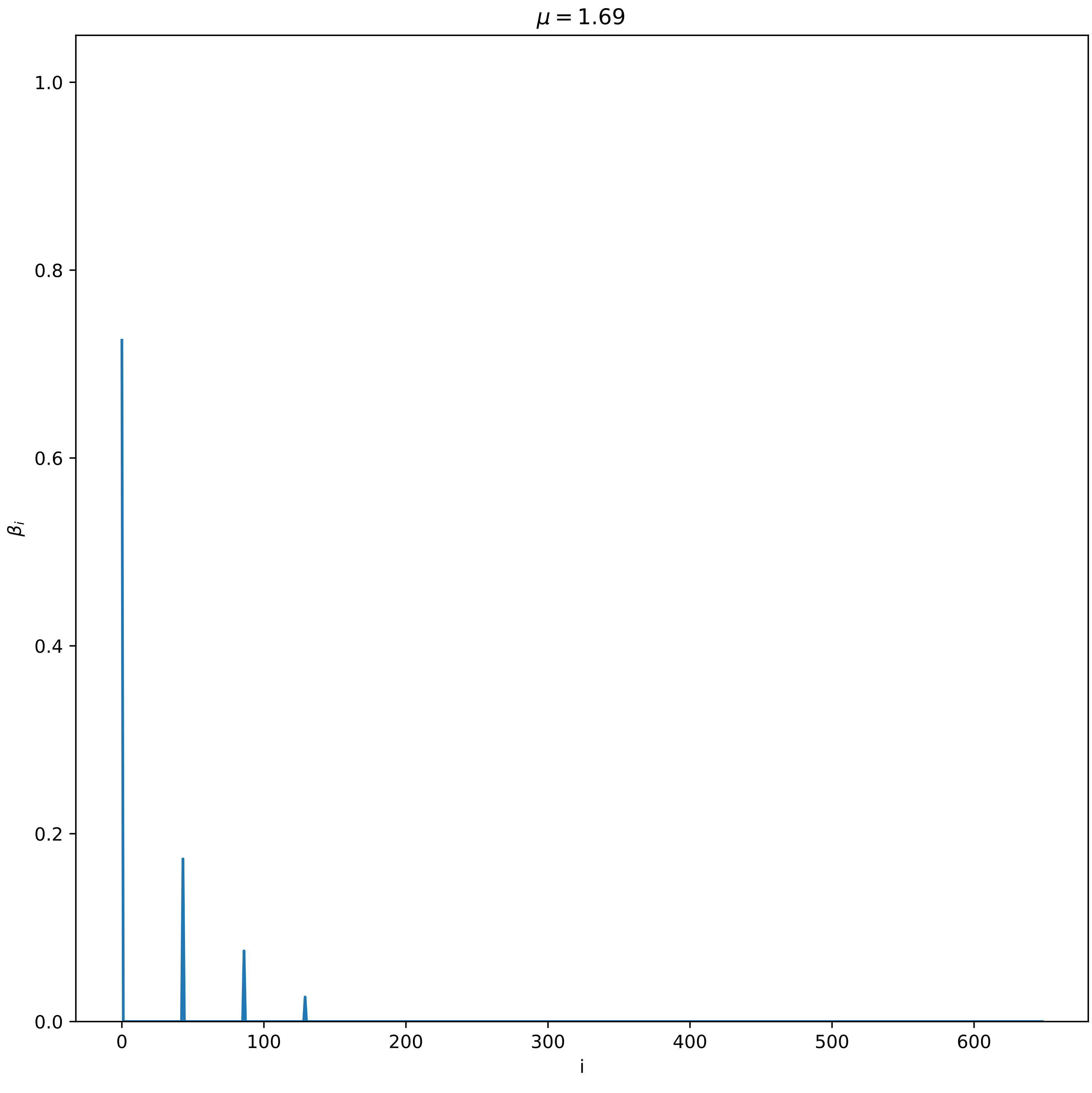


$\mu = 1.66$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

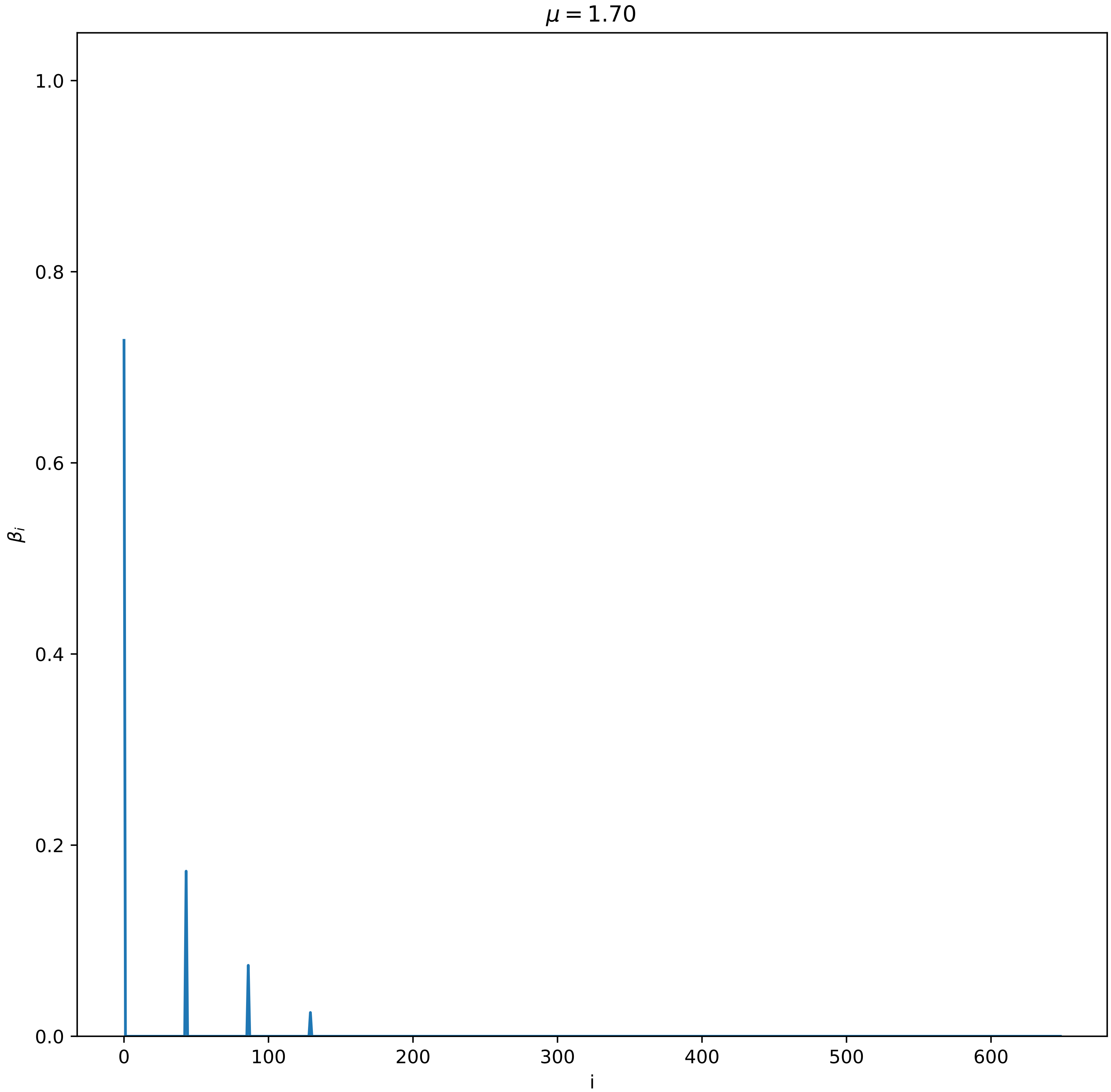

$\mu = 1.67$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

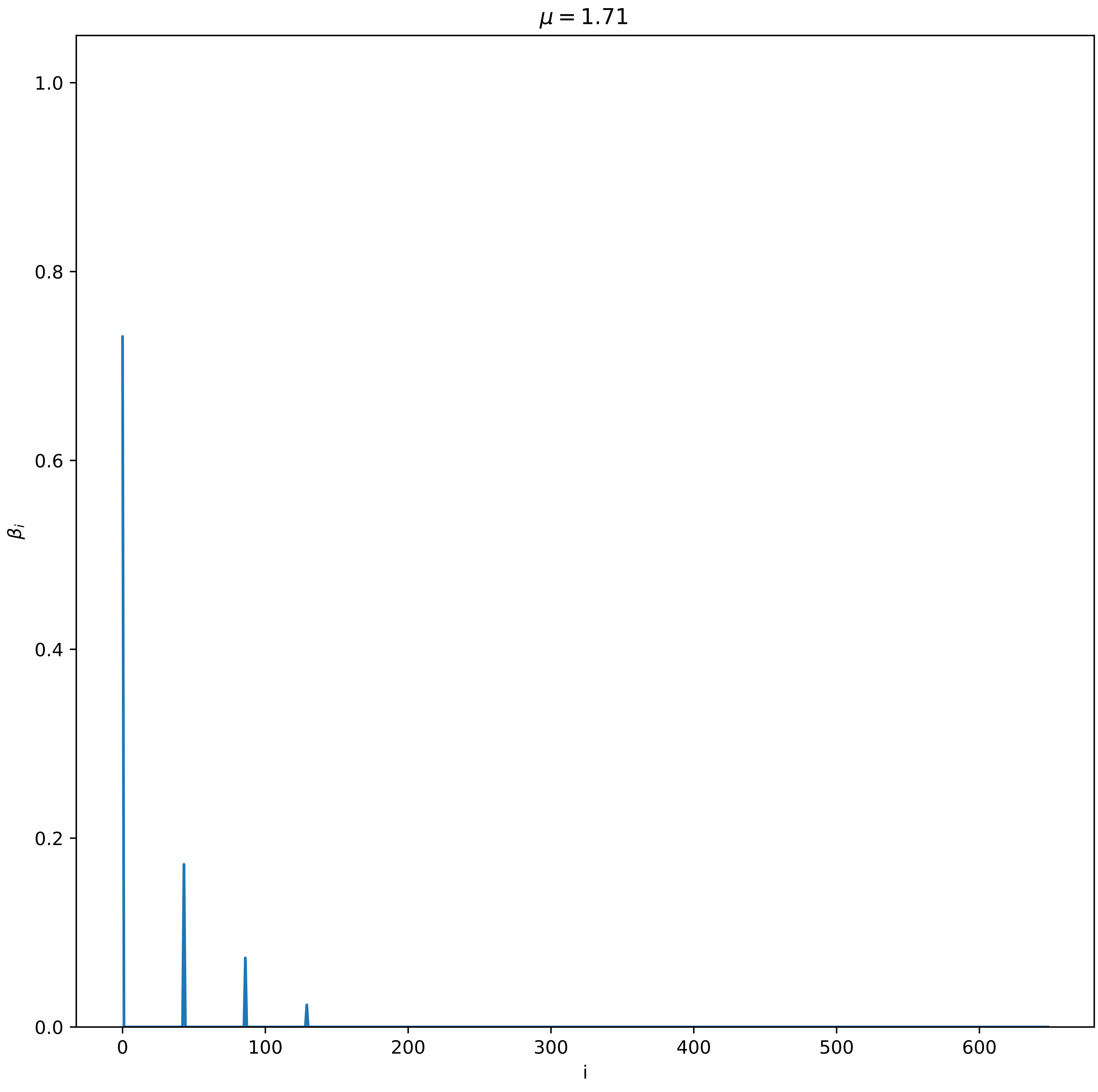


$\mu = 1.68$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

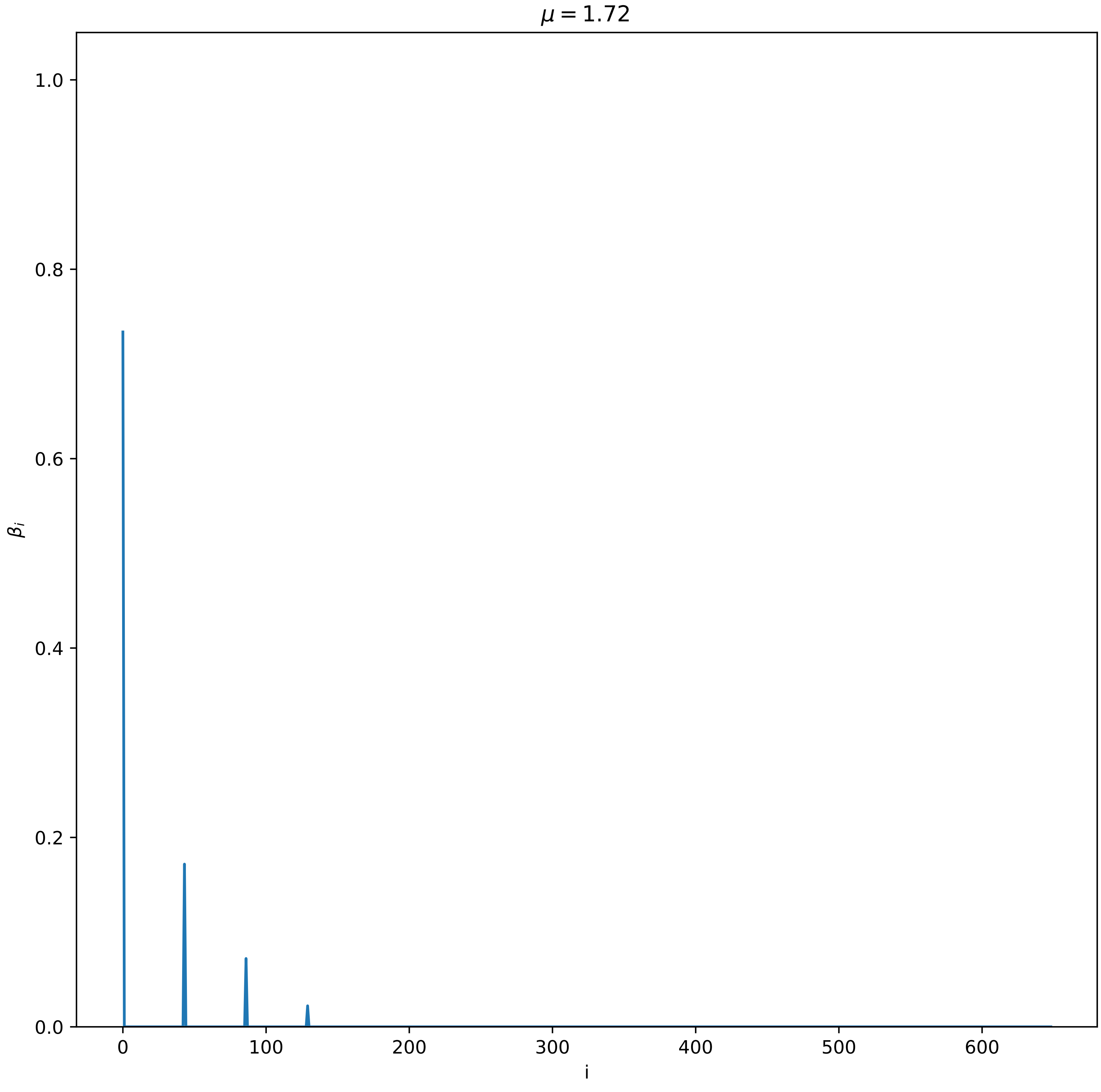


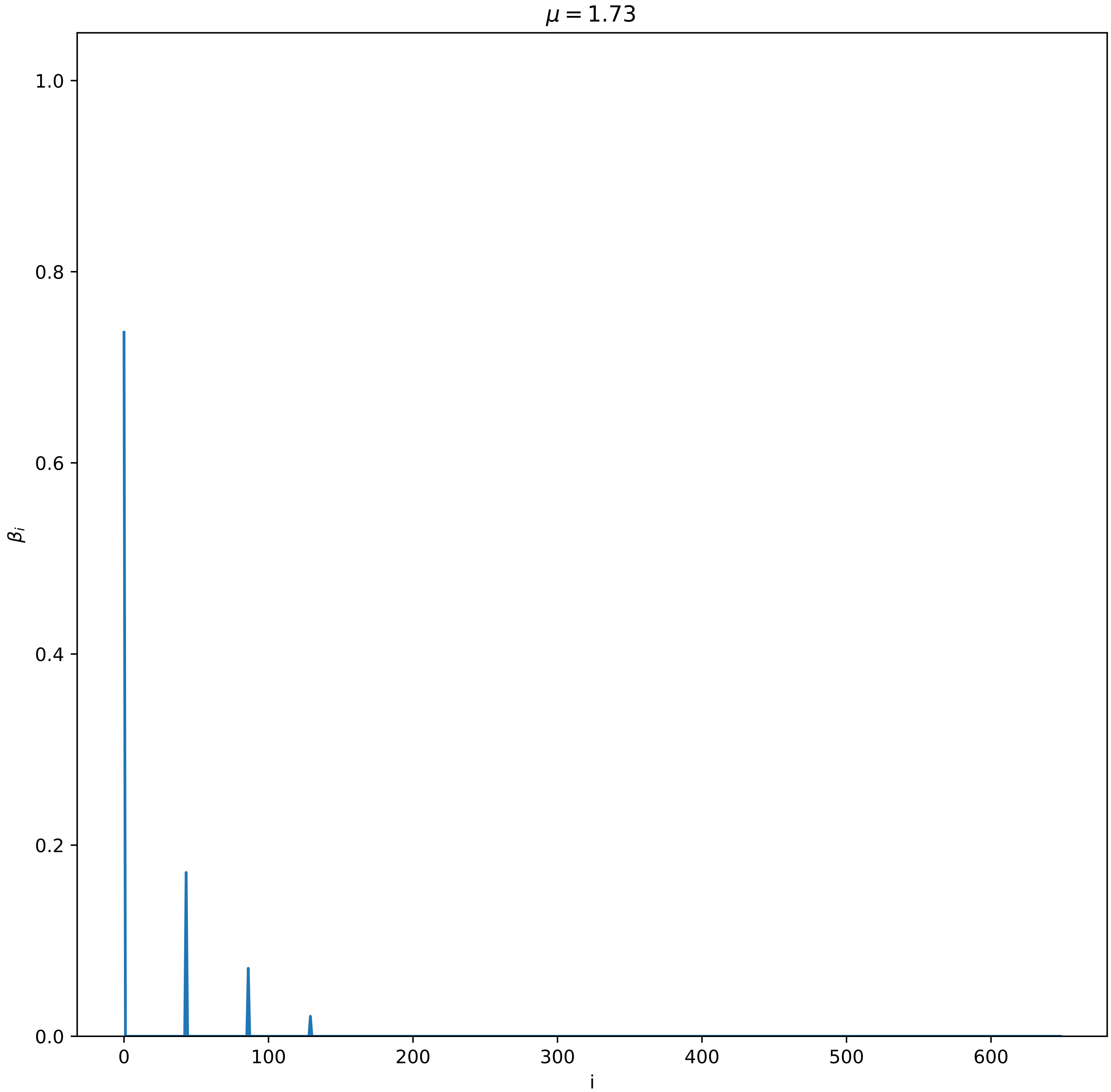
$\mu = 1.69$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.70$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

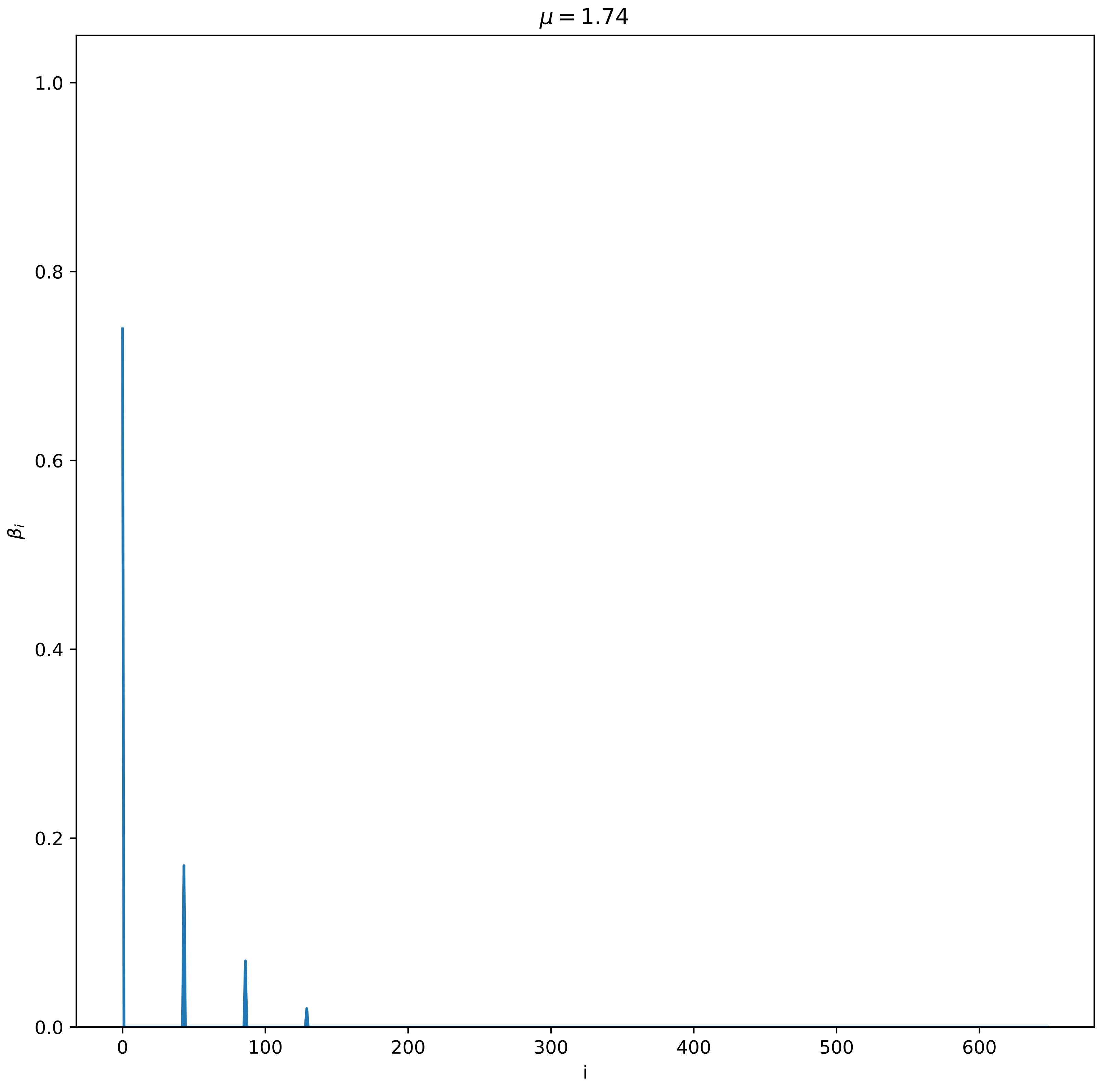


$\mu = 1.71$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

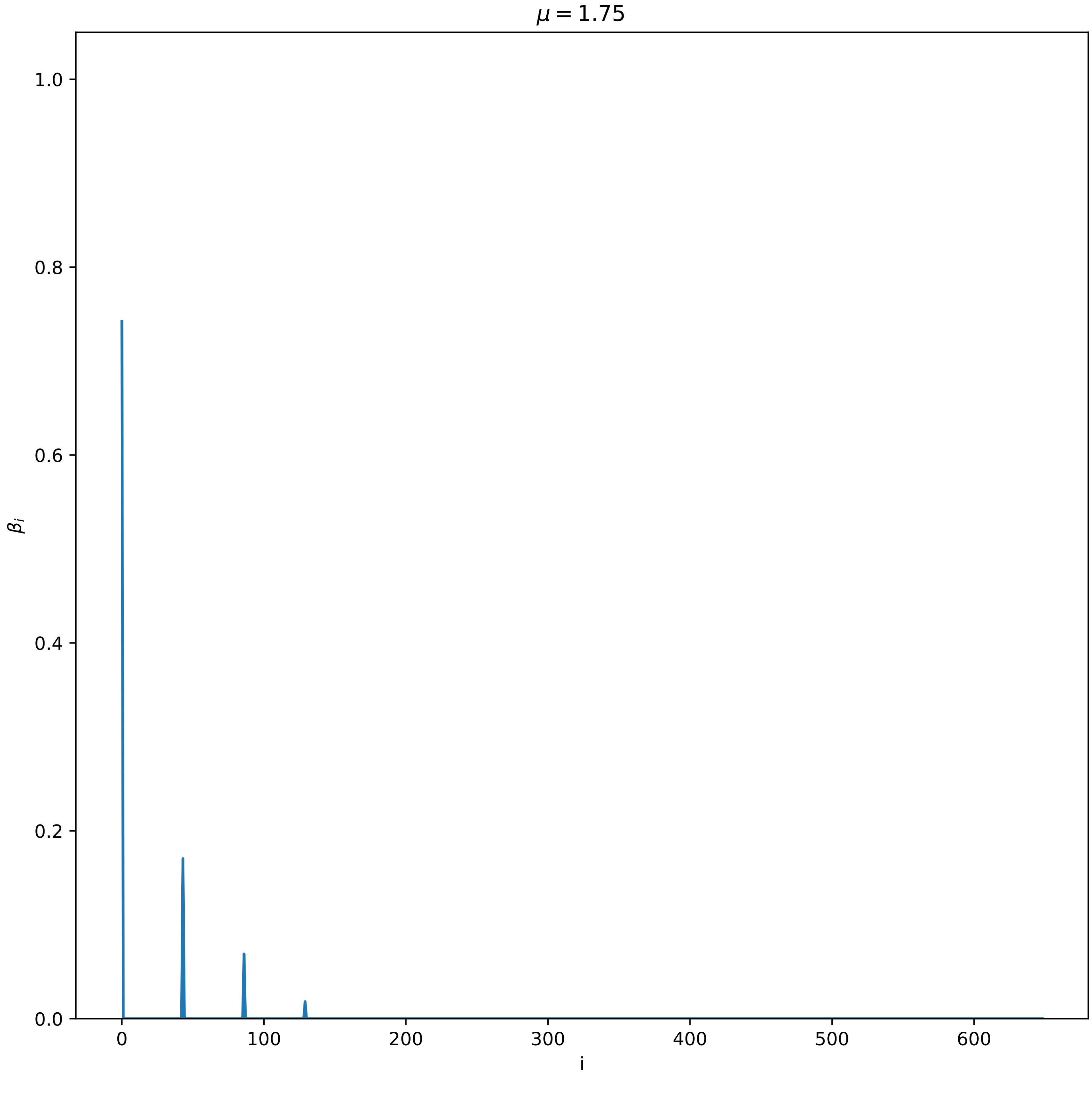

$\mu = 1.72$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



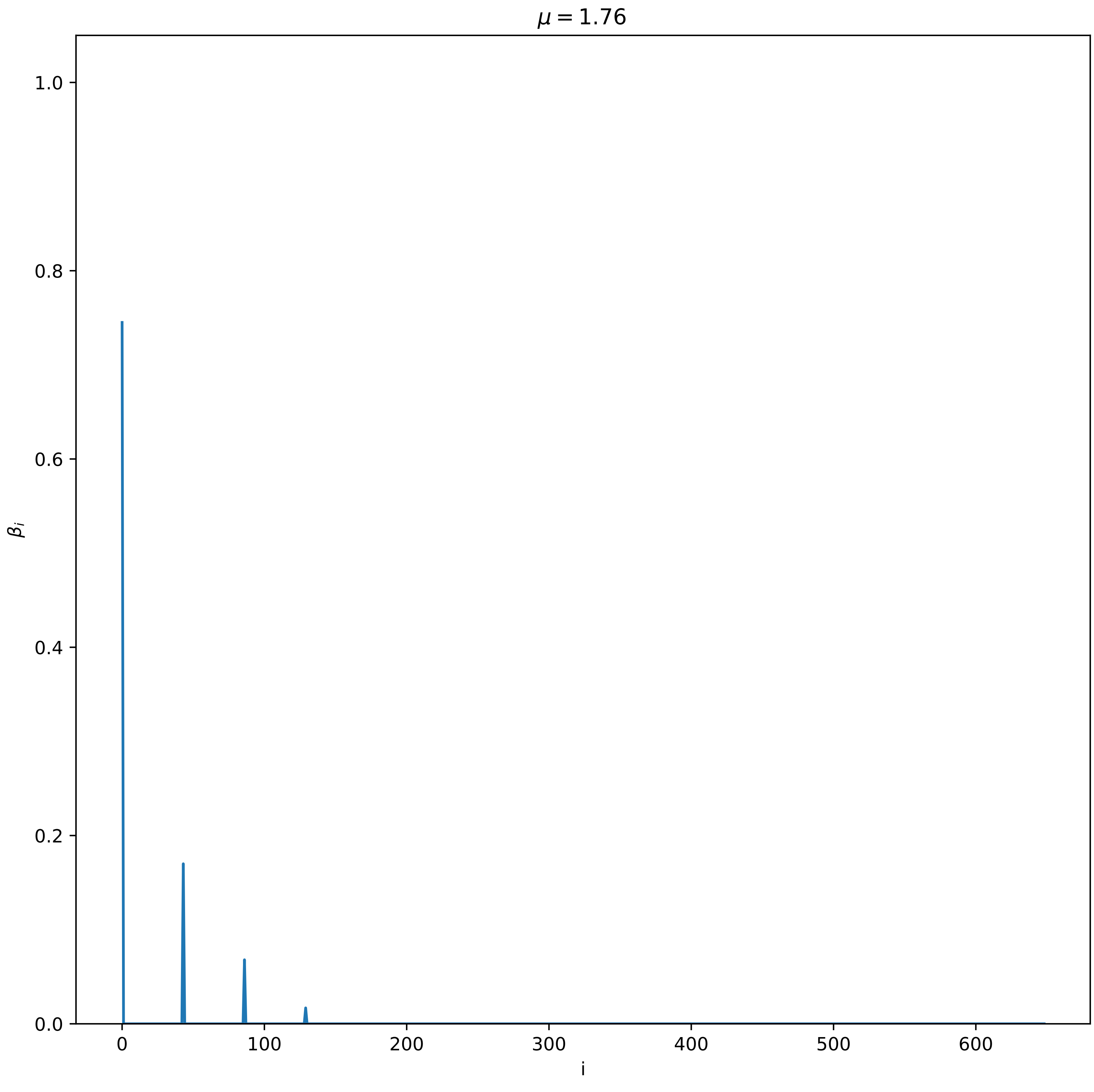
$\mu = 1.73$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

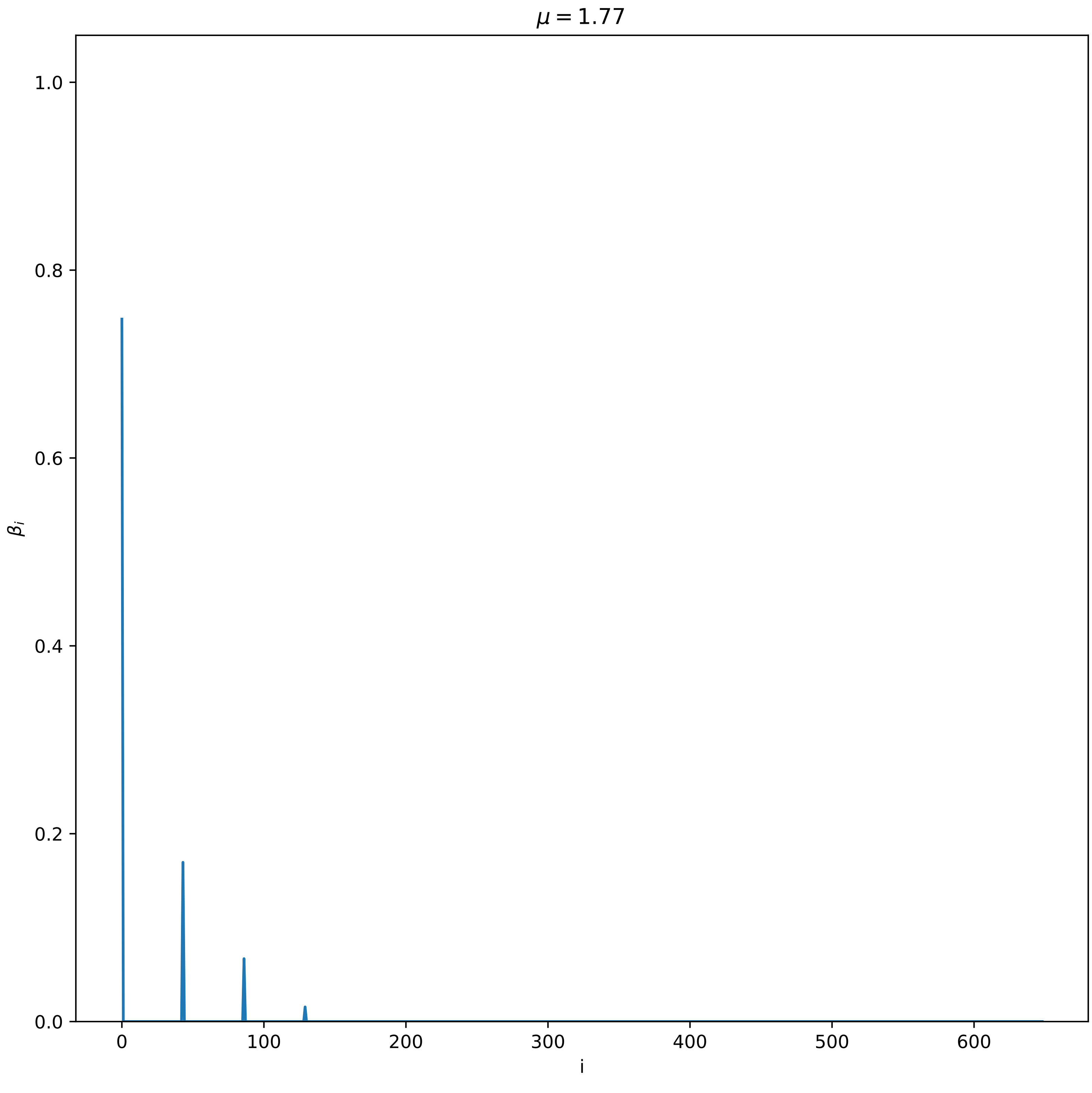


$\mu = 1.74$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

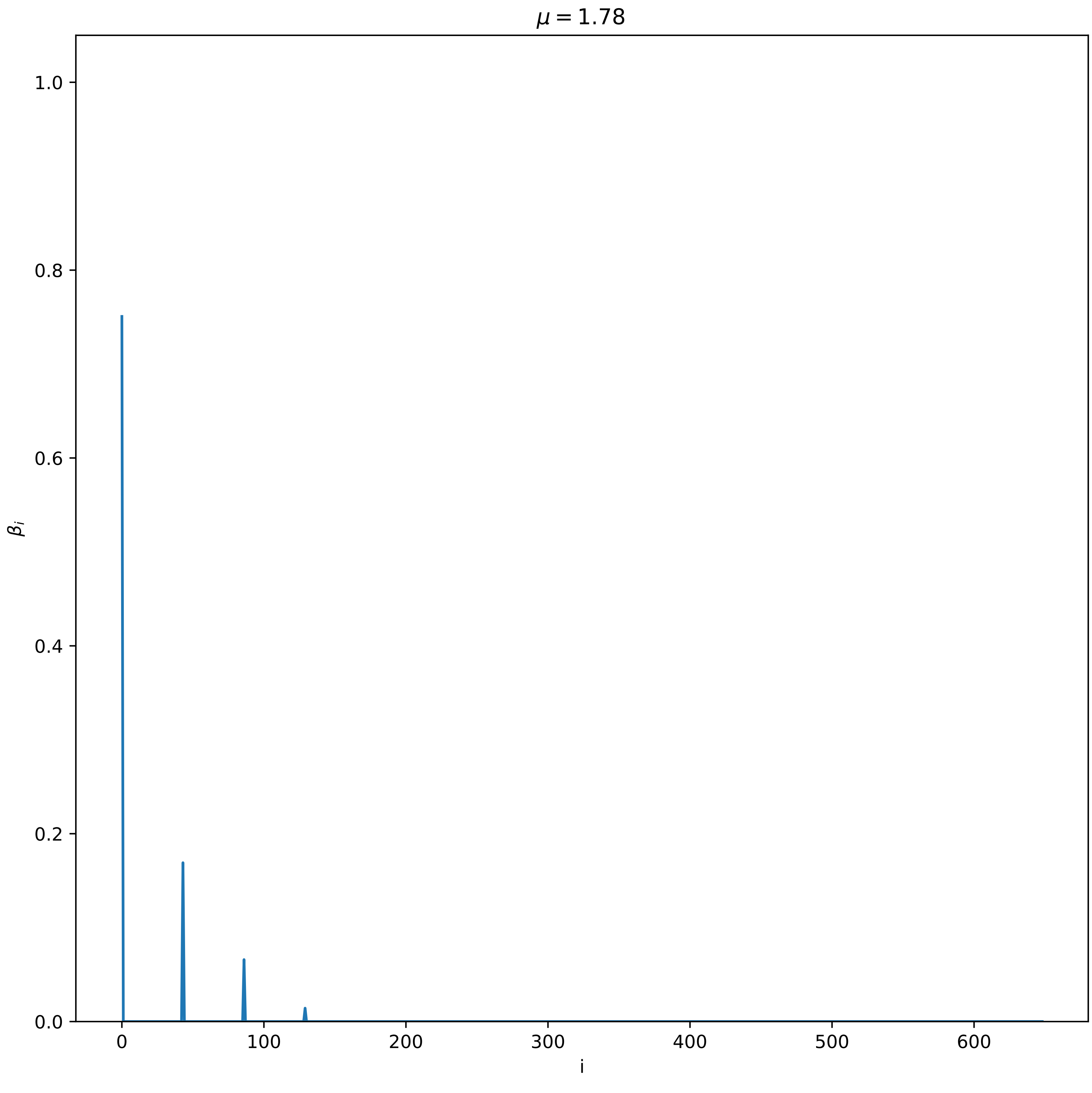


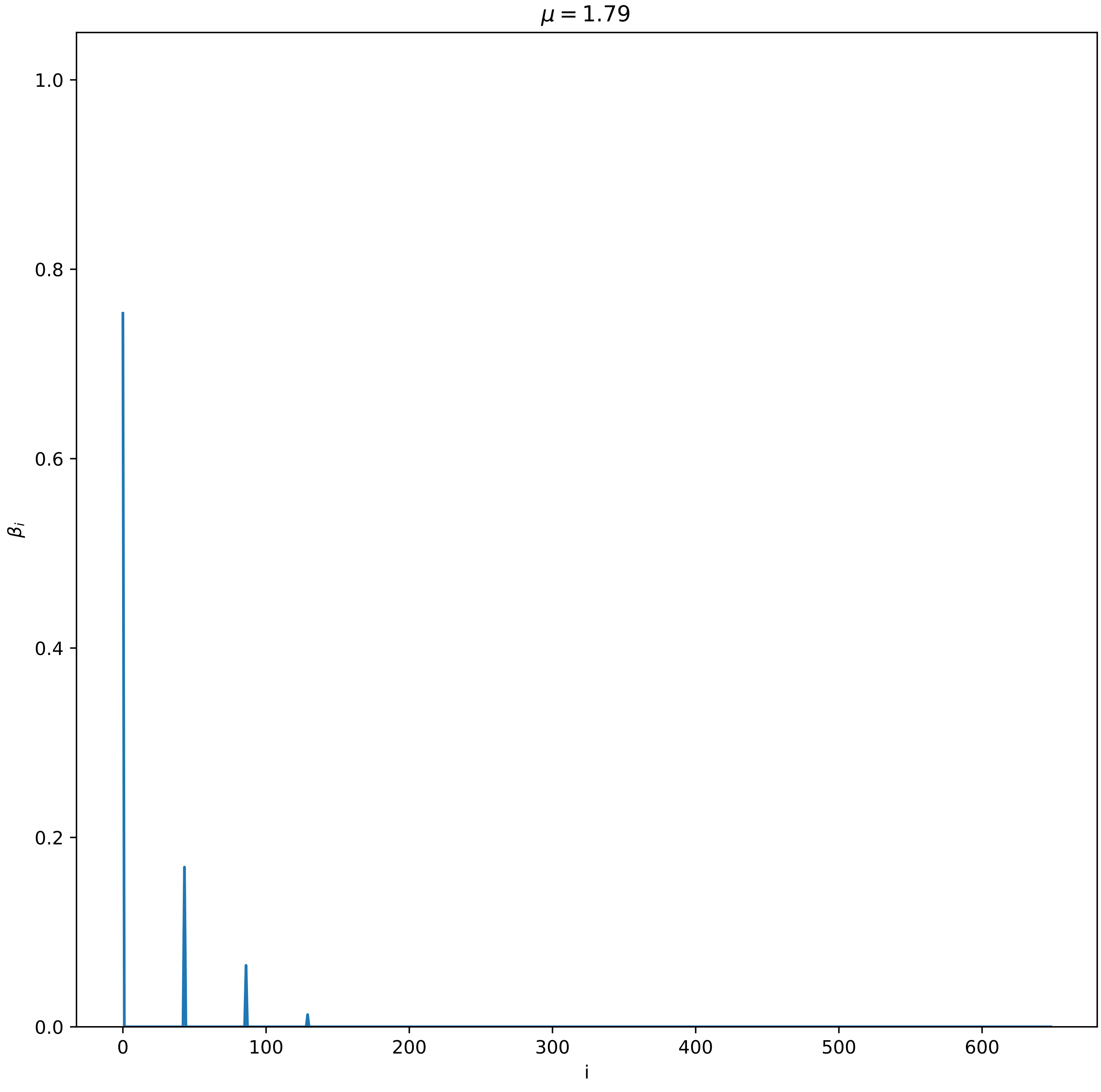
$\mu = 1.75$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.76$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

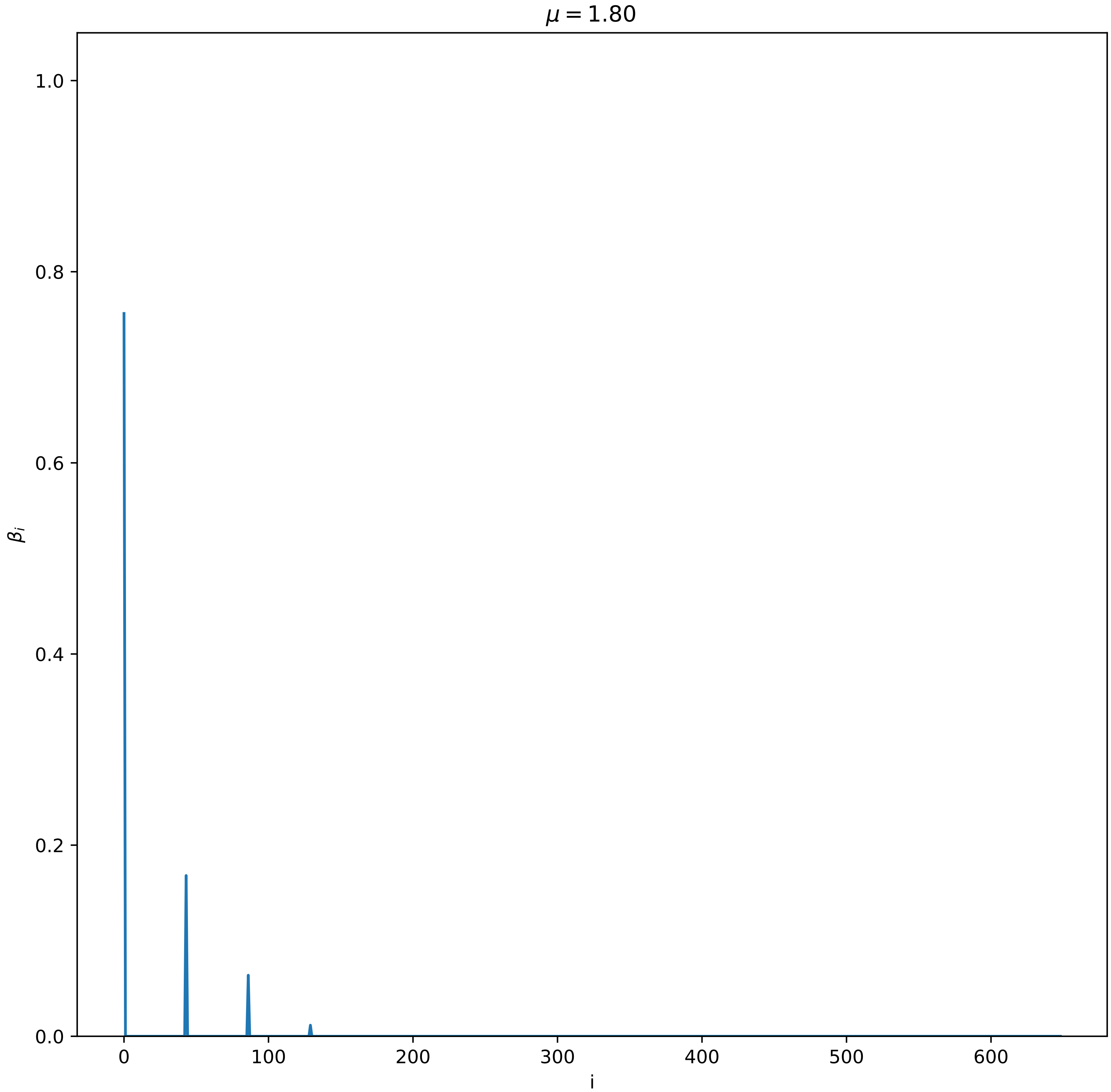


$\mu = 1.77$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

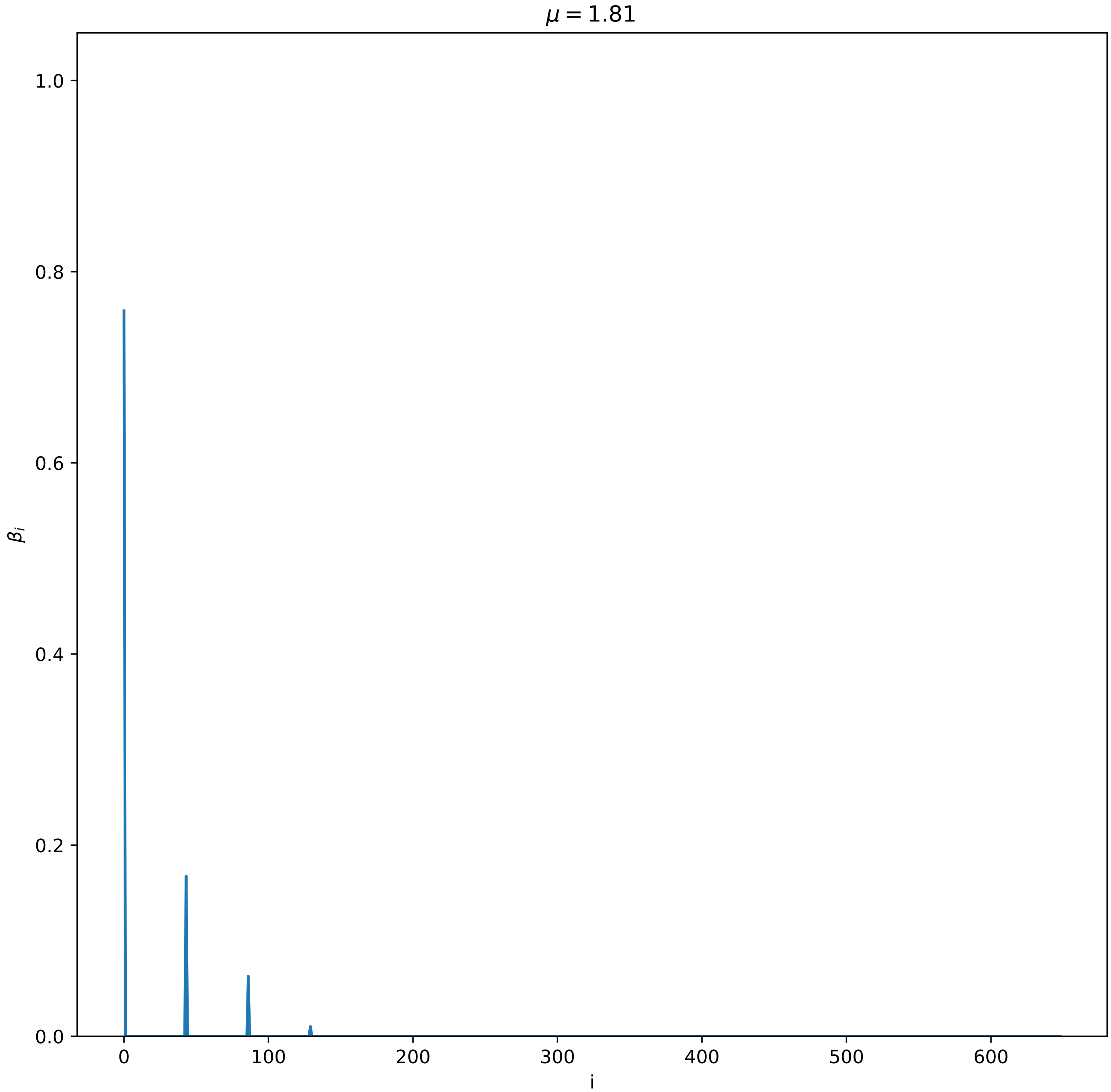

$\mu = 1.78$
β_i
i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600



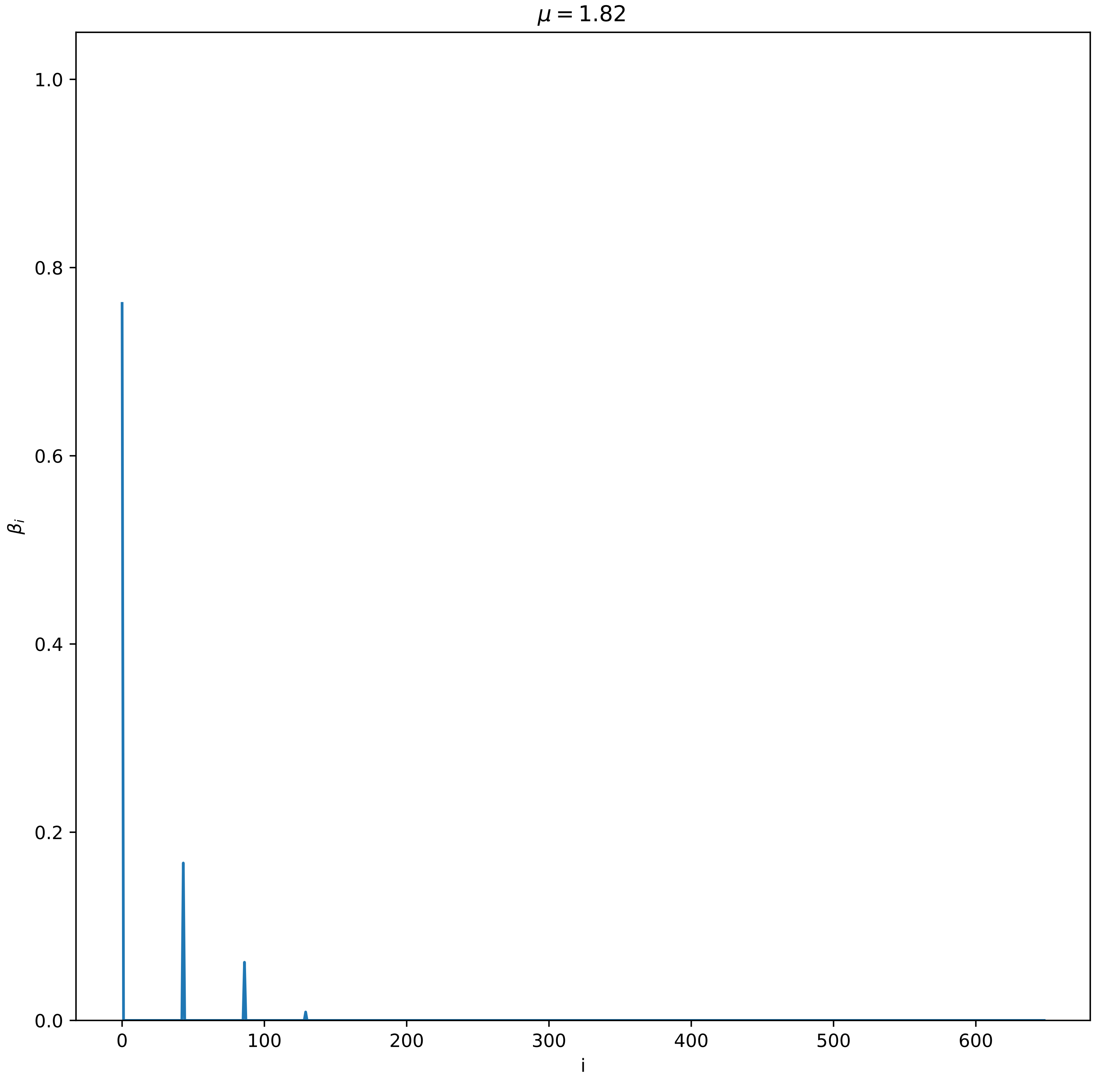
$\mu = 1.79$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



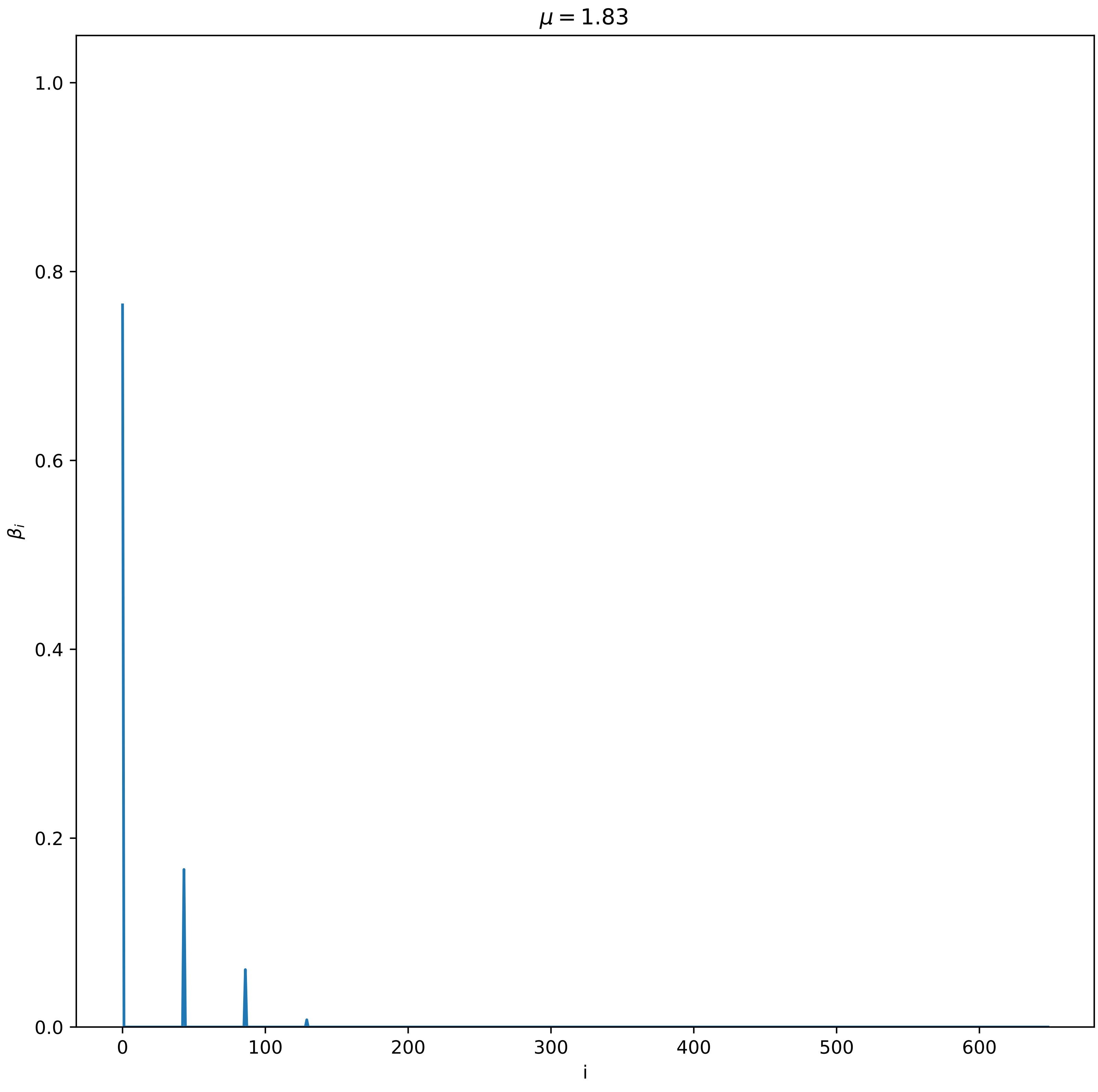
$\mu = 1.80$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



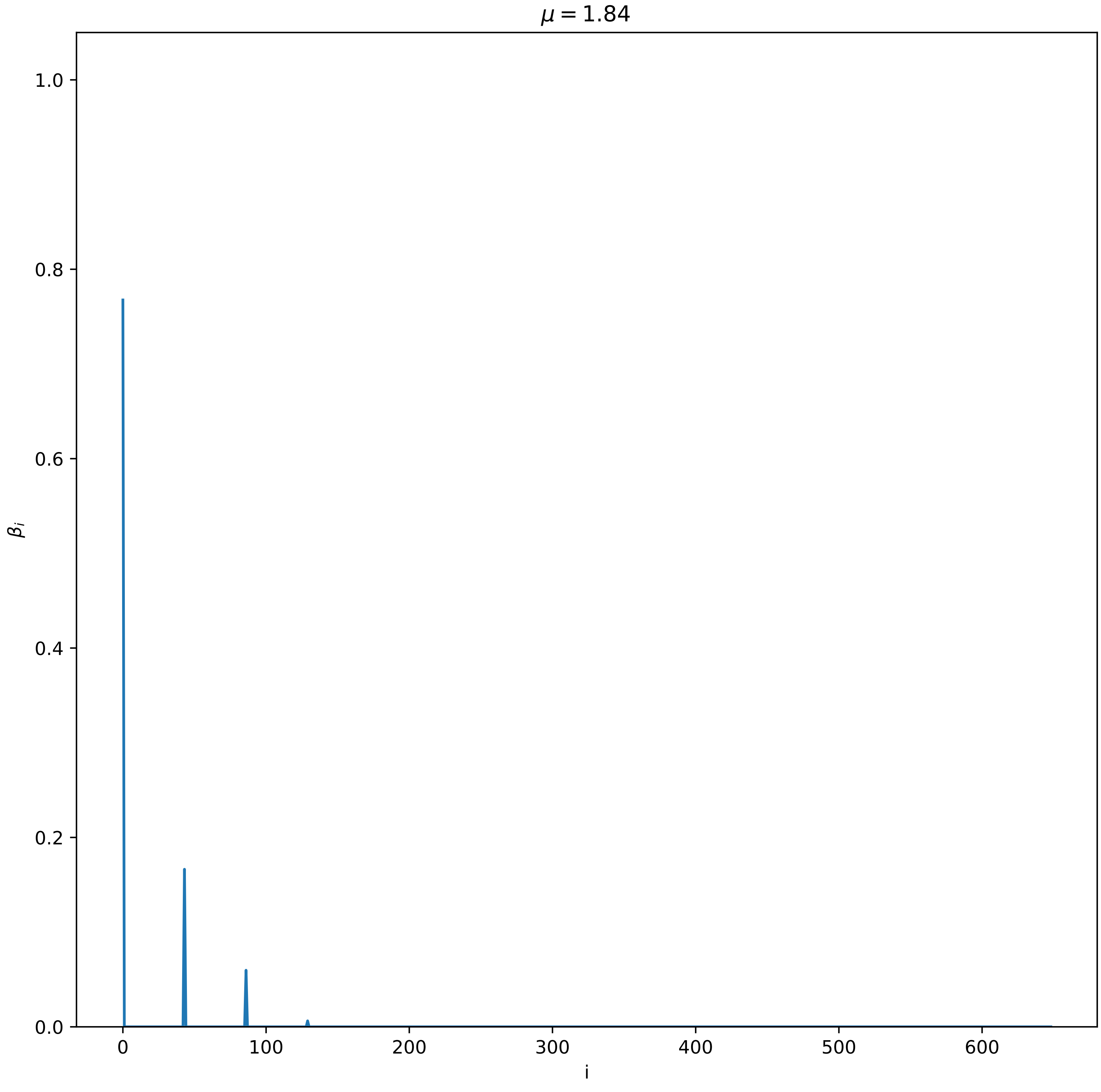
$\mu = 1.81$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



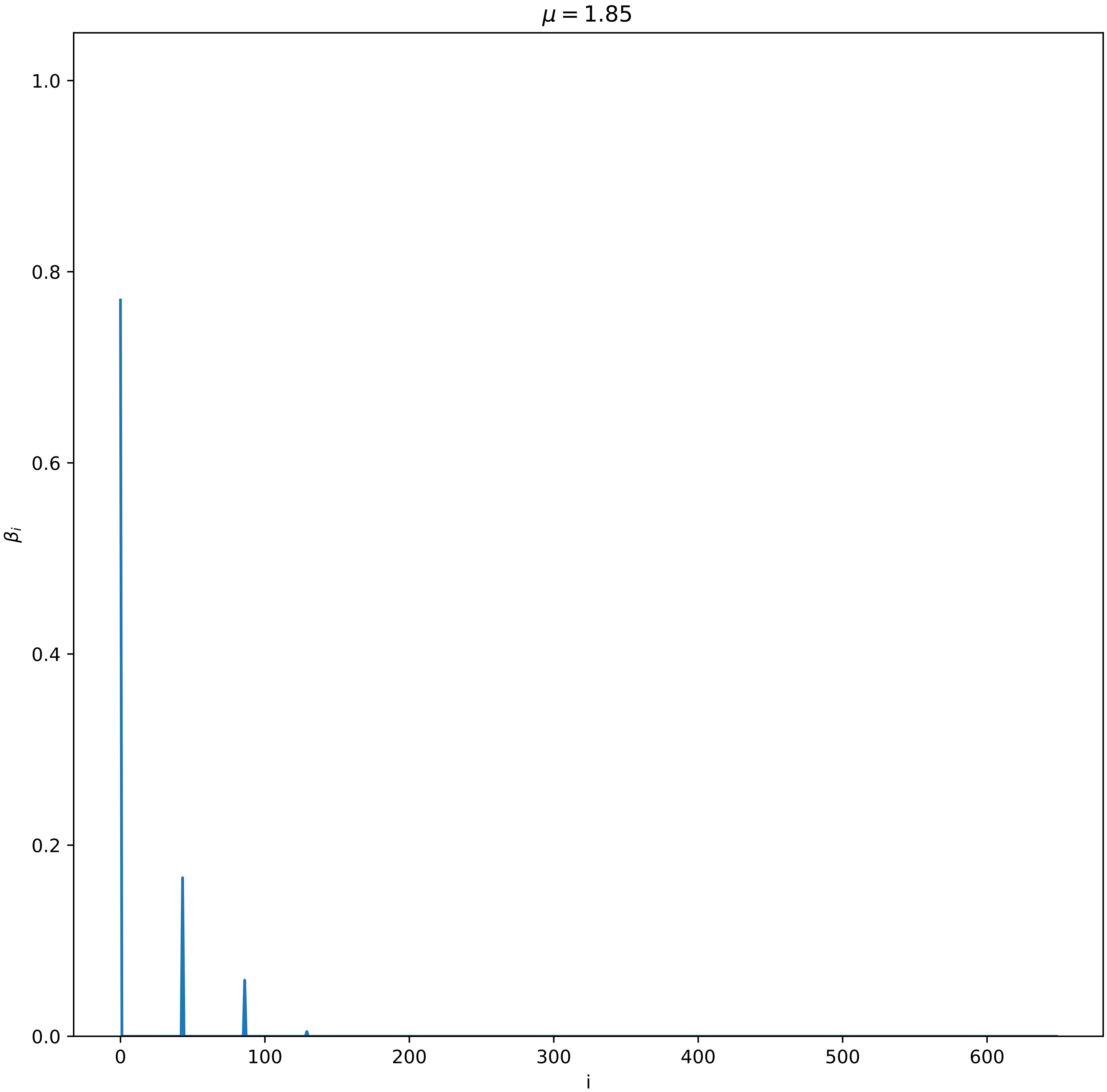
$\mu = 1.82$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



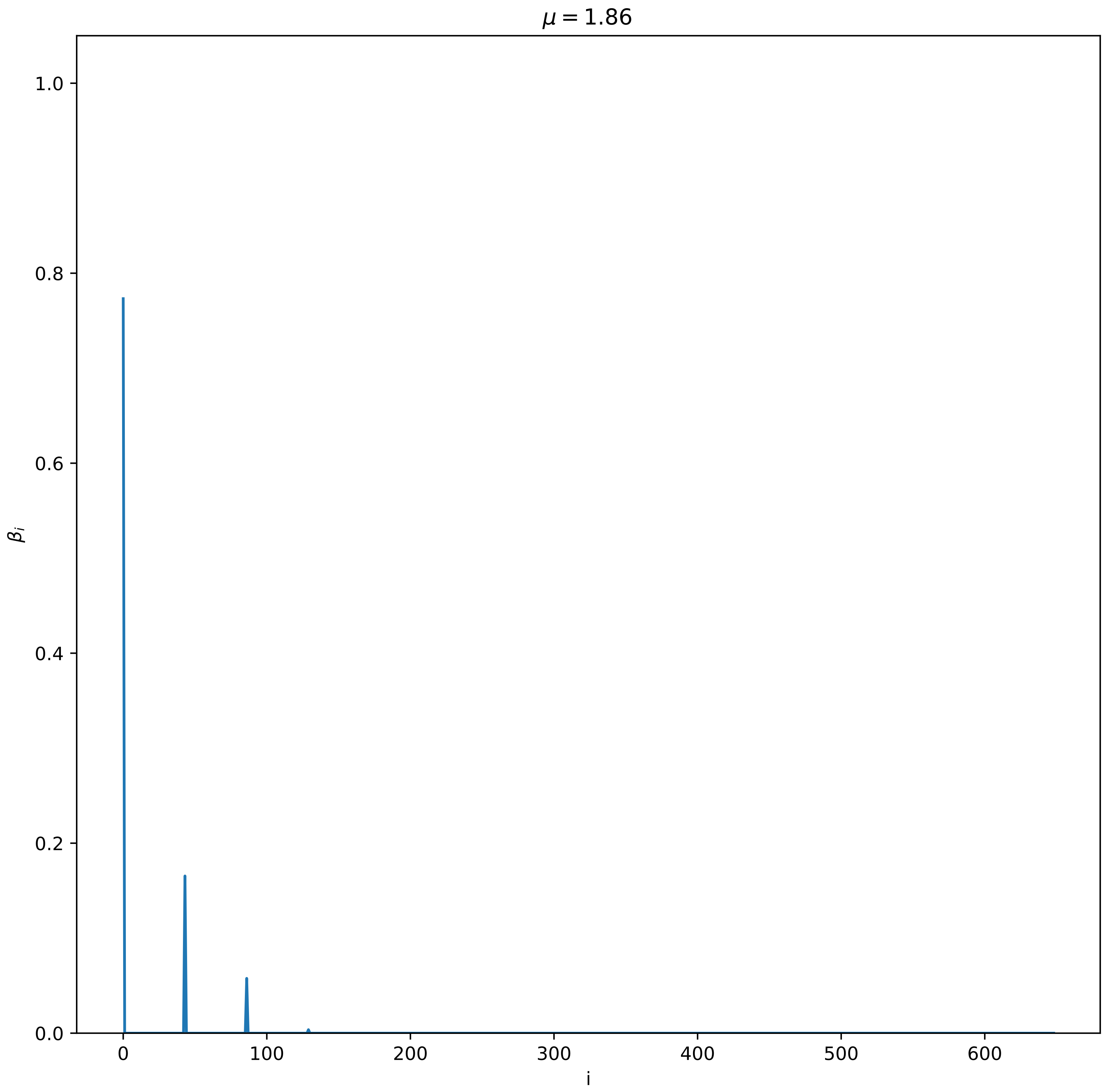
$\mu = 1.83$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



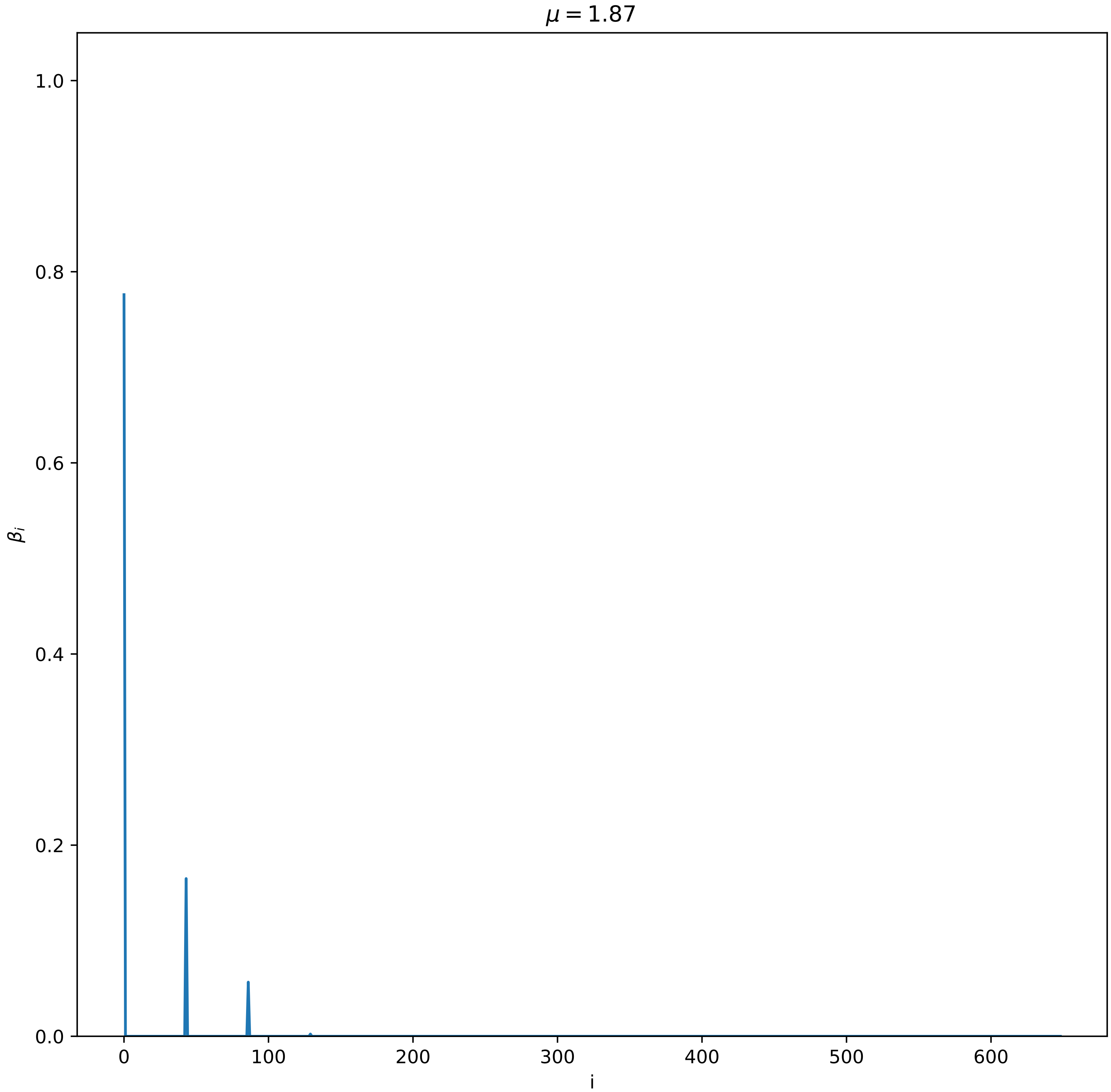
$\mu = 1.84$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



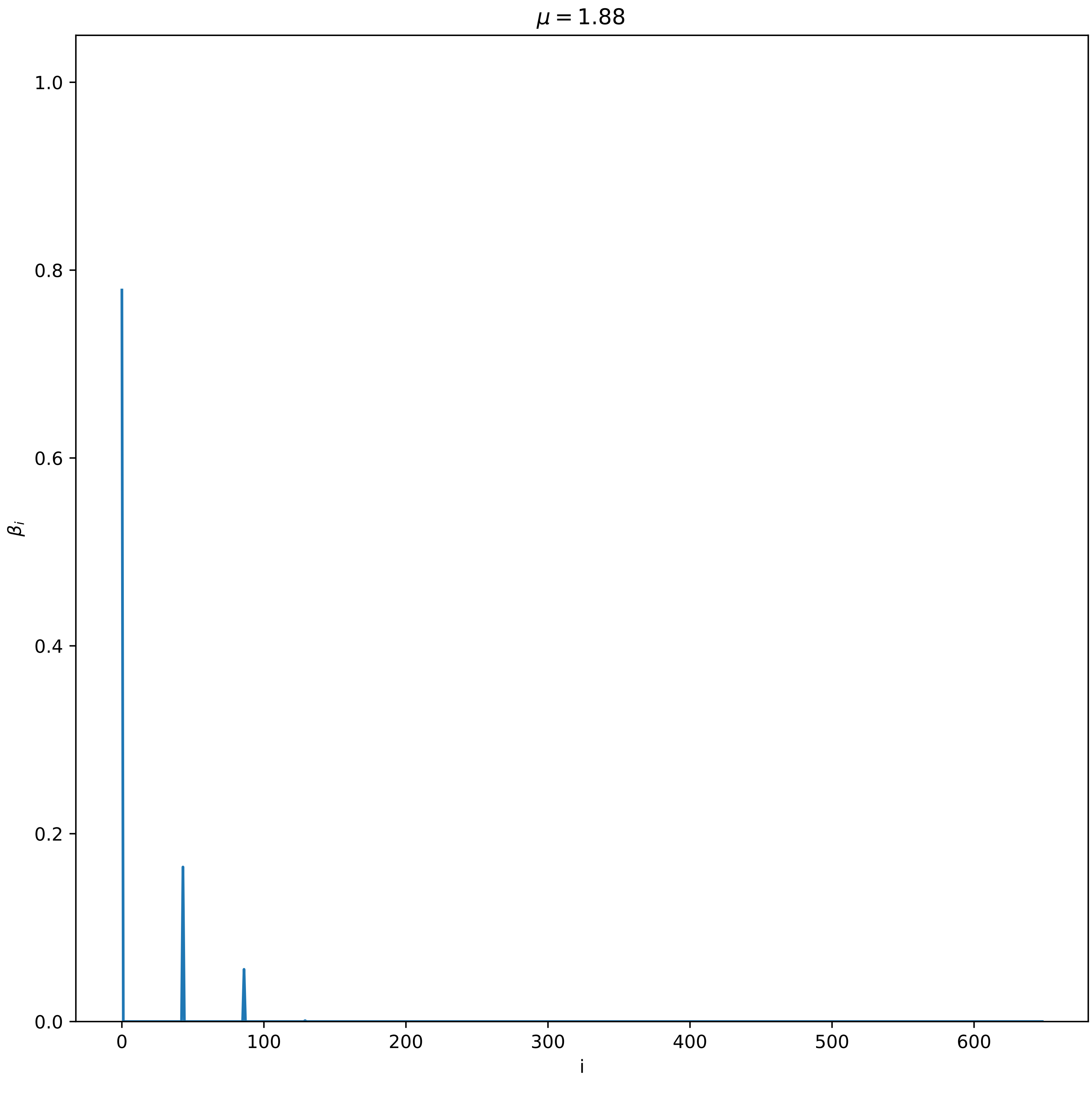
$\mu = 1.85$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

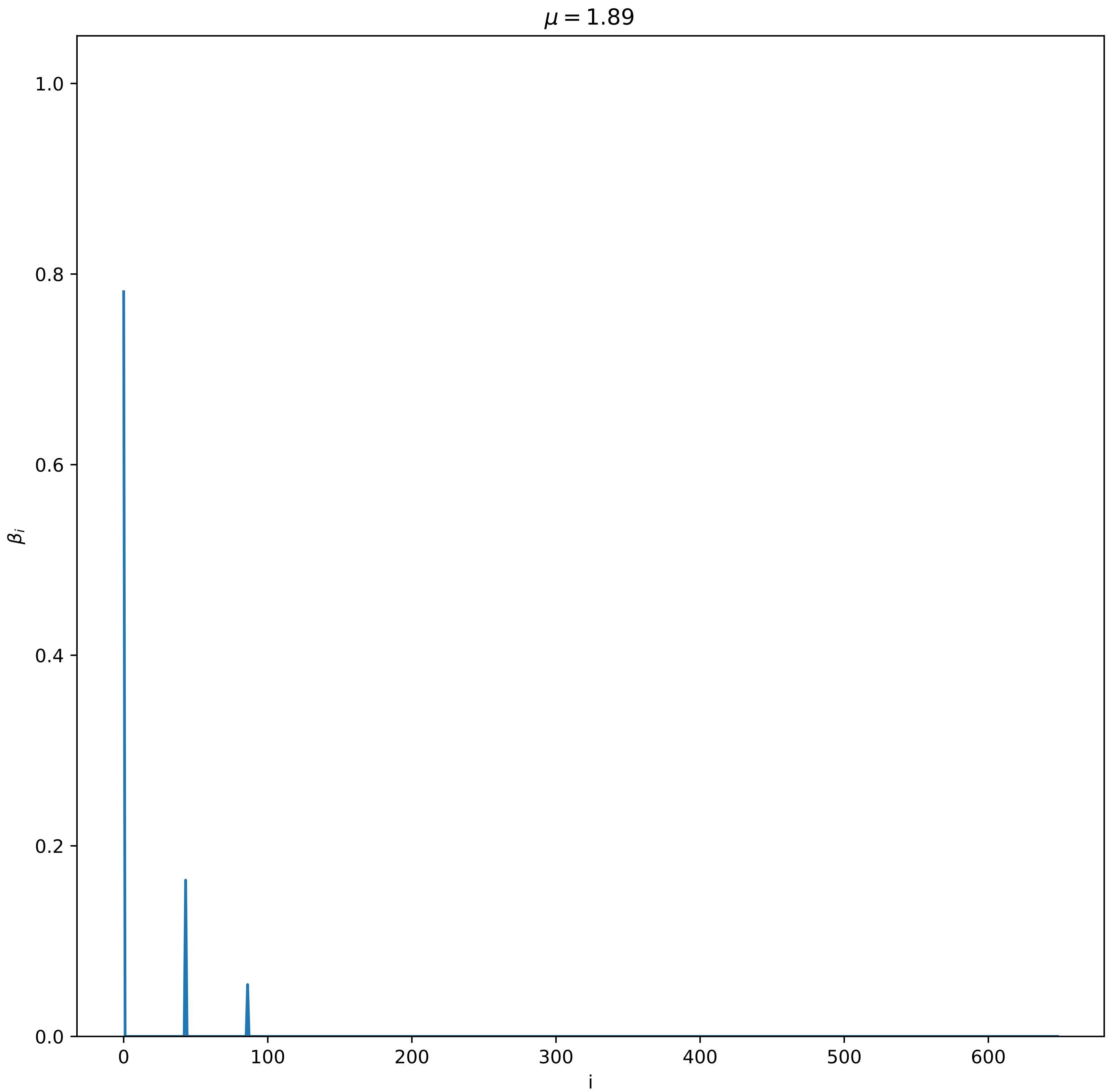


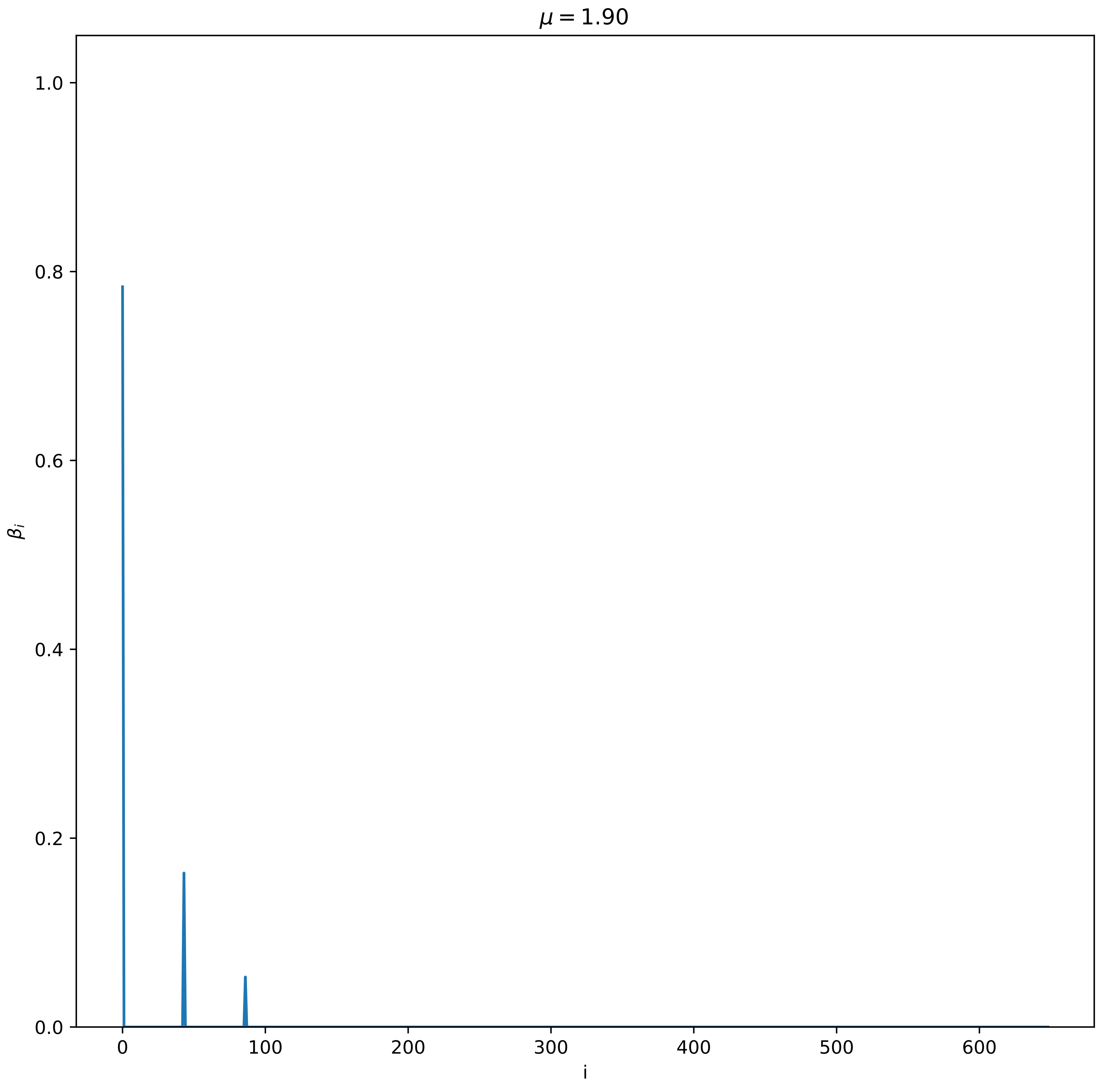
$\mu = 1.86$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



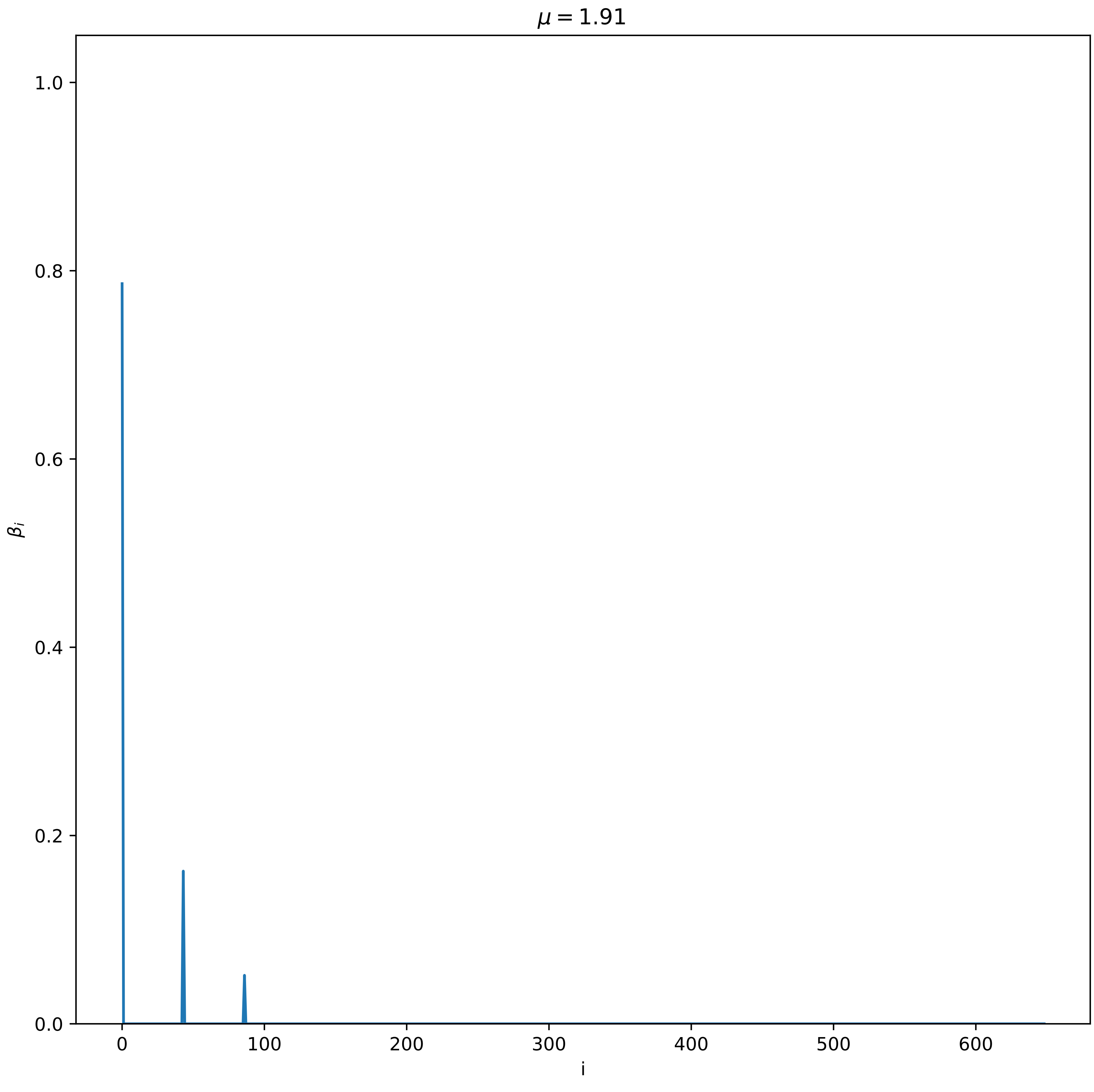
$\mu = 1.87$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 1.88$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

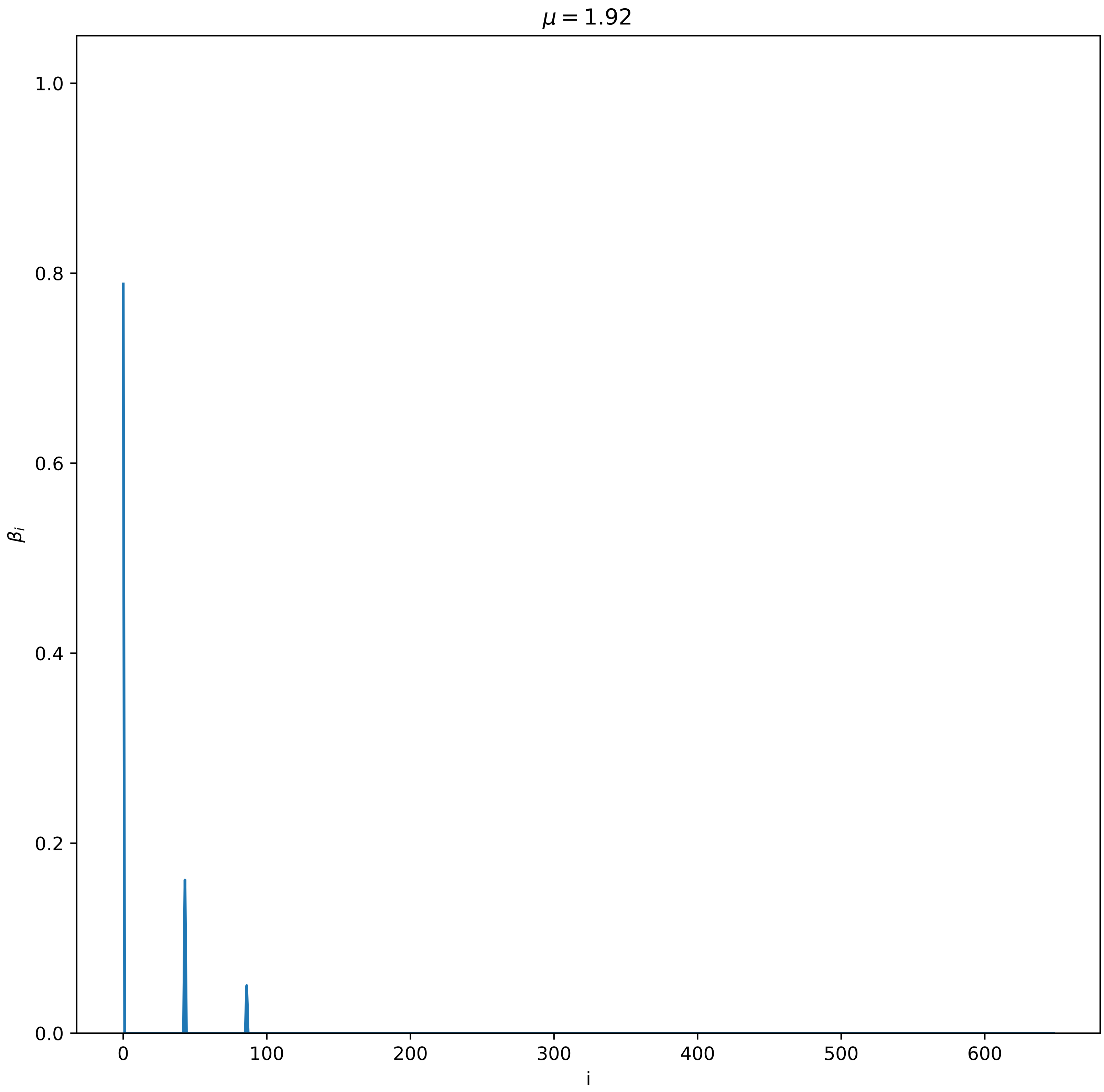

$\mu = 1.89$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



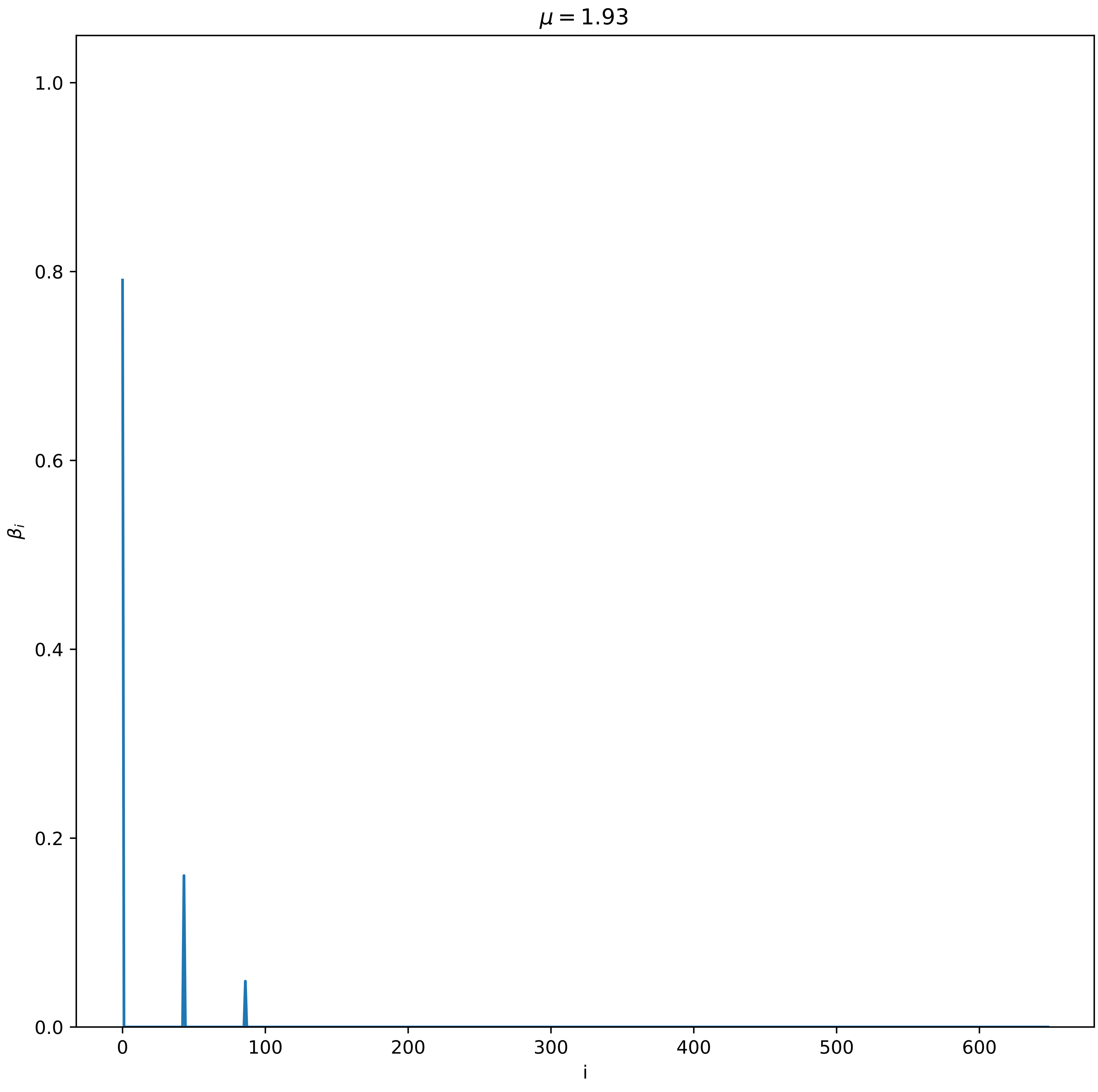
$\mu = 1.90$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



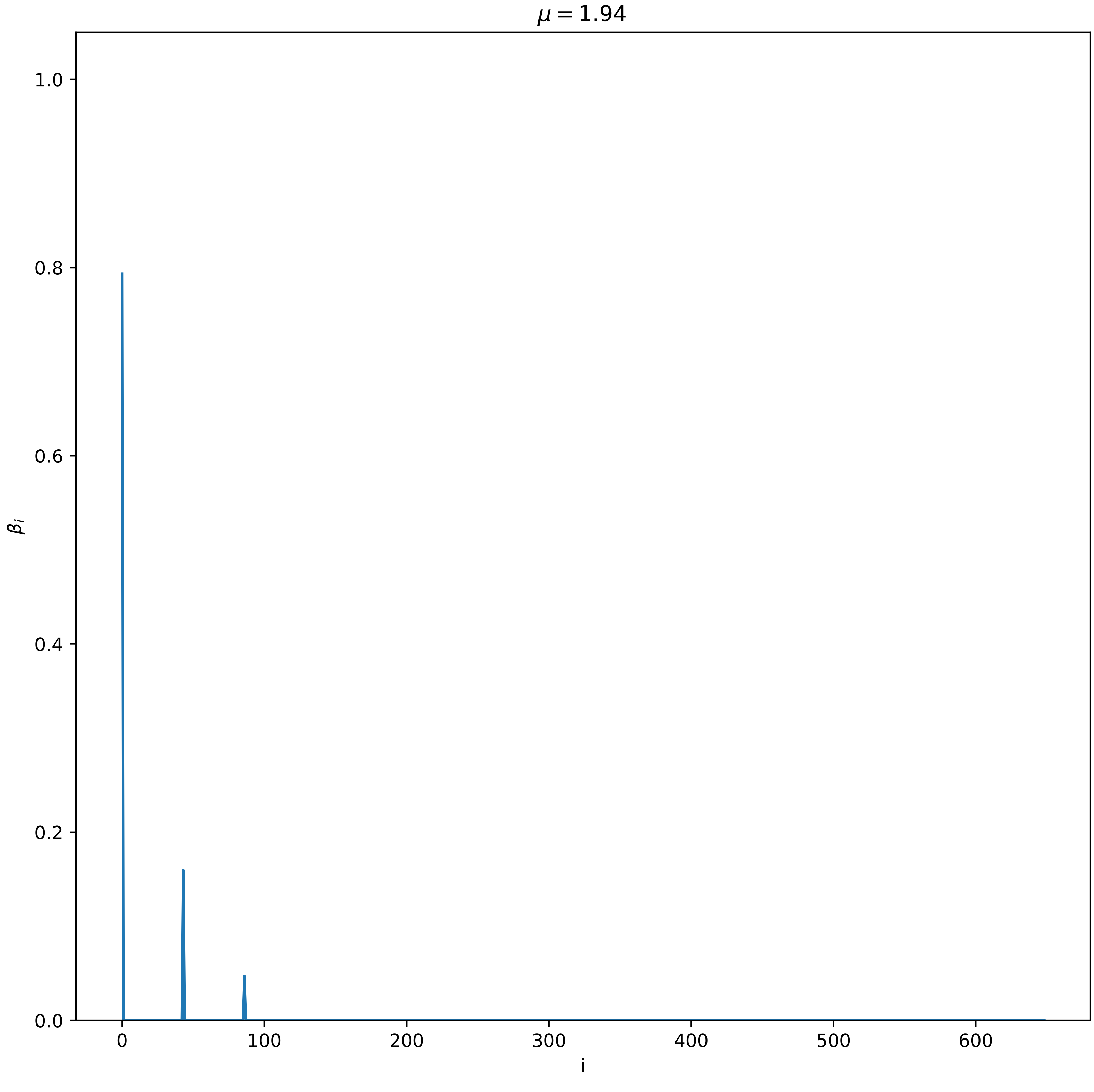
$\mu = 1.91$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



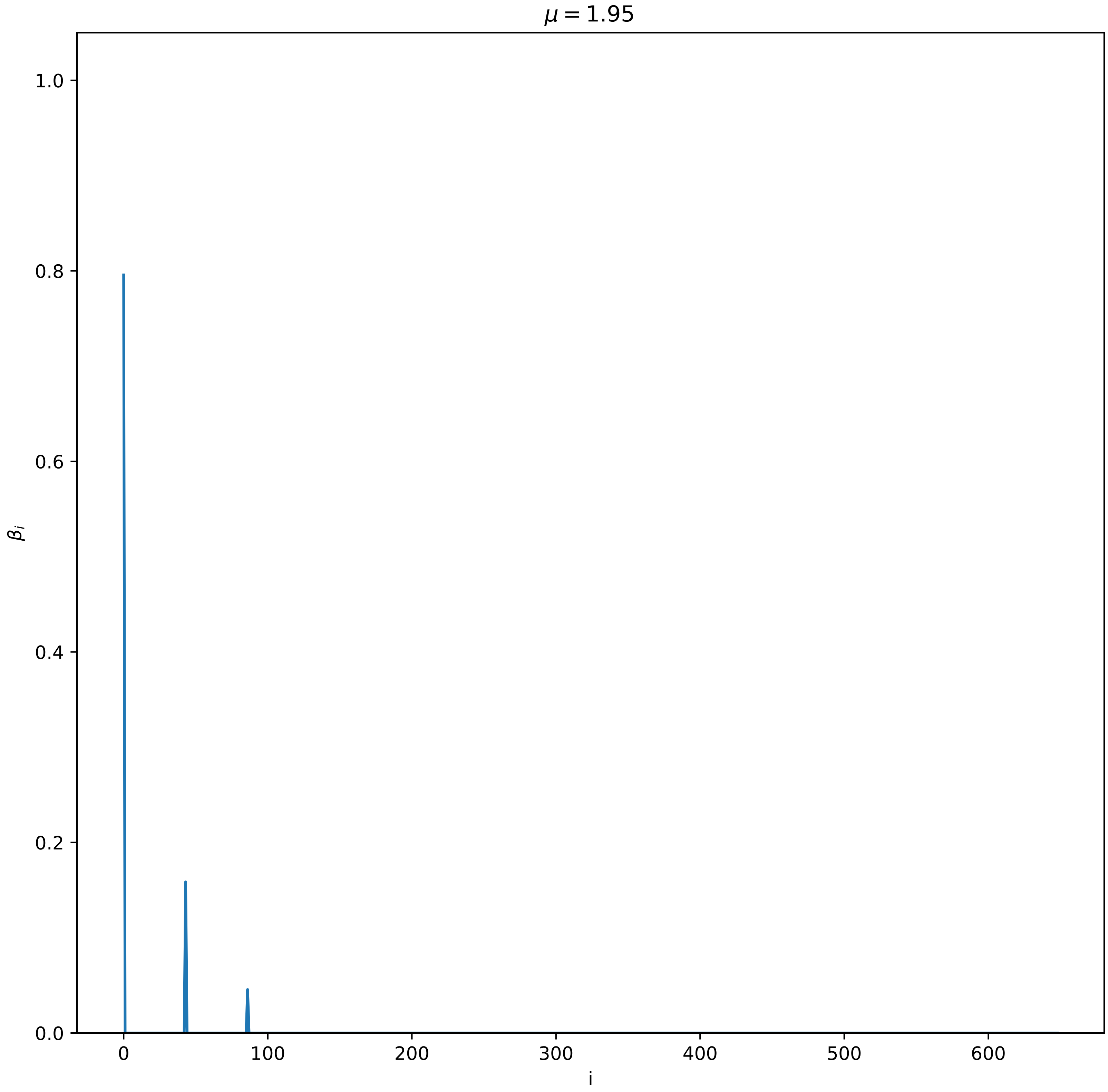
$\mu = 1.92$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



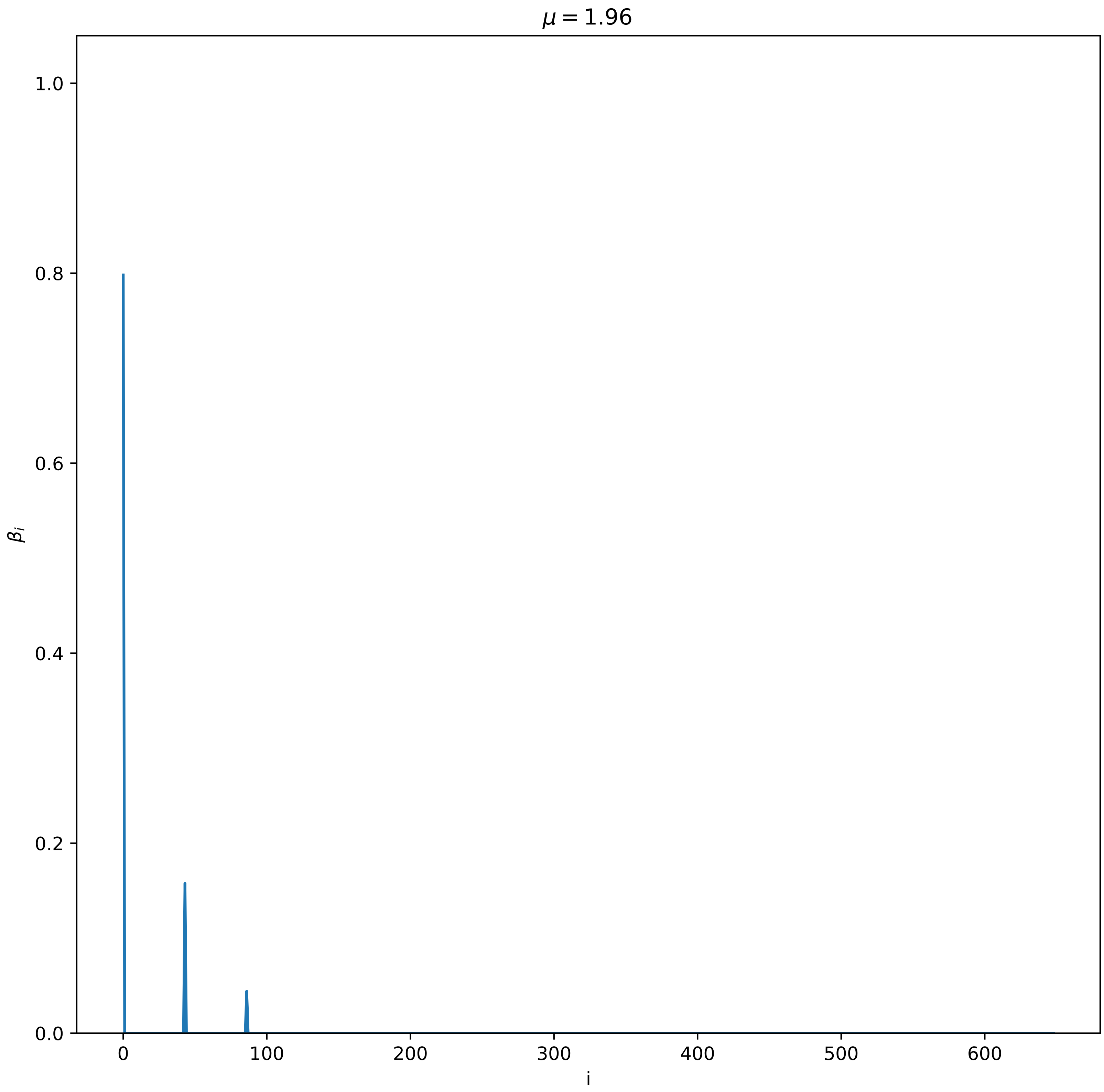
$\mu = 1.93$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

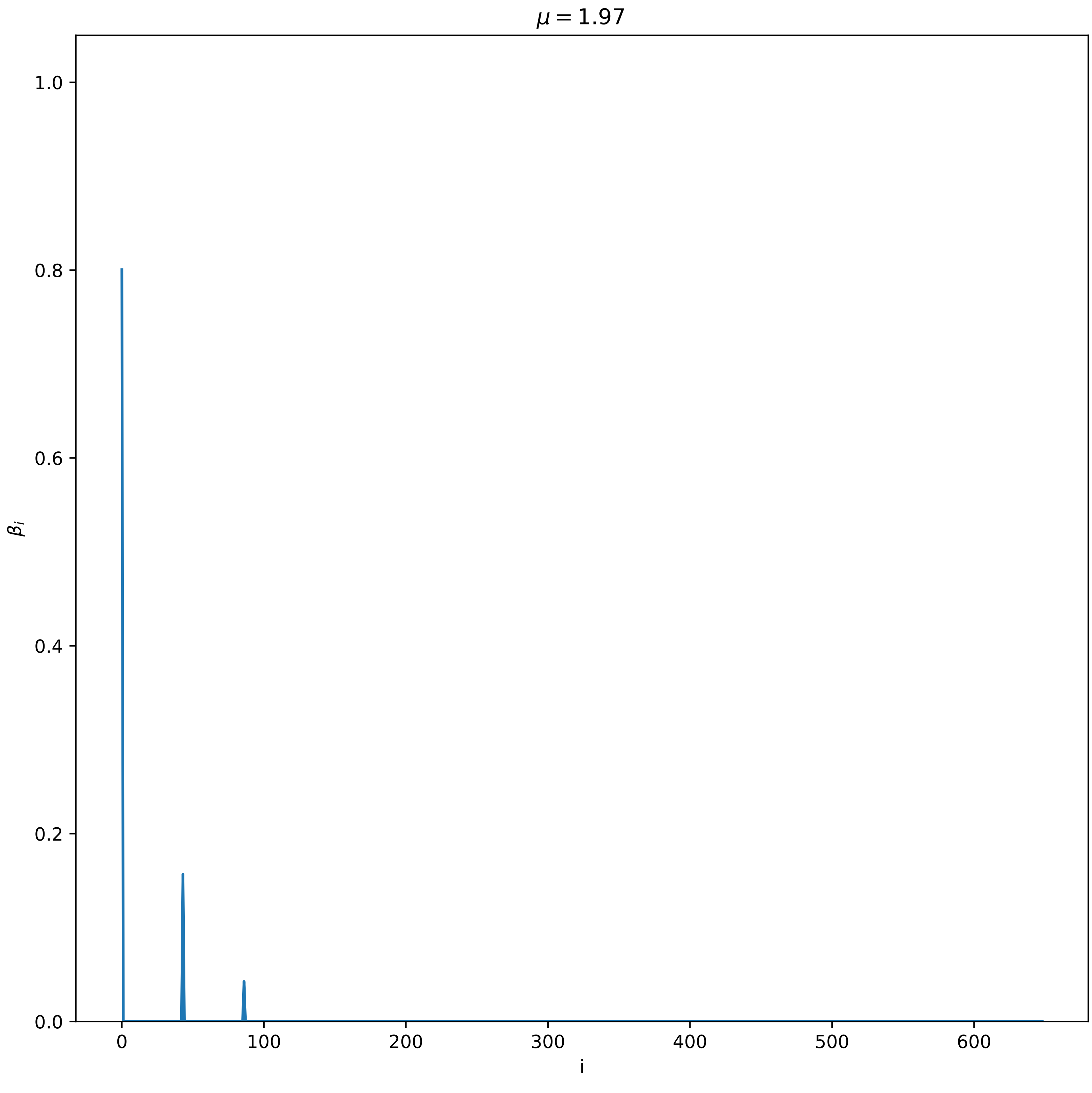


$\mu = 1.94$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

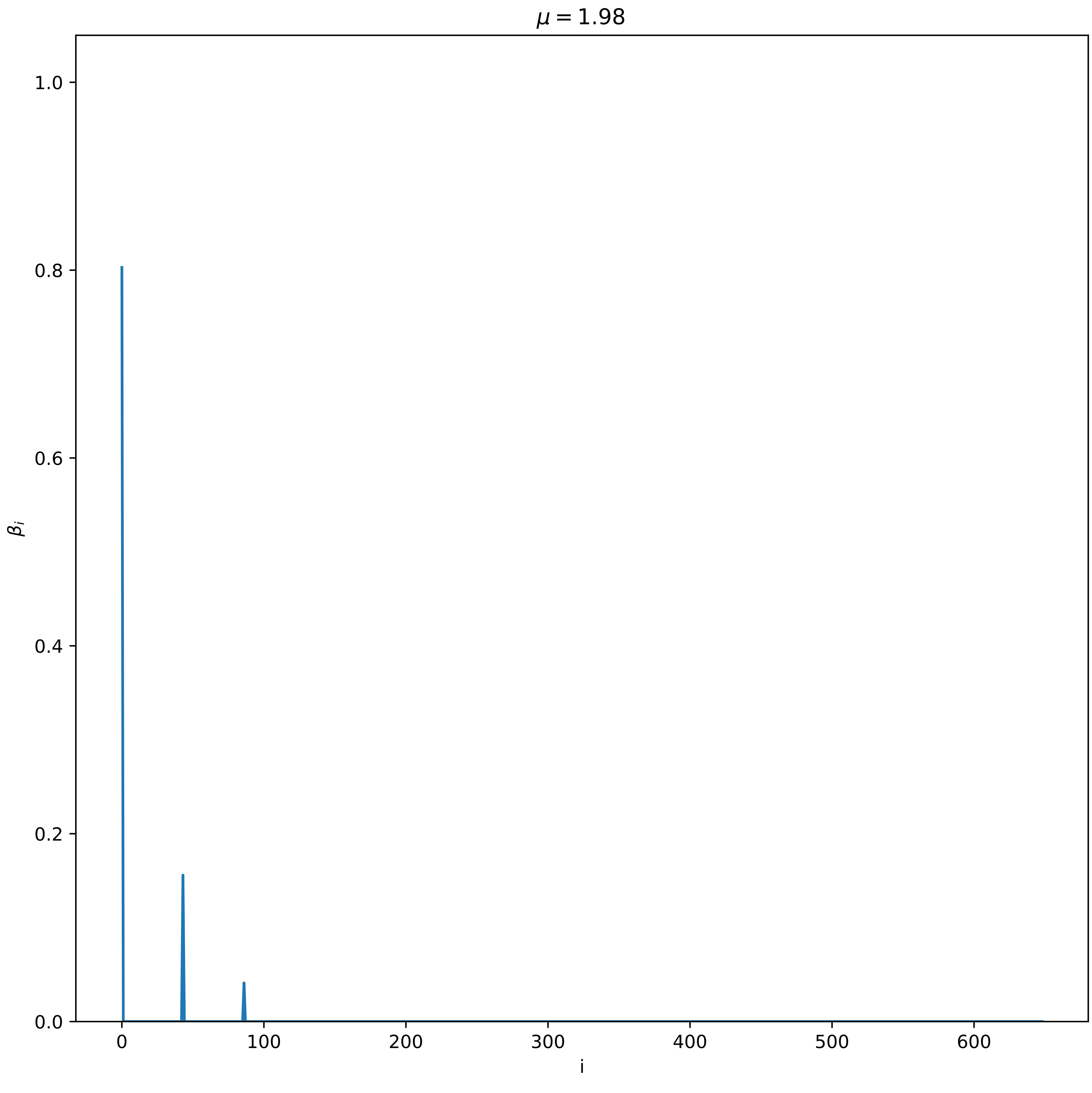


$\mu = 1.95$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

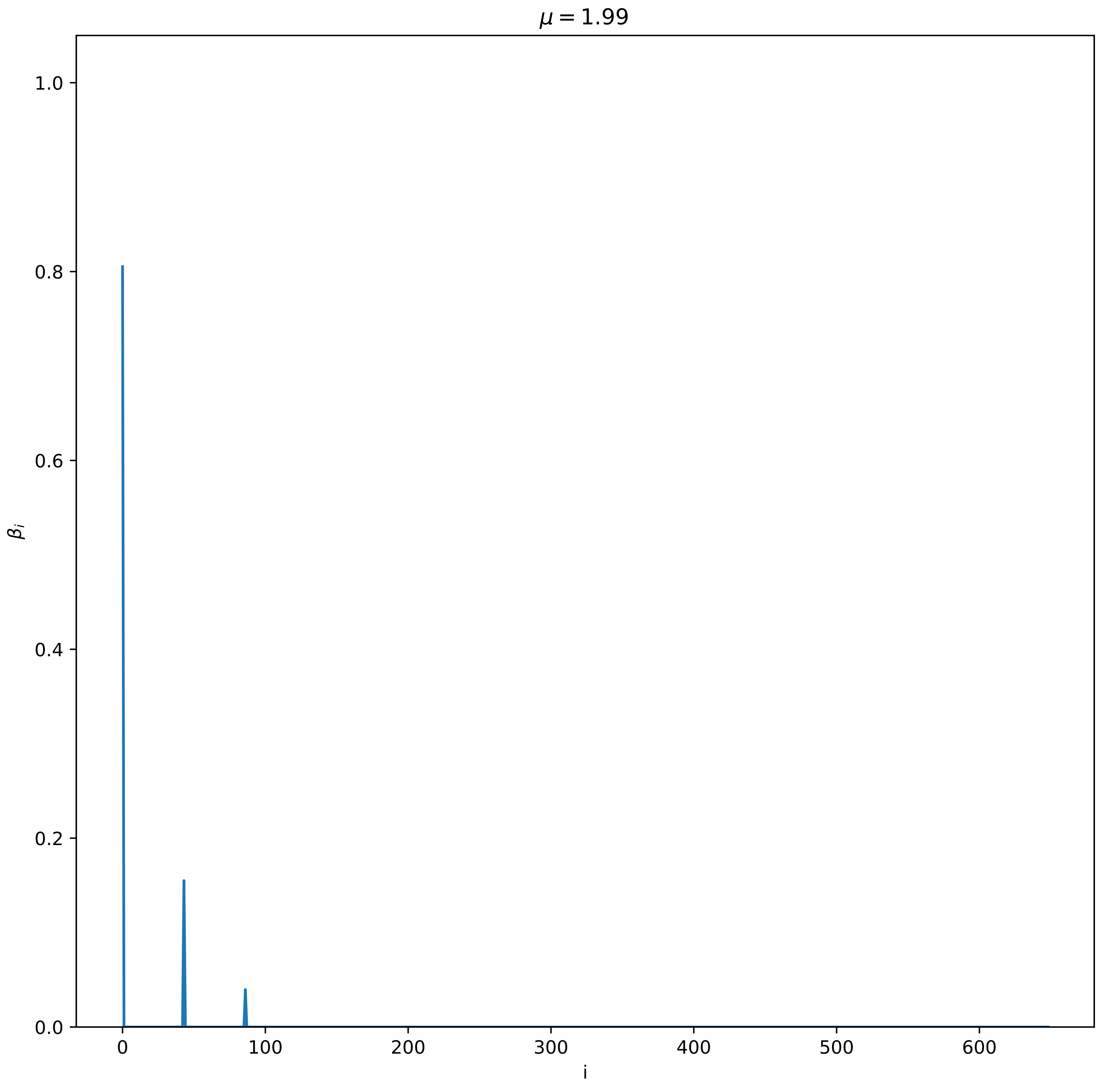

$\mu = 1.96$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



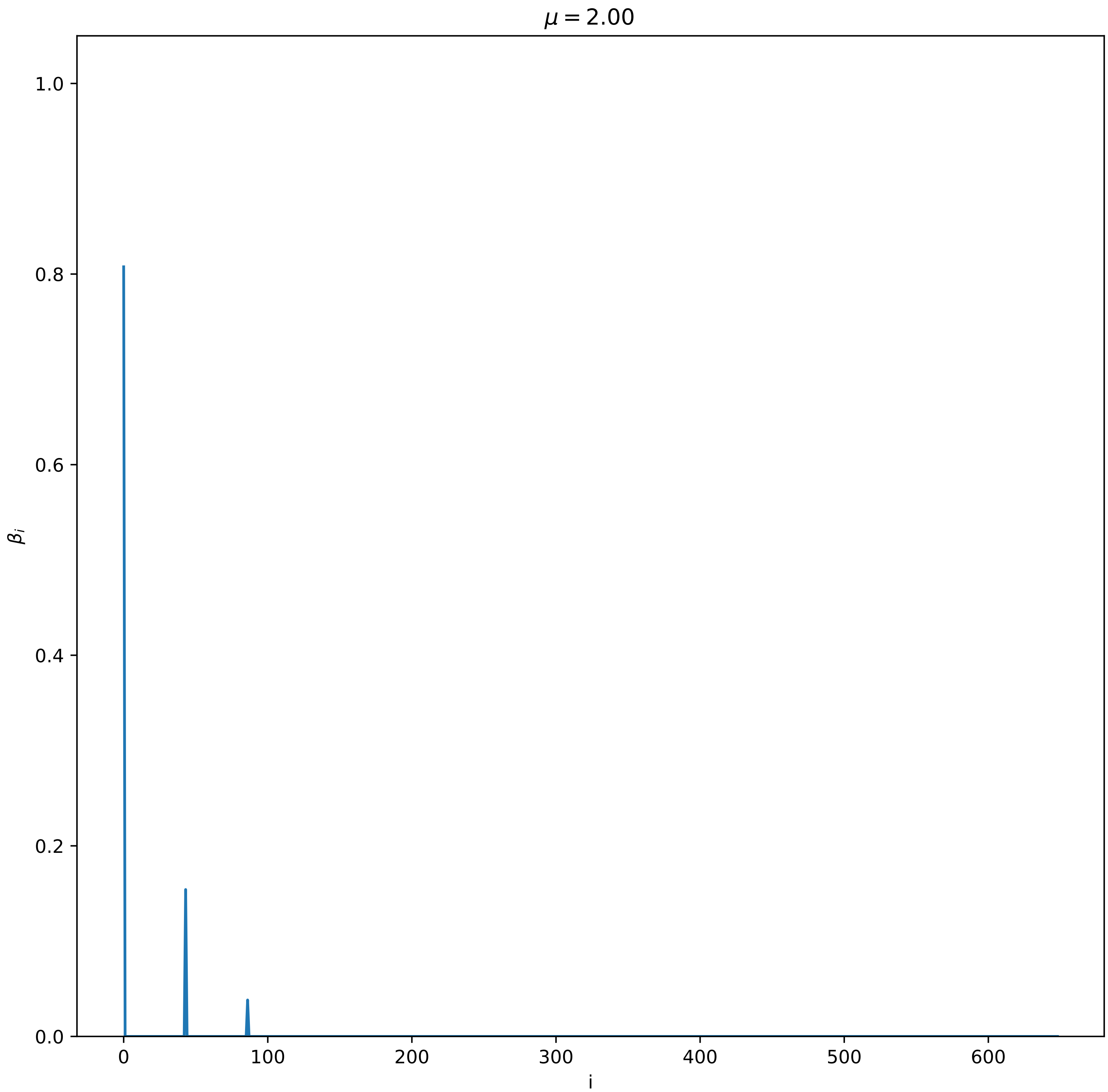
$\mu = 1.97$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



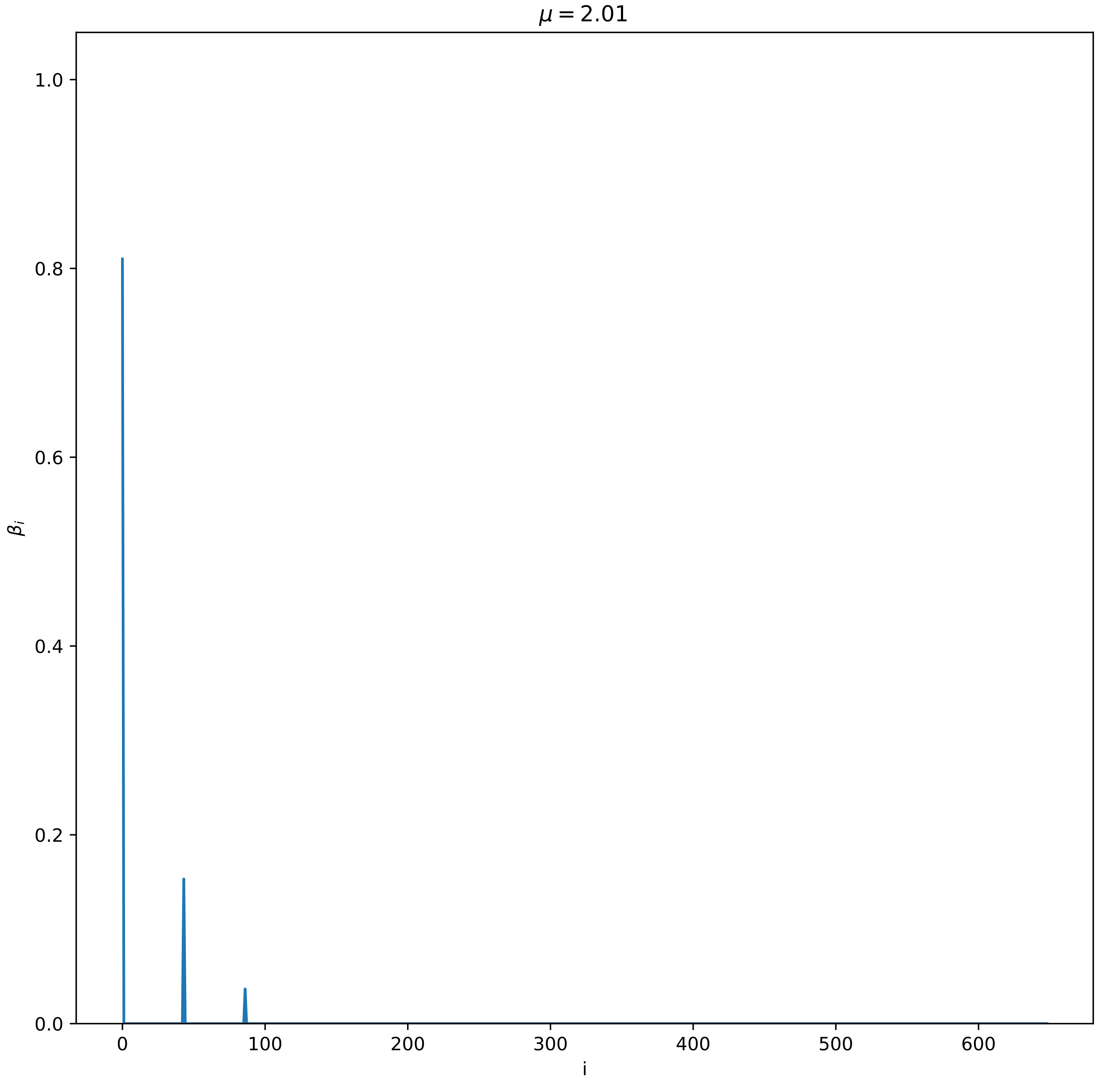
$\mu = 1.98$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



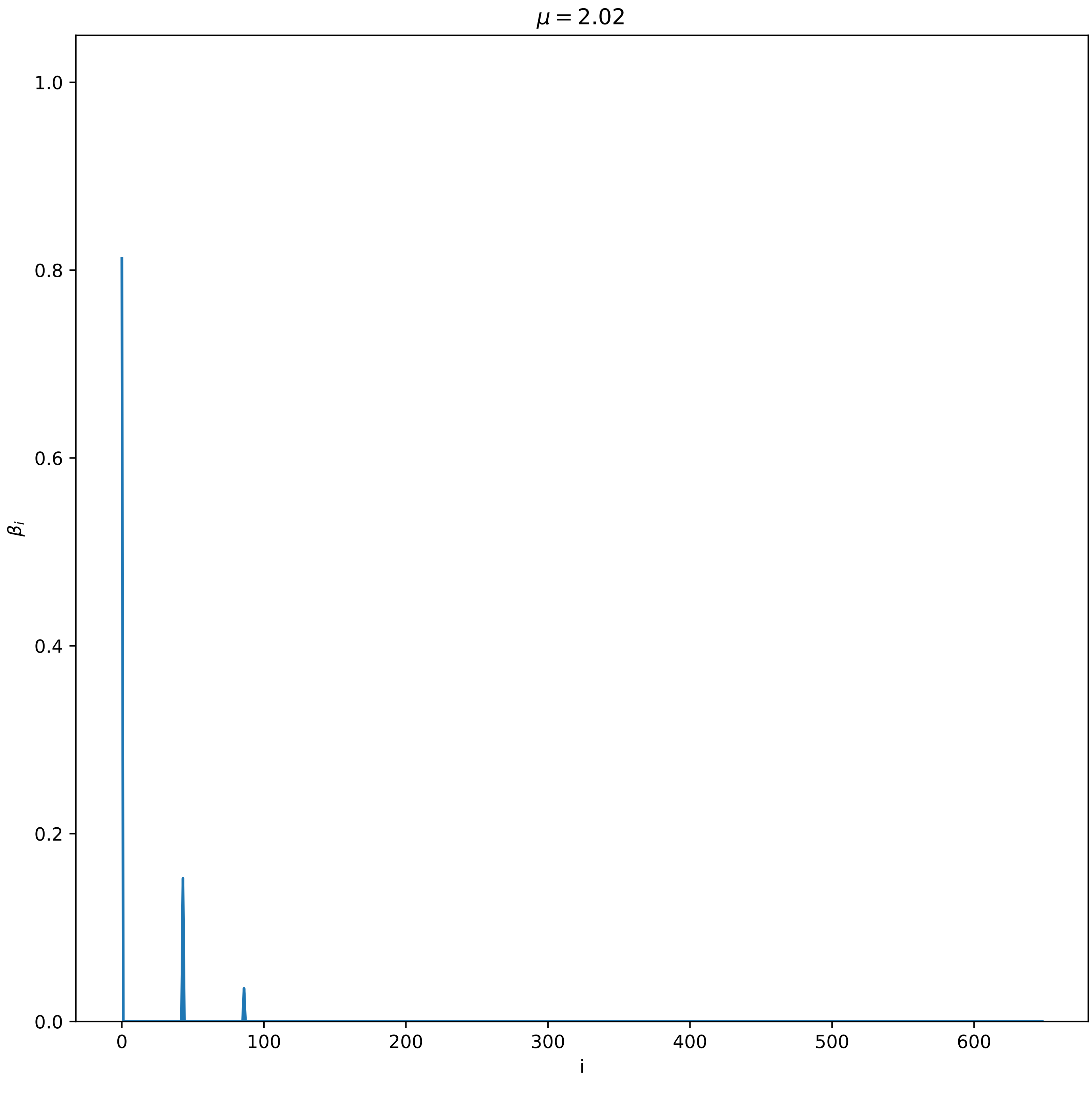
$\mu = 1.99$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



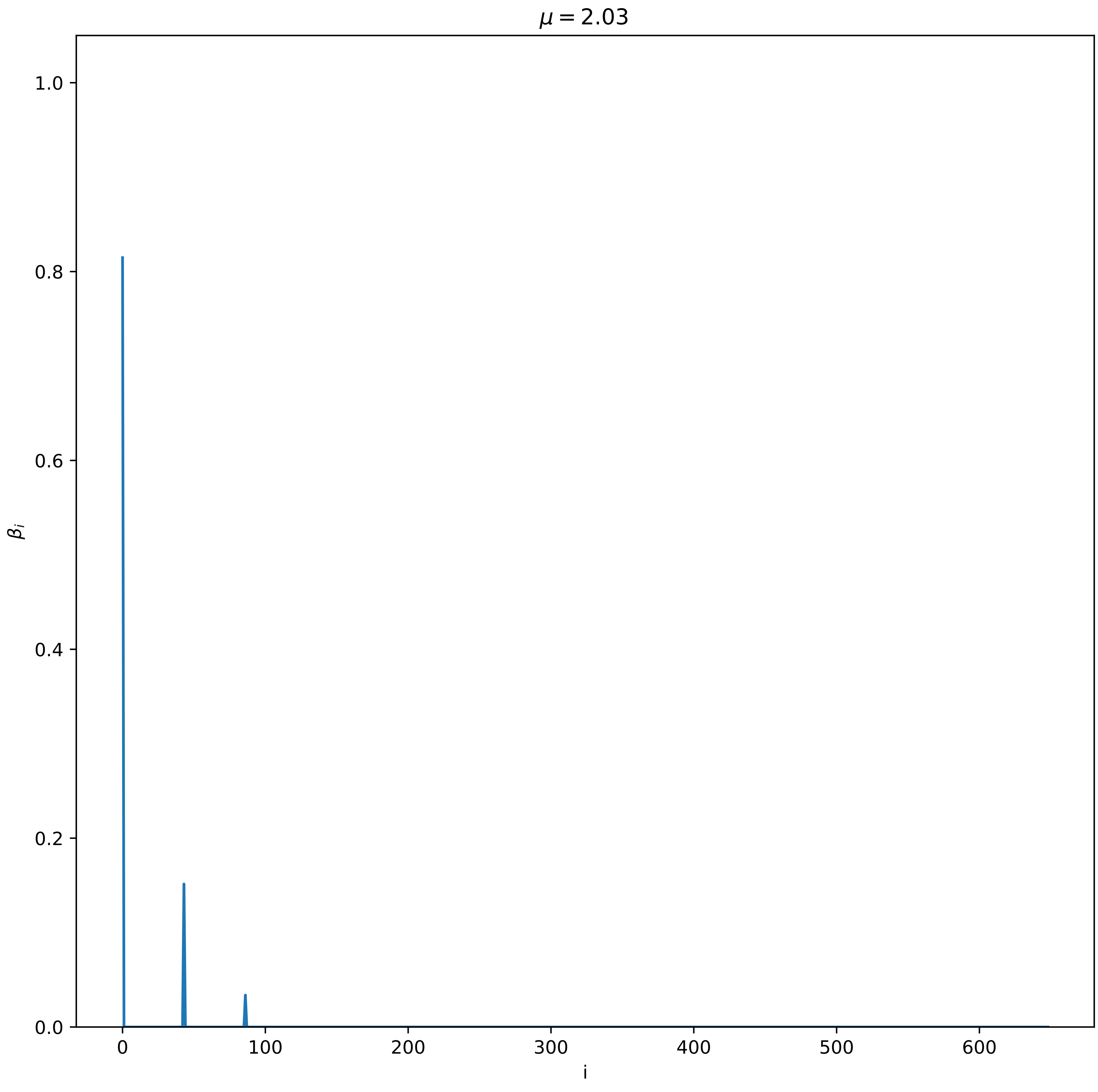
$\mu = 2.00$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



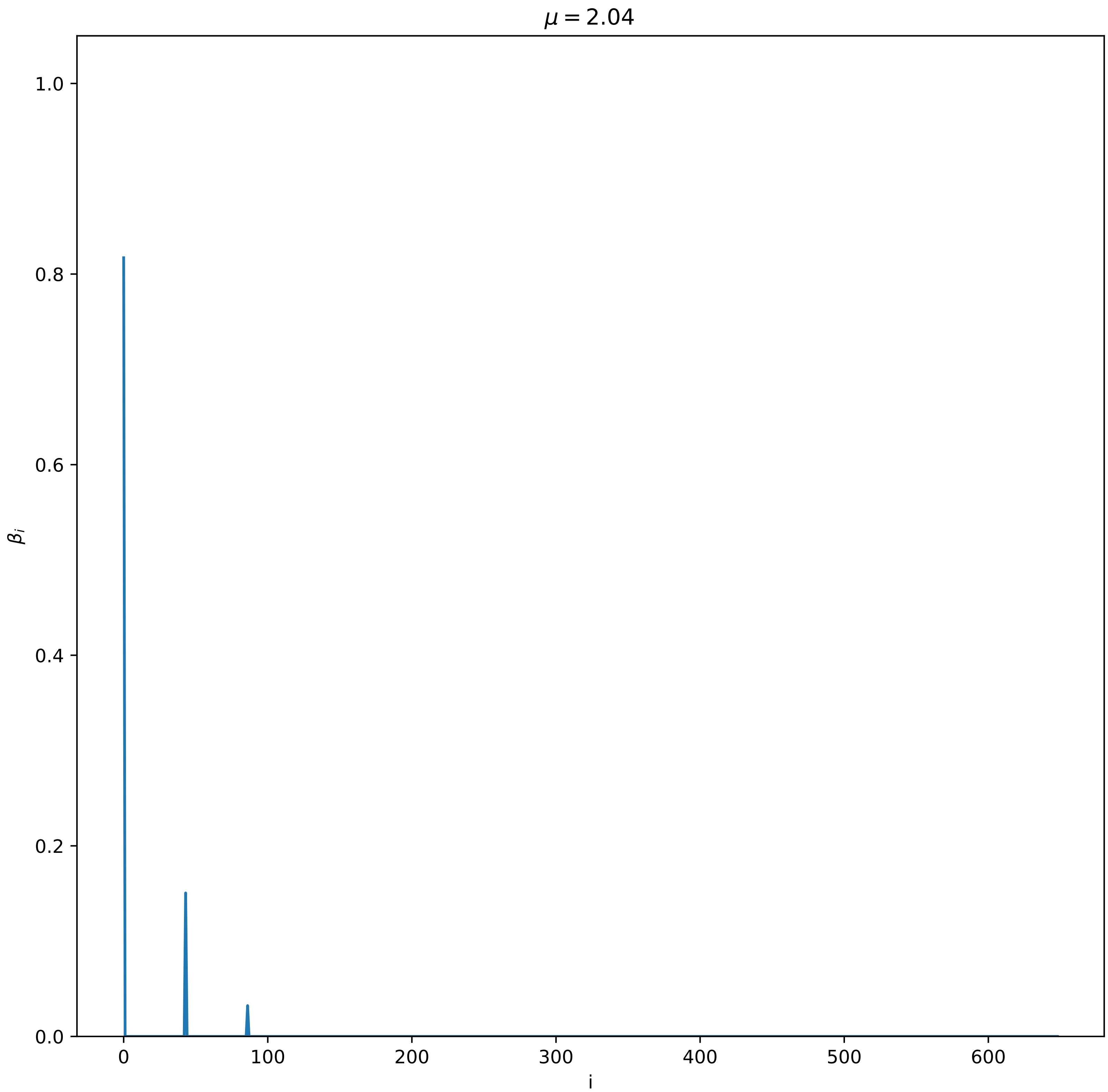
$\mu = 2.01$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



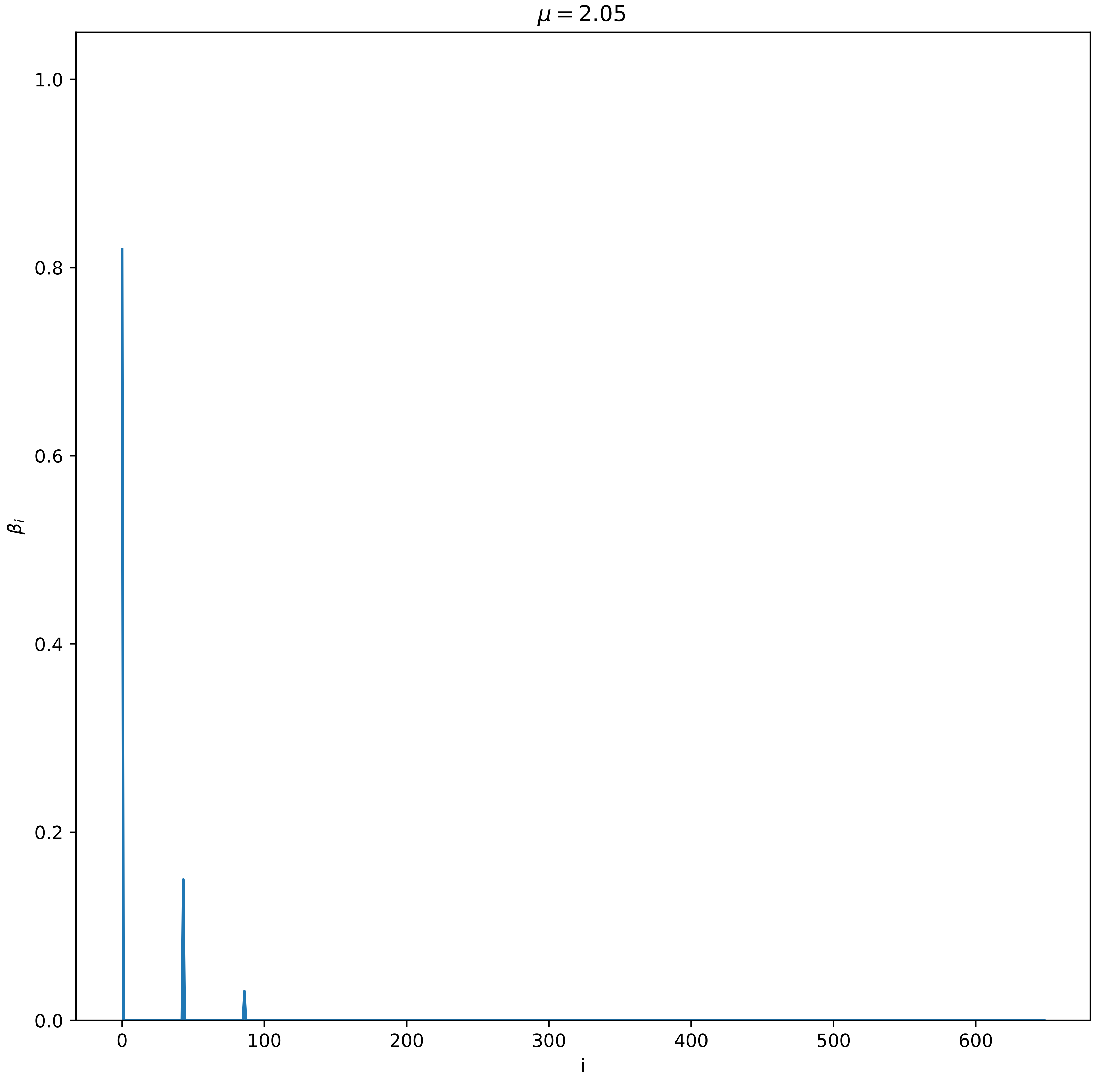
$\mu = 2.02$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



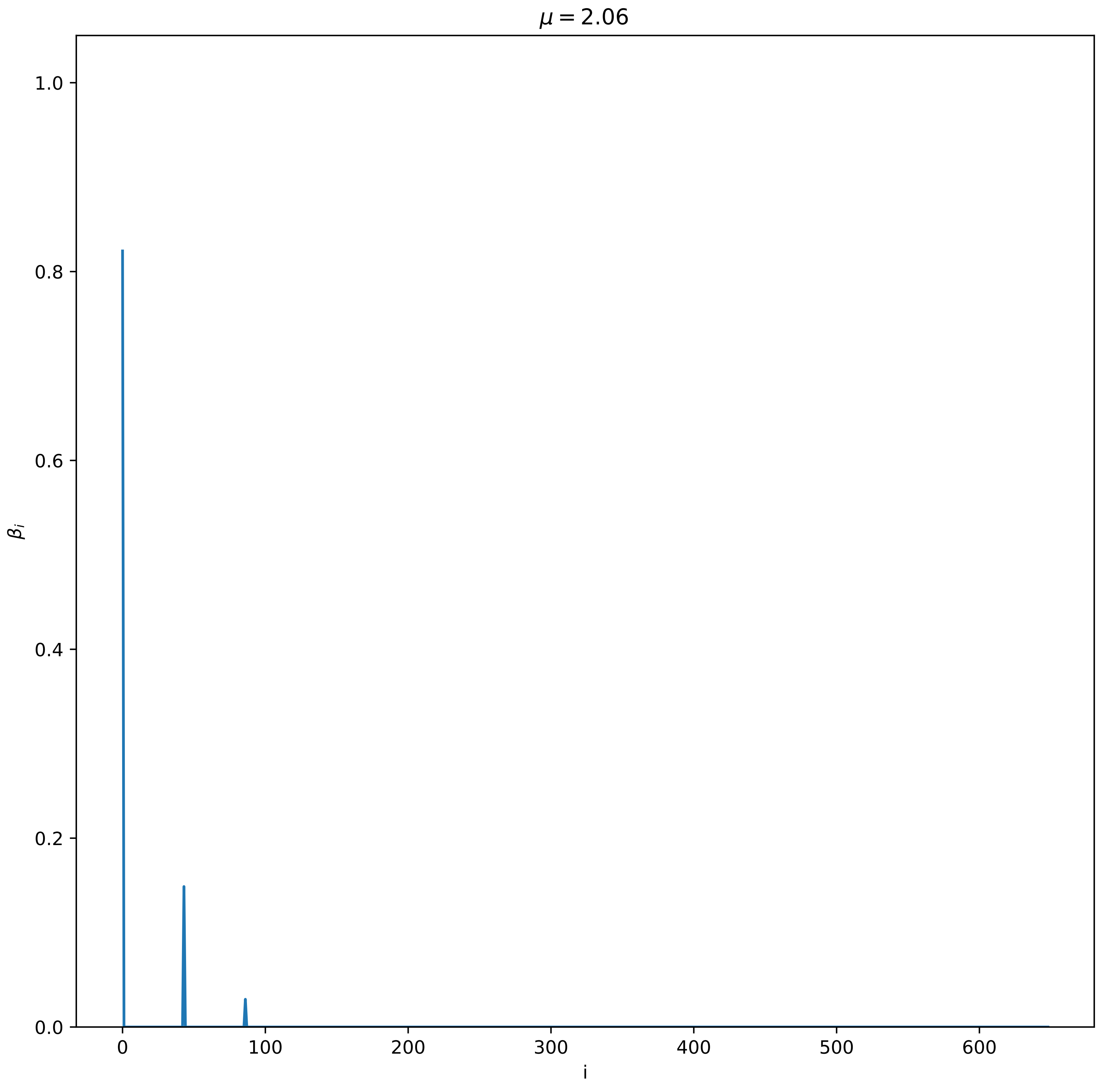
$\mu = 2.03$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



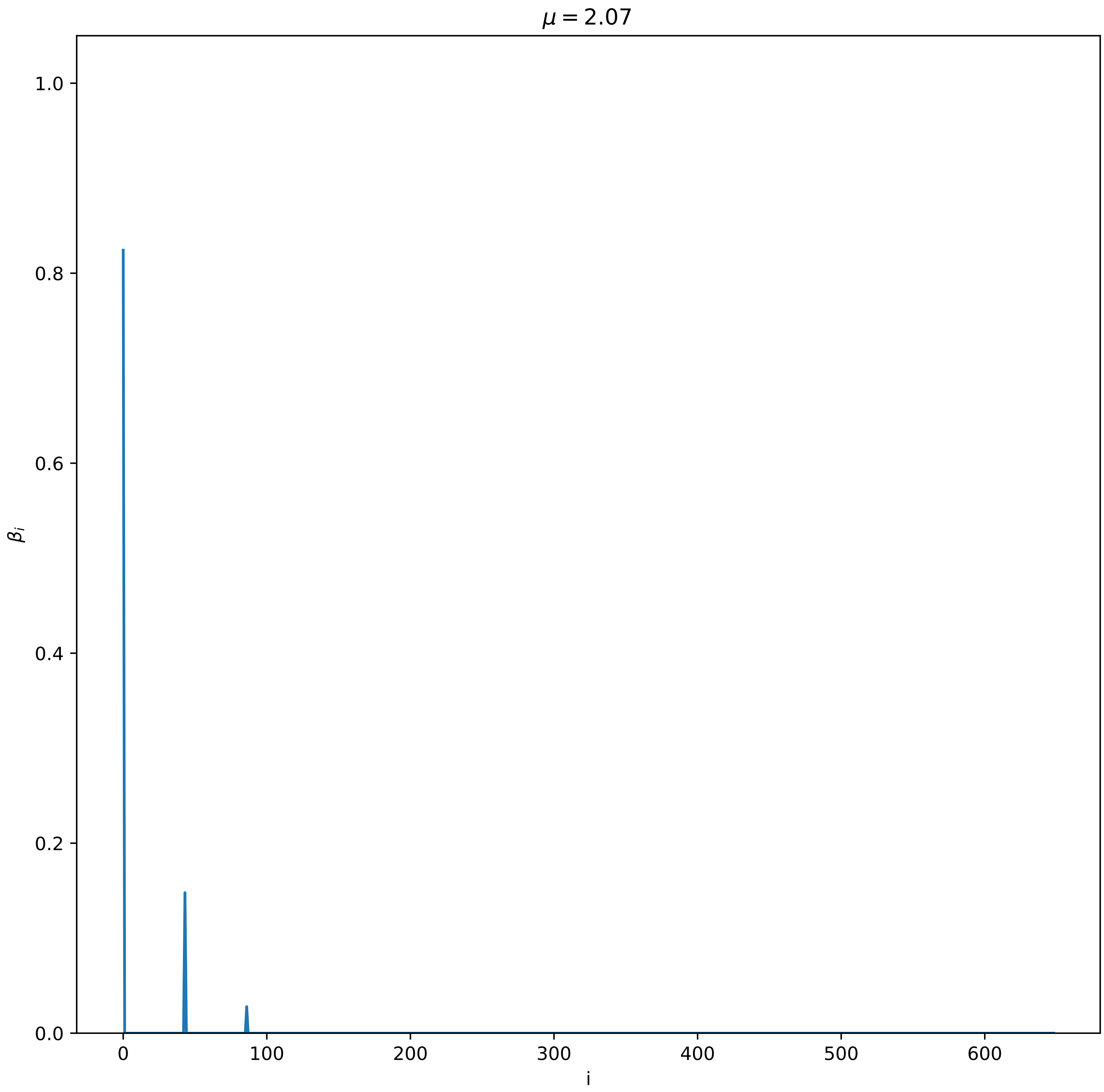
$\mu = 2.04$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



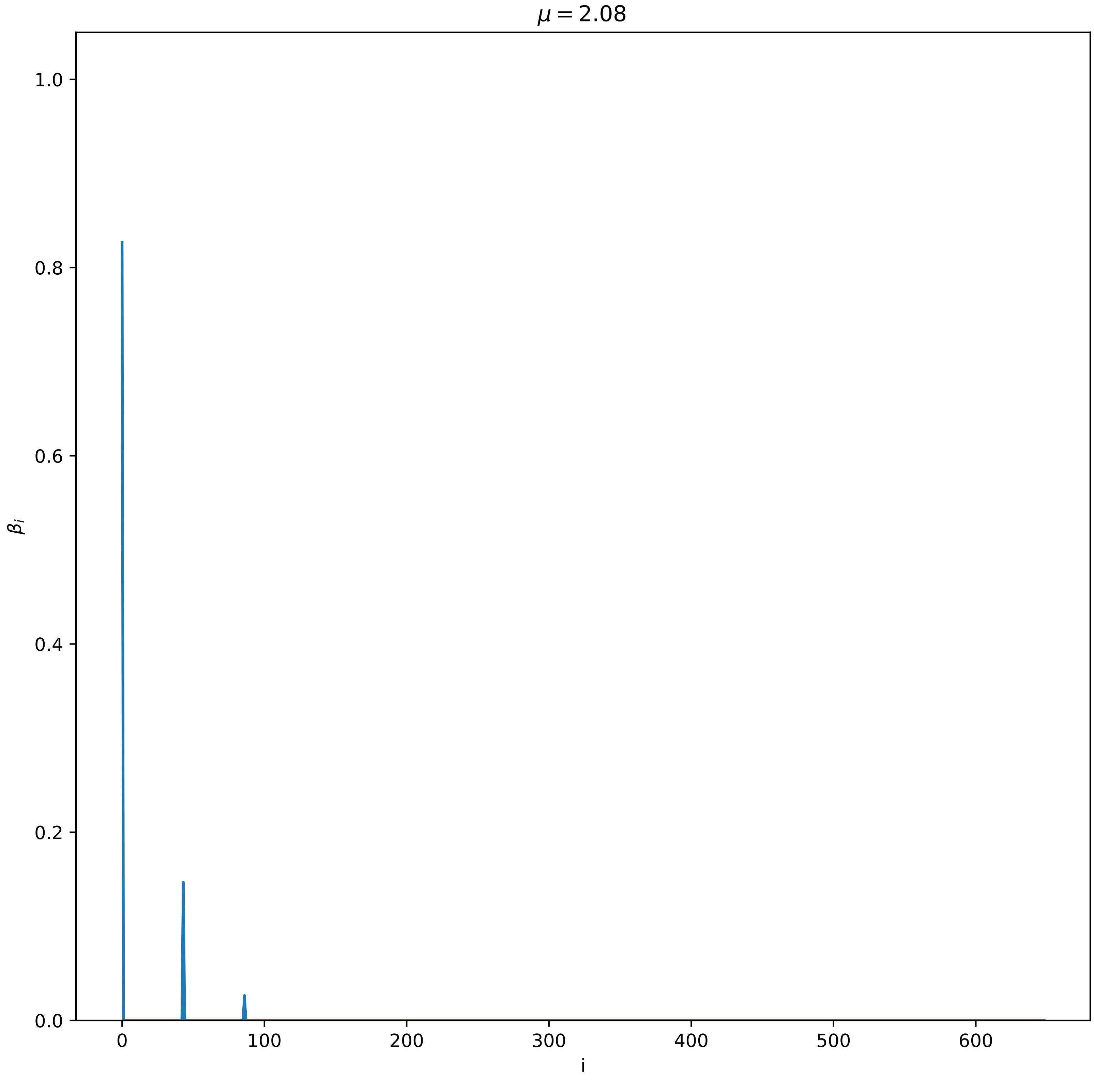
$\mu = 2.05$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



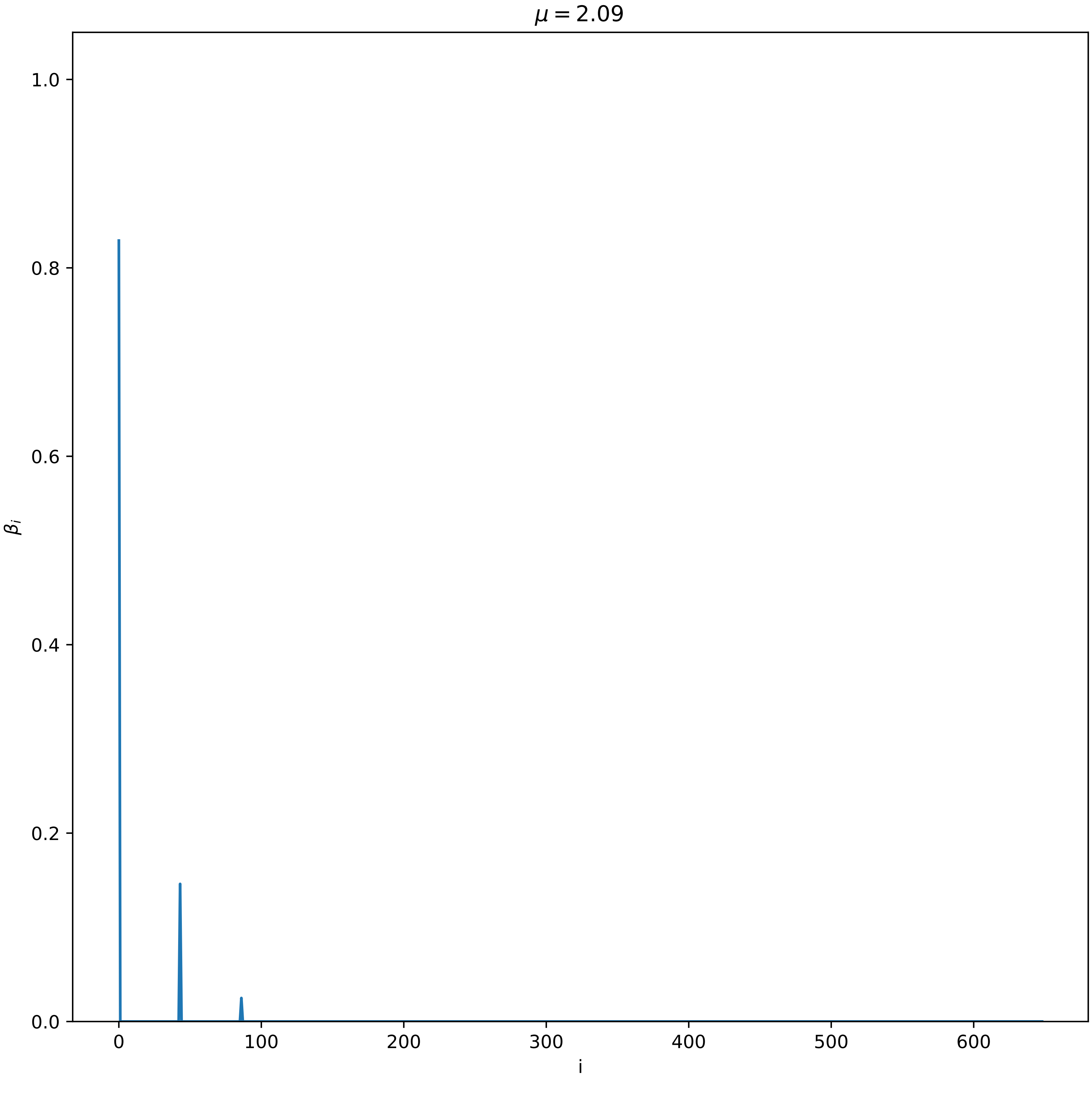
$\mu = 2.06$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



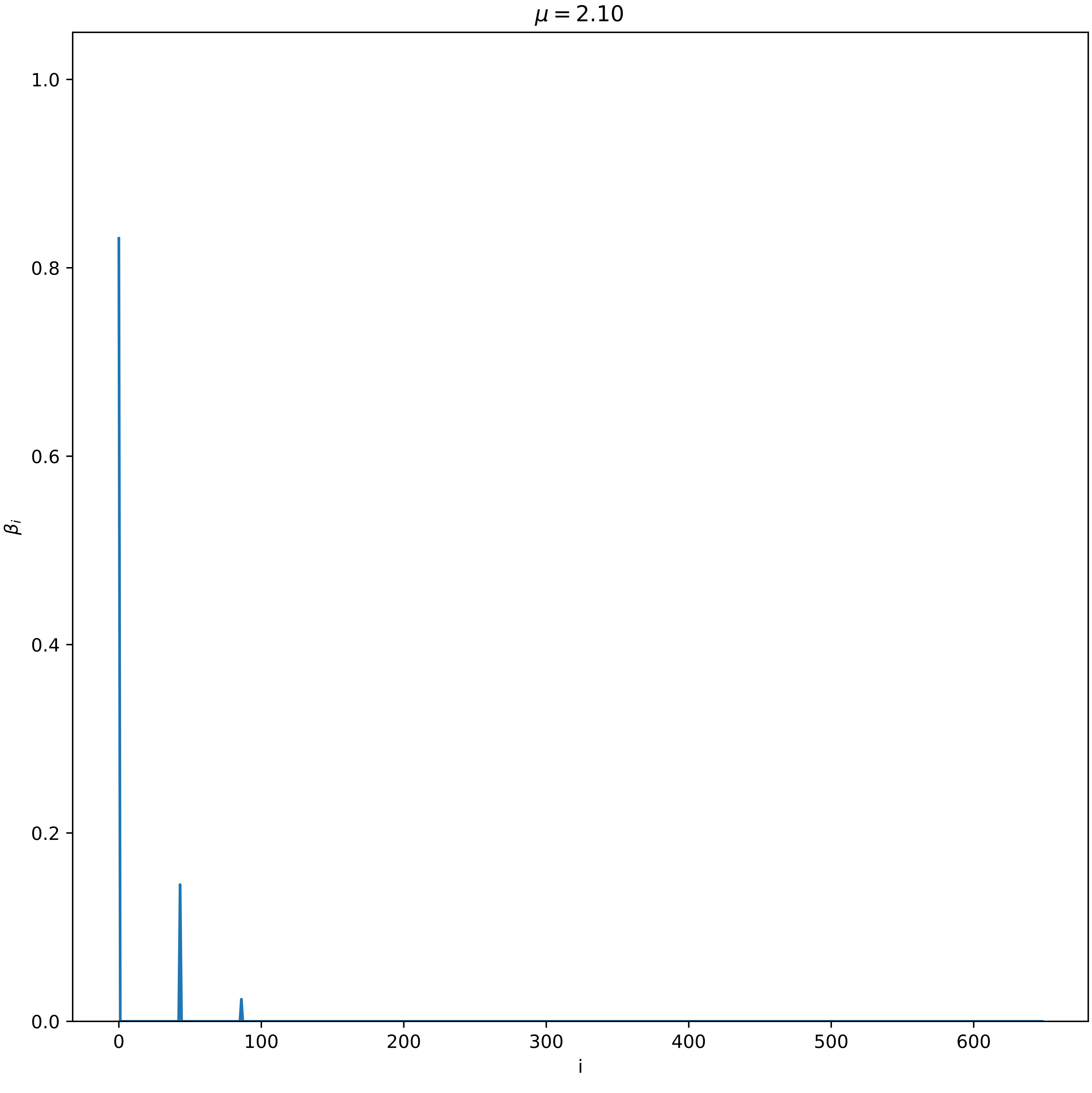
$\mu = 2.07$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



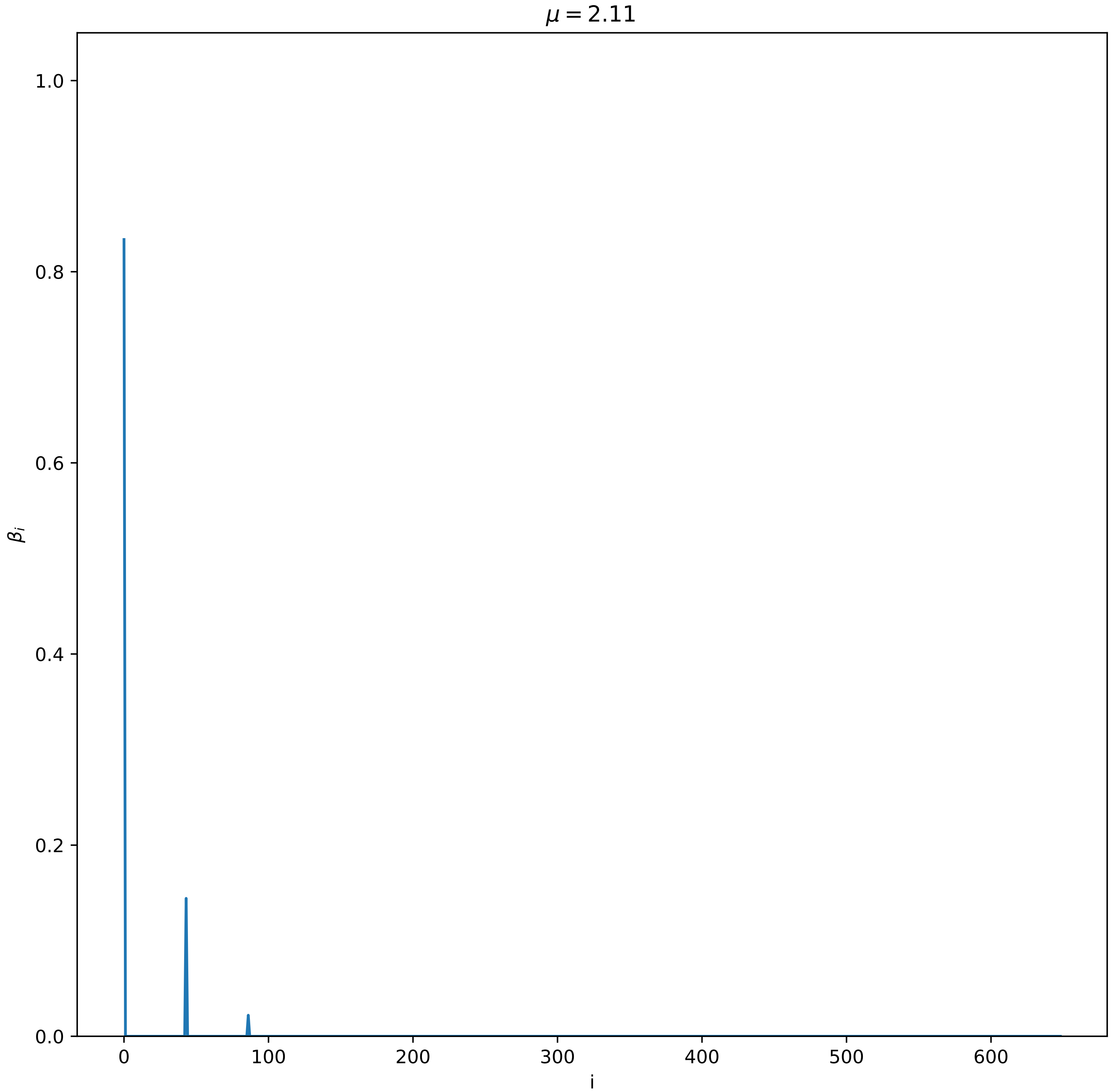
$\mu = 2.08$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



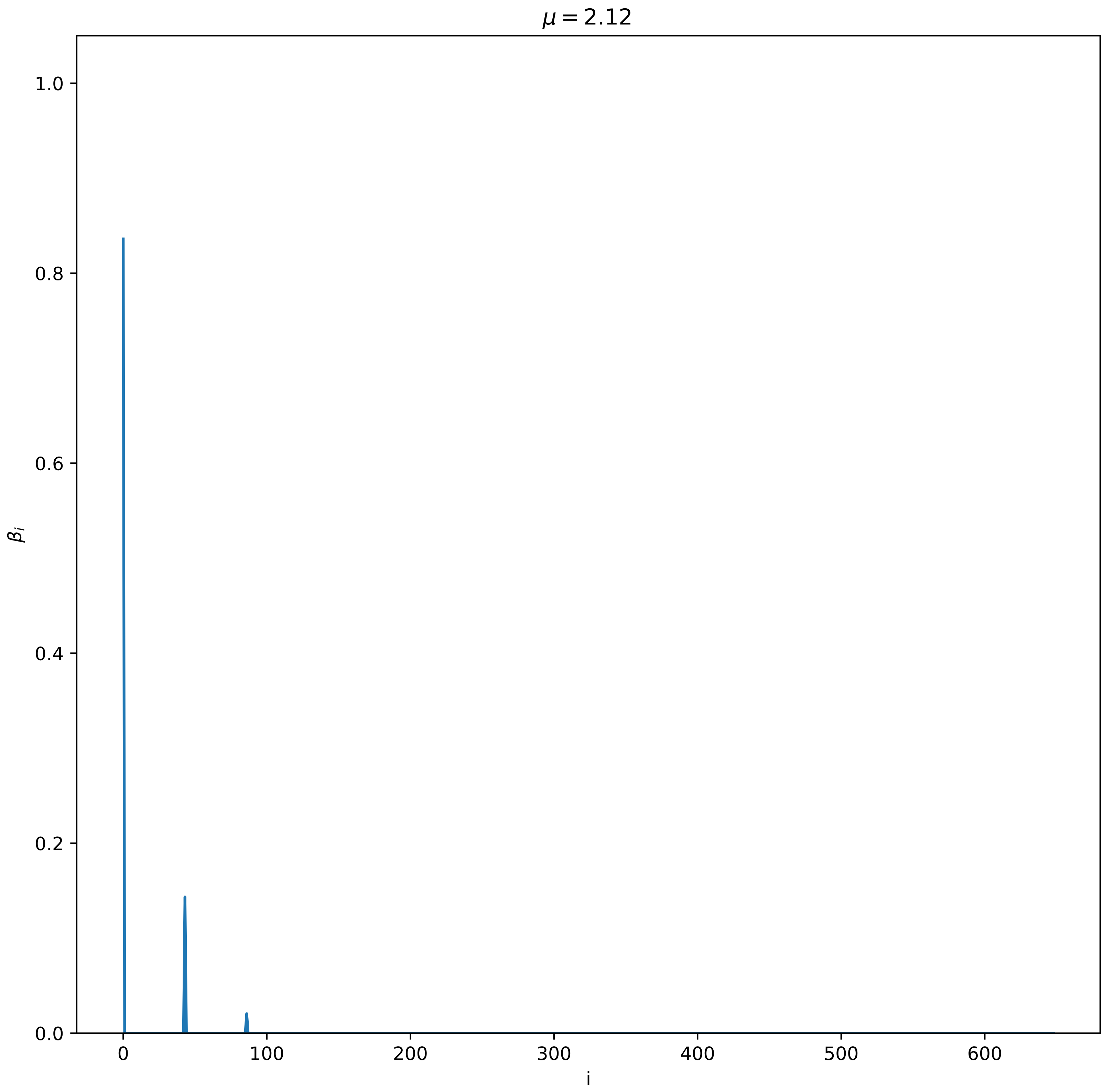
$\mu = 2.09$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



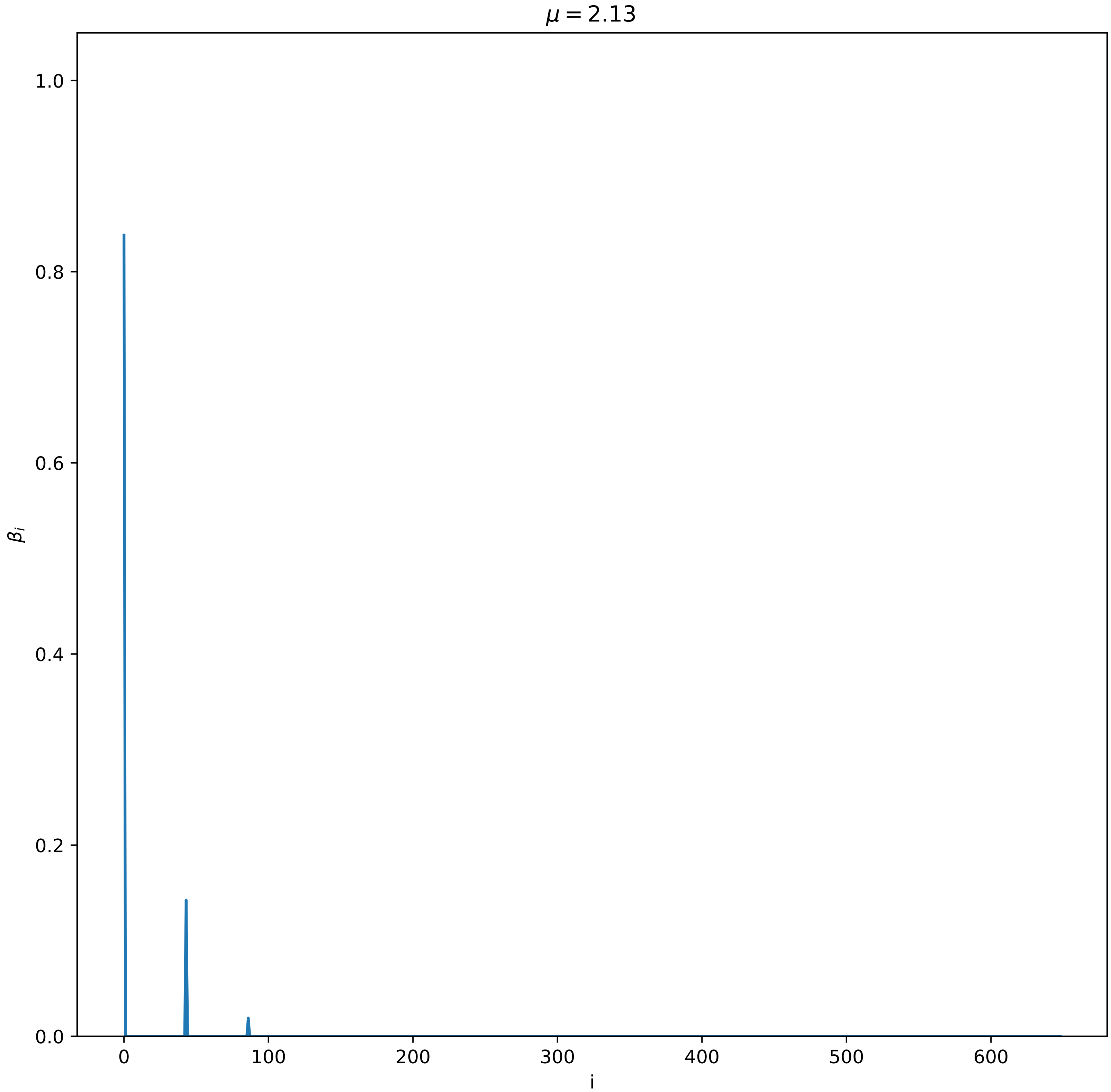
$\mu = 2.10$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



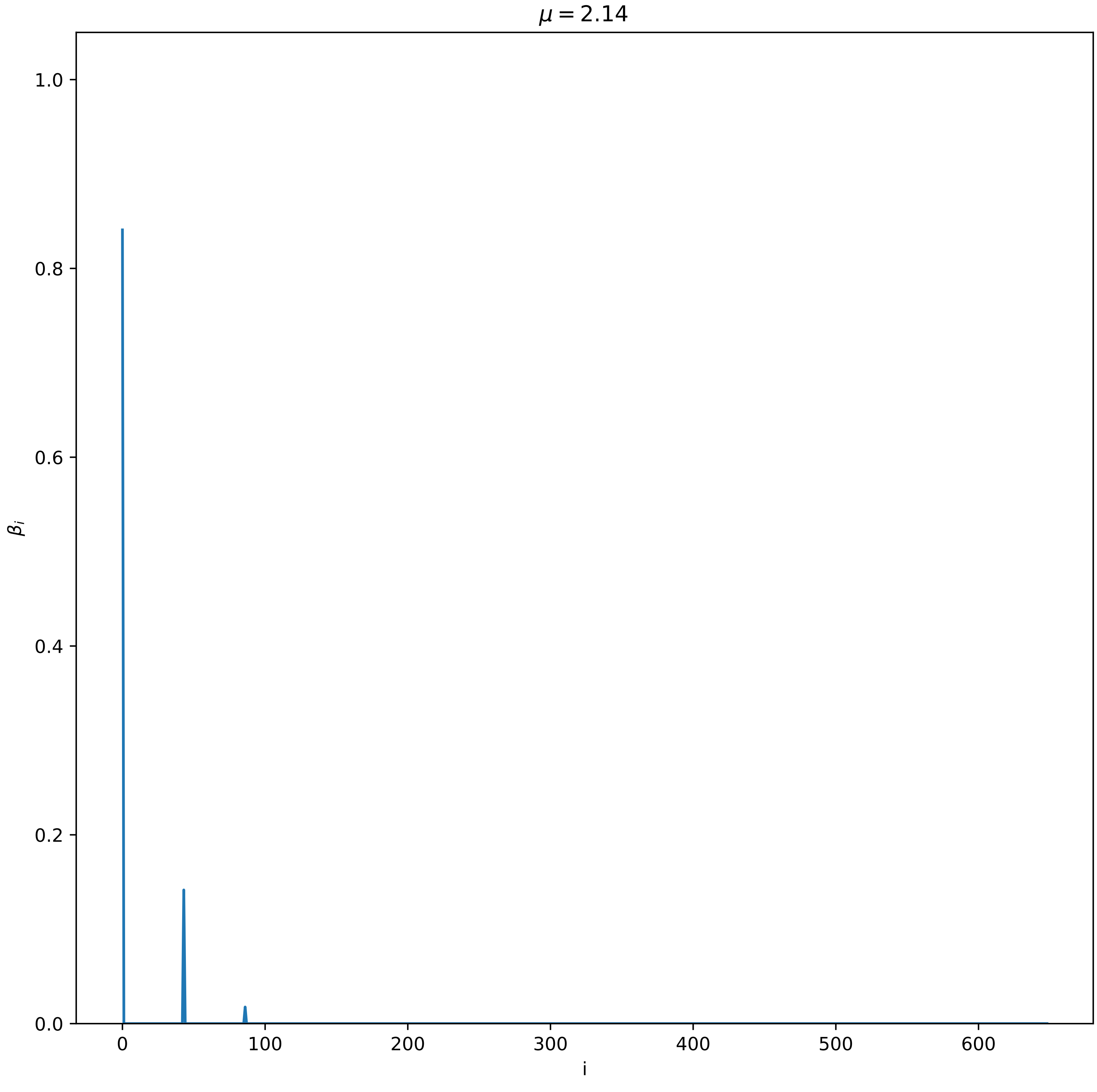
$\mu = 2.11$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



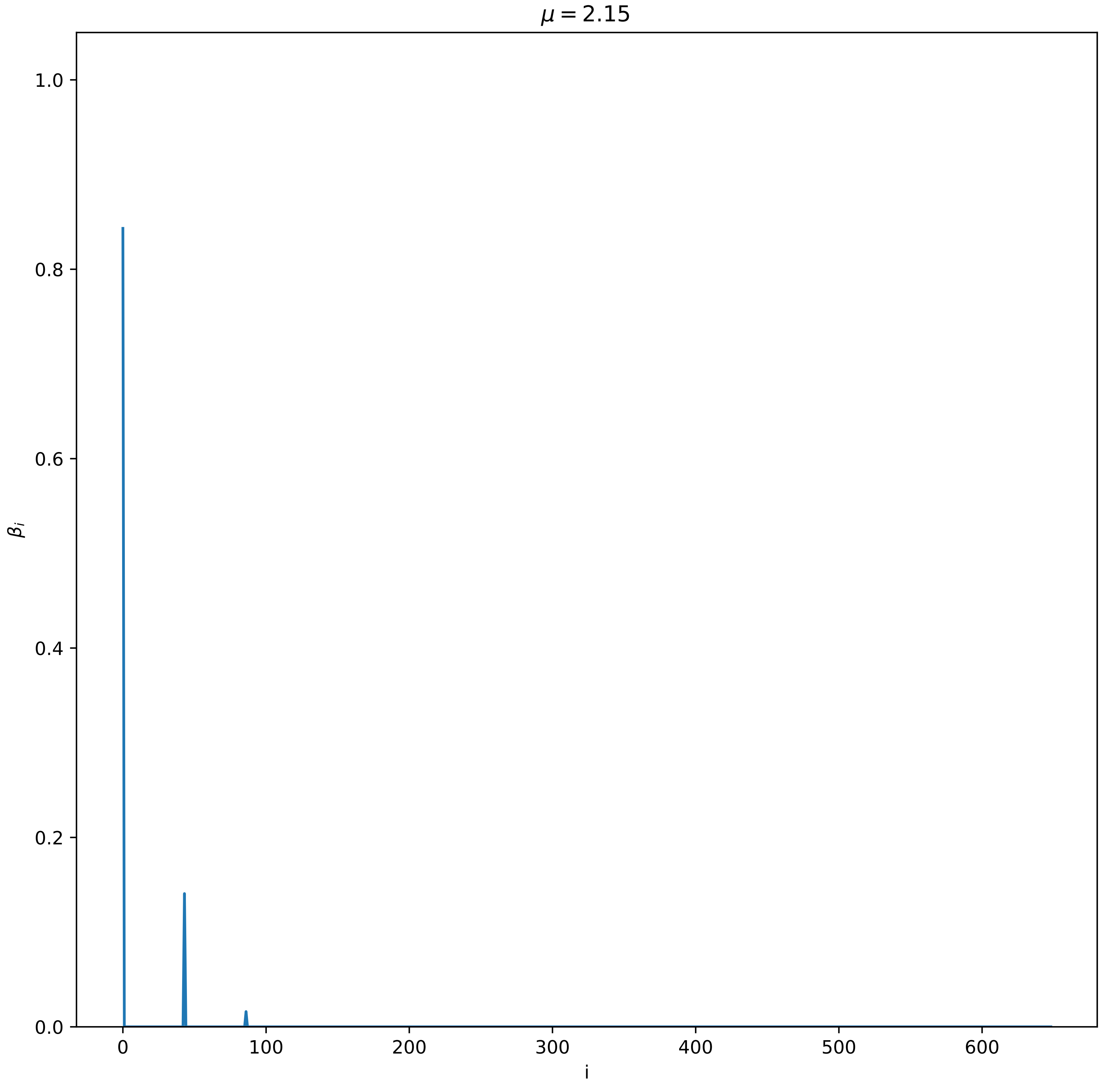
$\mu = 2.12$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

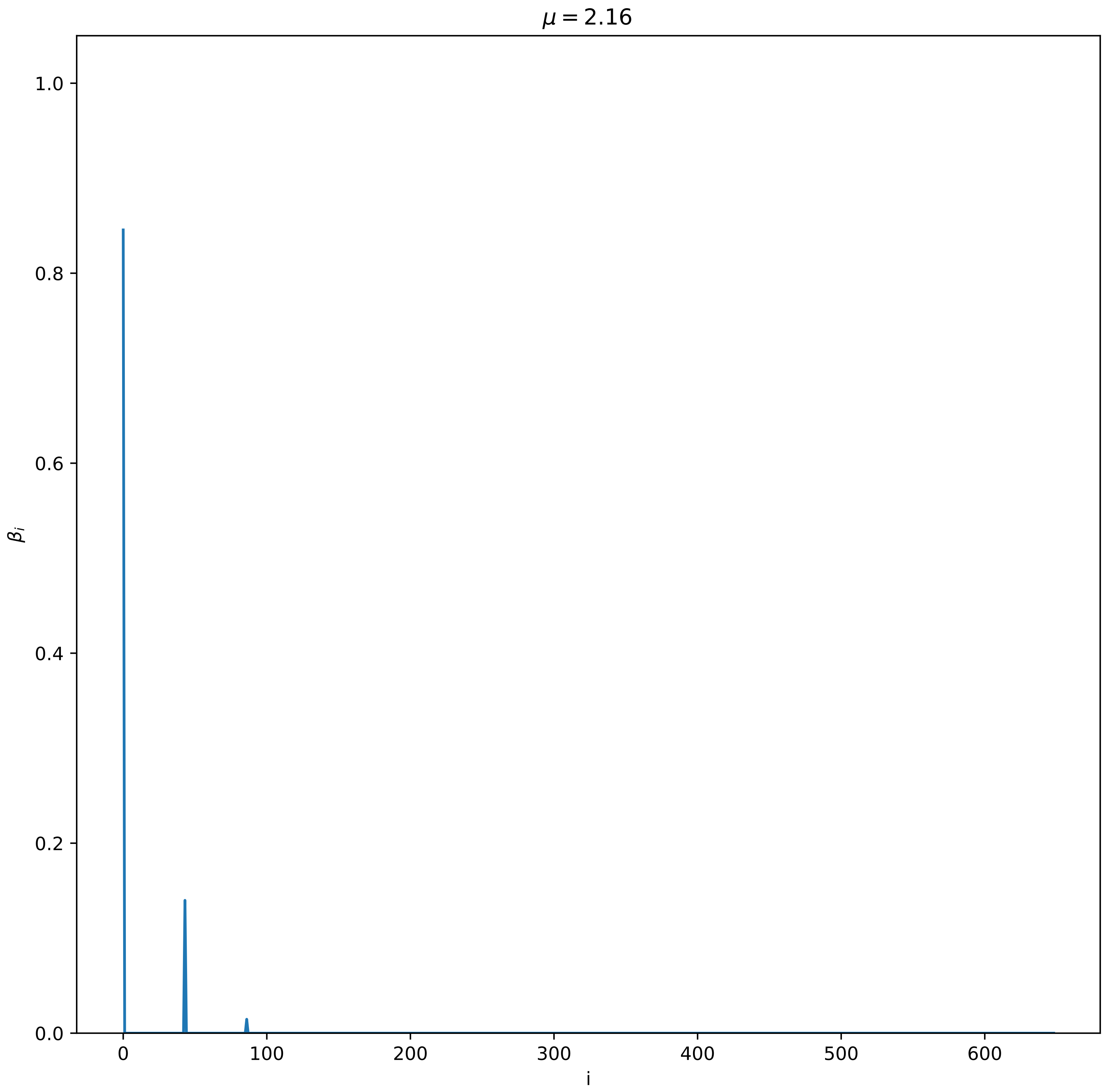


$\mu = 2.13$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

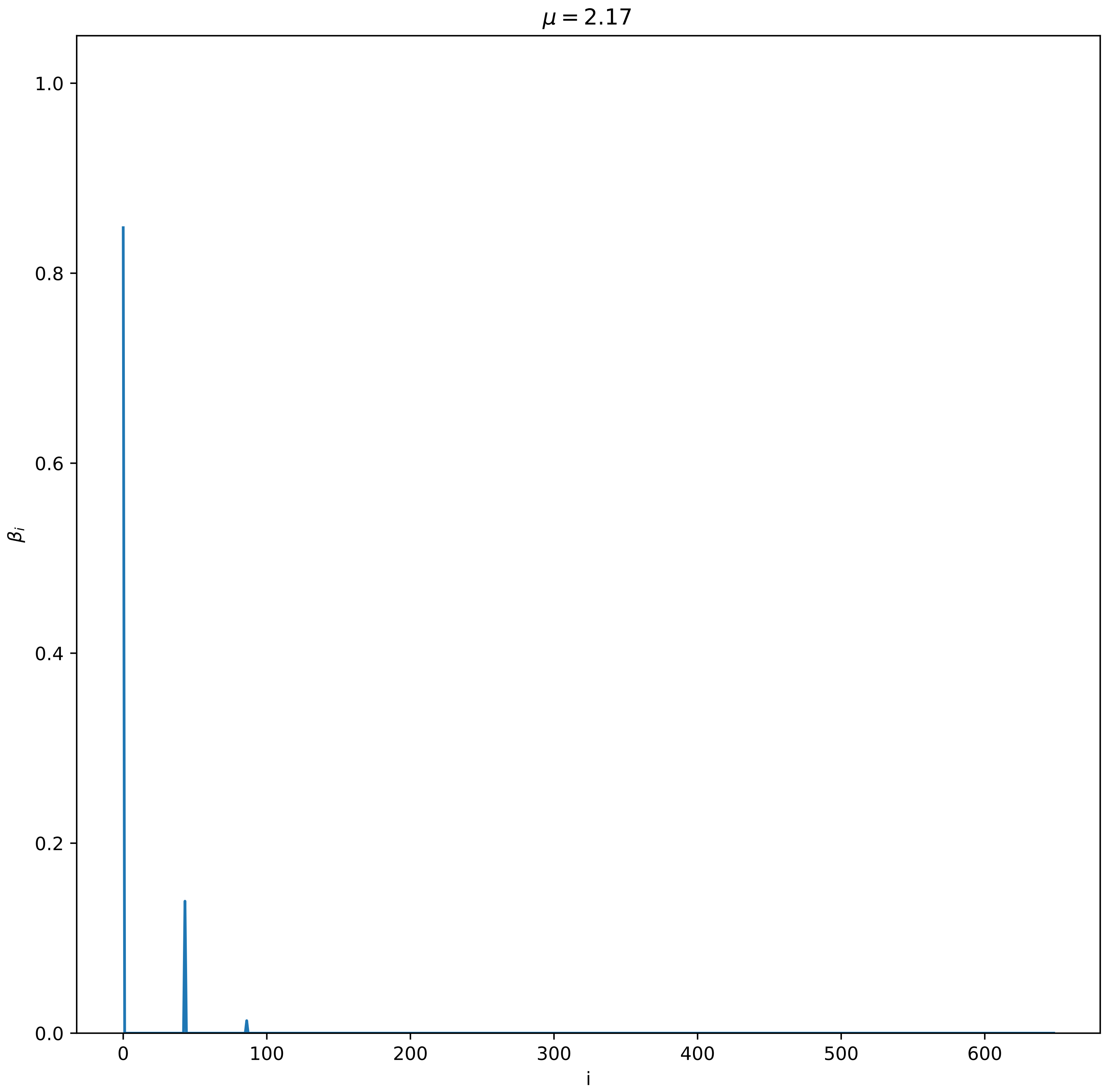


$\mu = 2.14$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

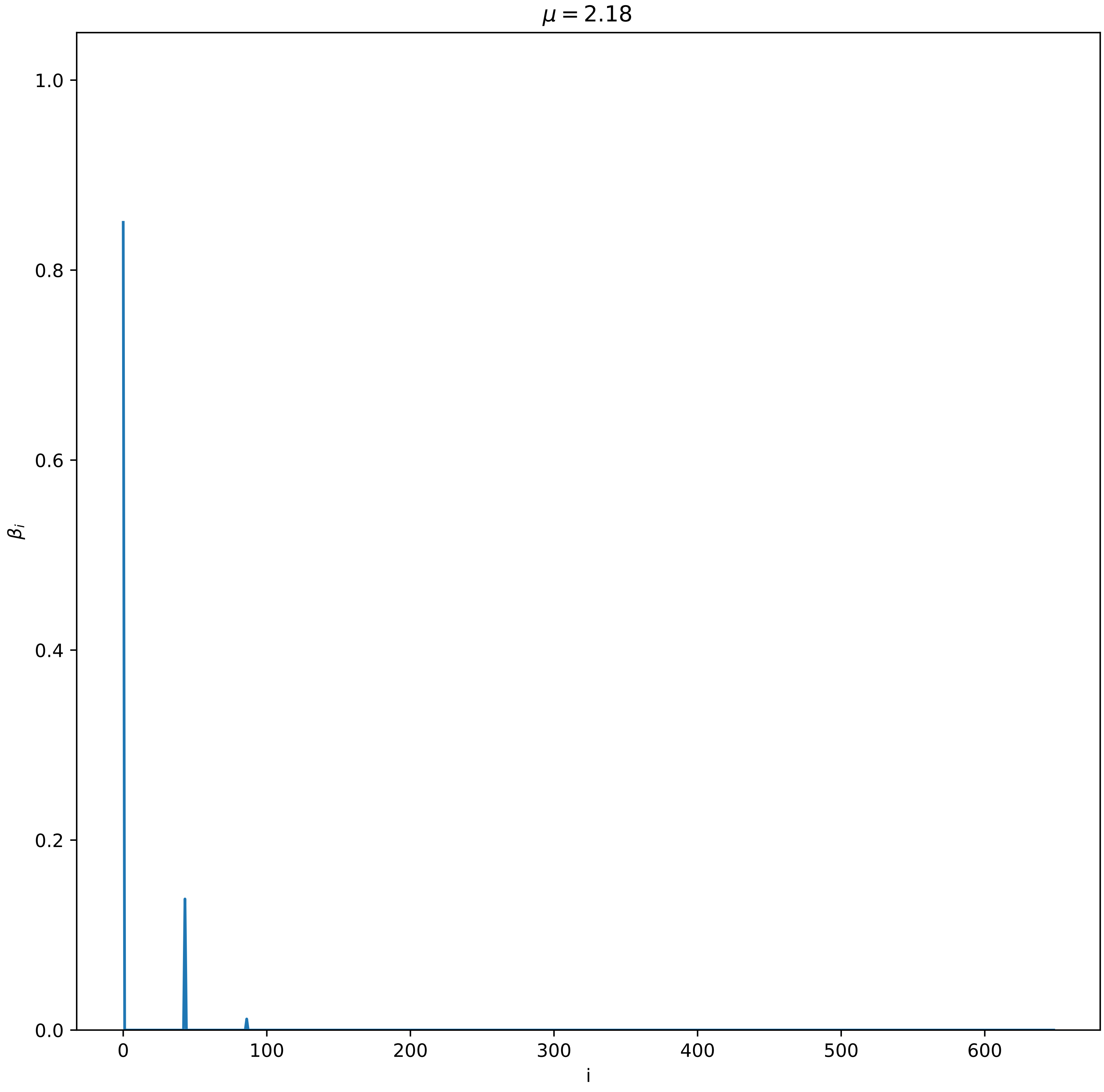

$\mu = 2.15$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



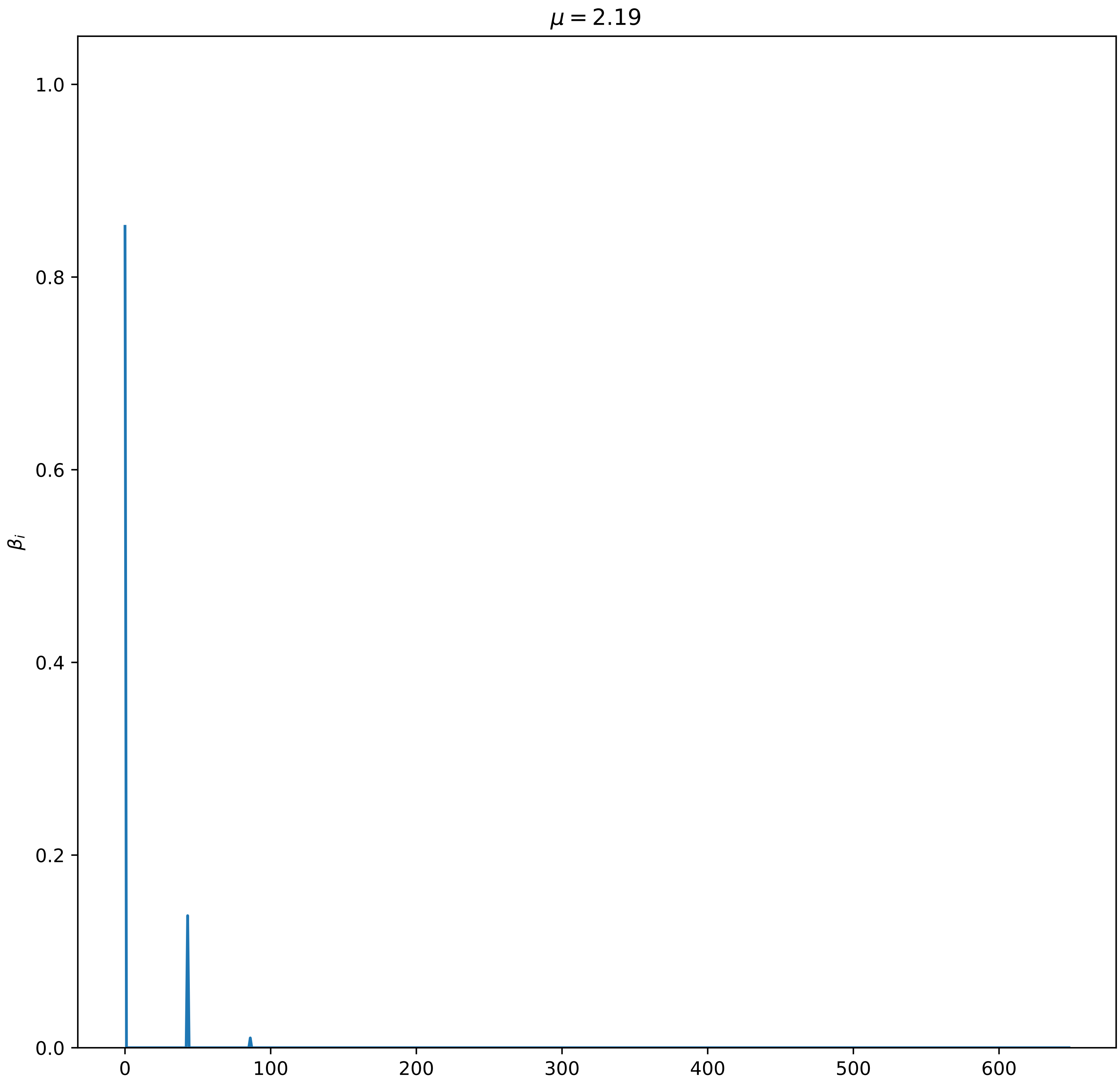
$\mu = 2.16$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



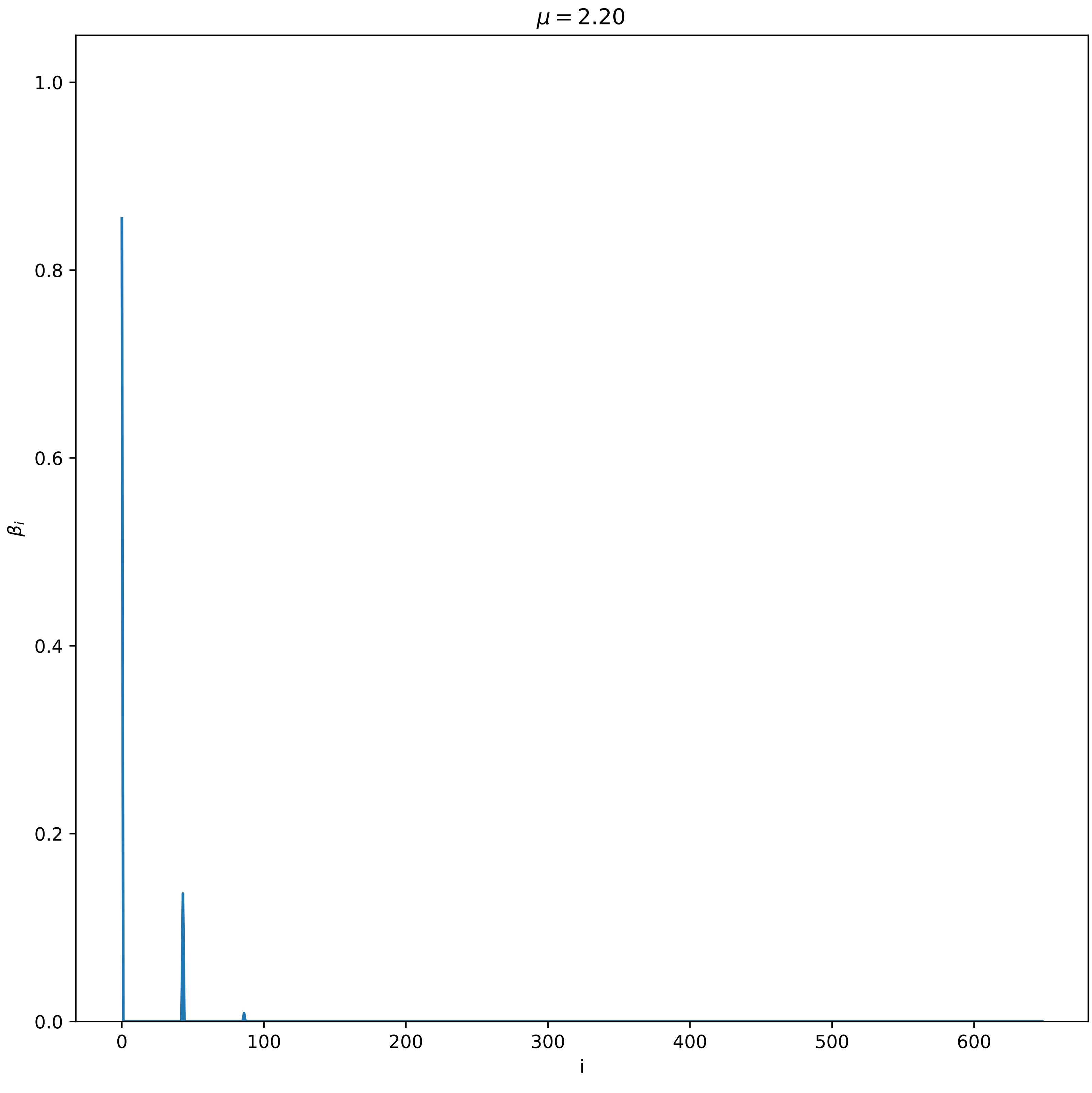
$\mu = 2.17$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

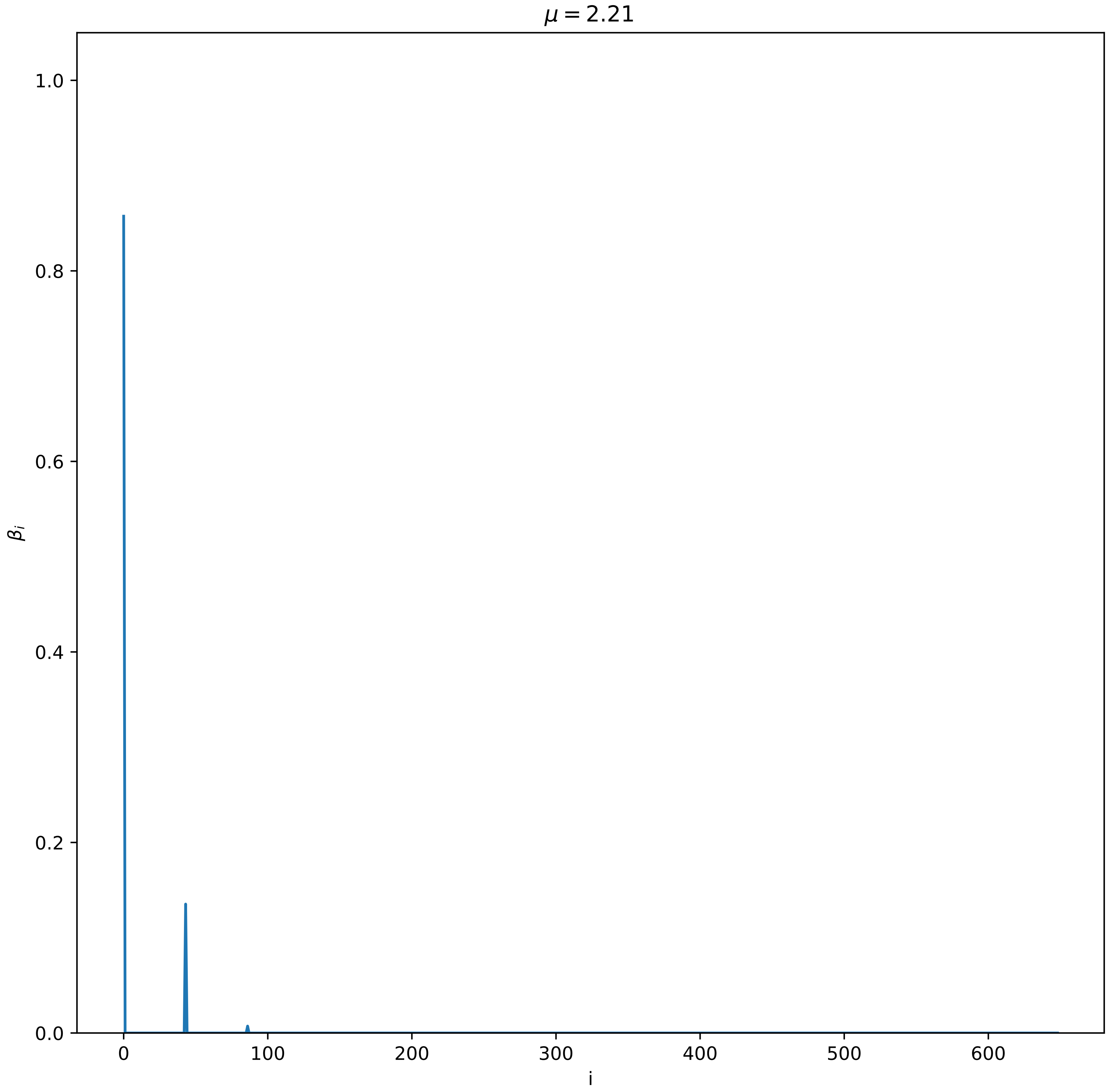


$\mu = 2.18$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

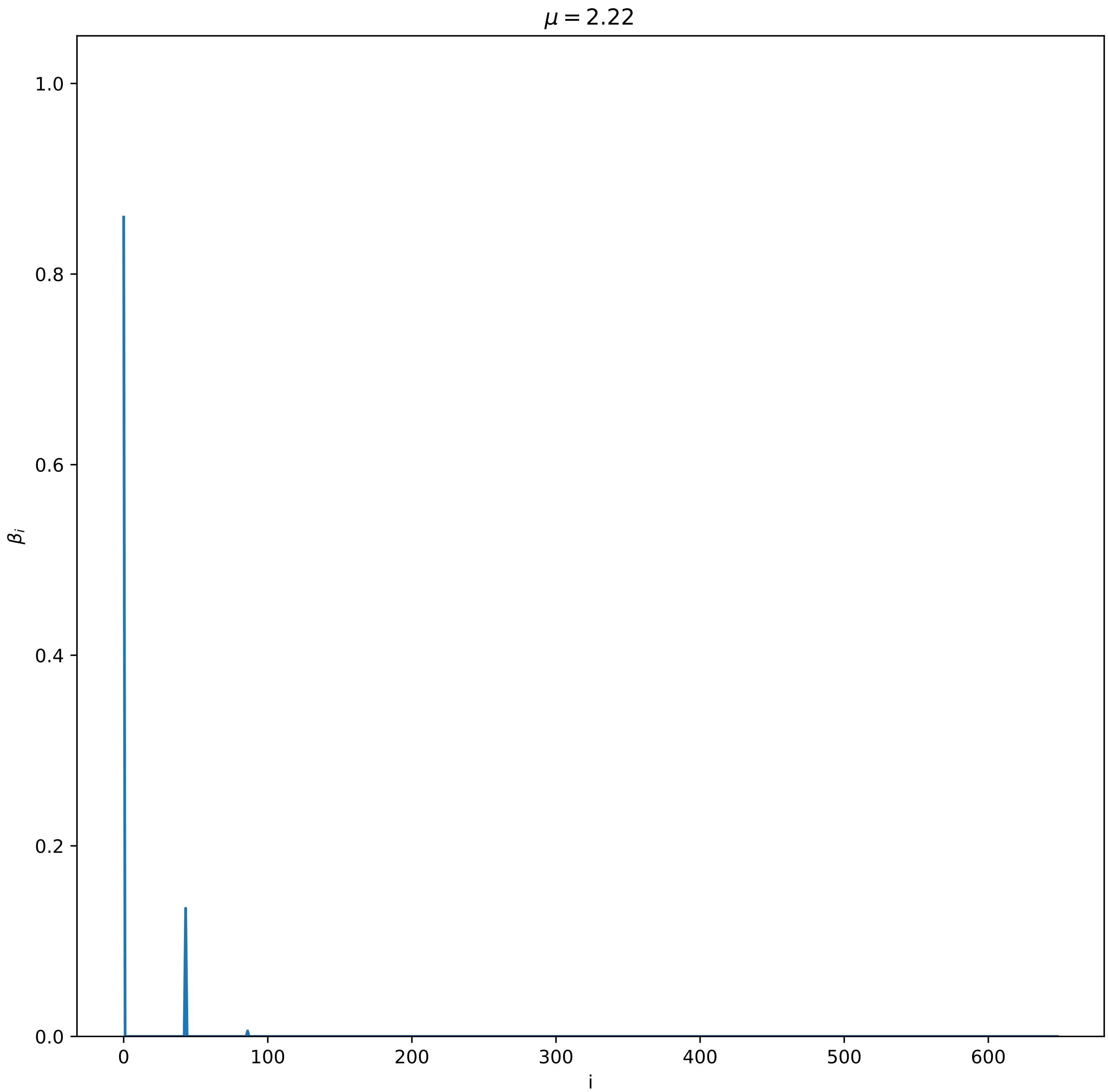


$\mu = 2.19$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

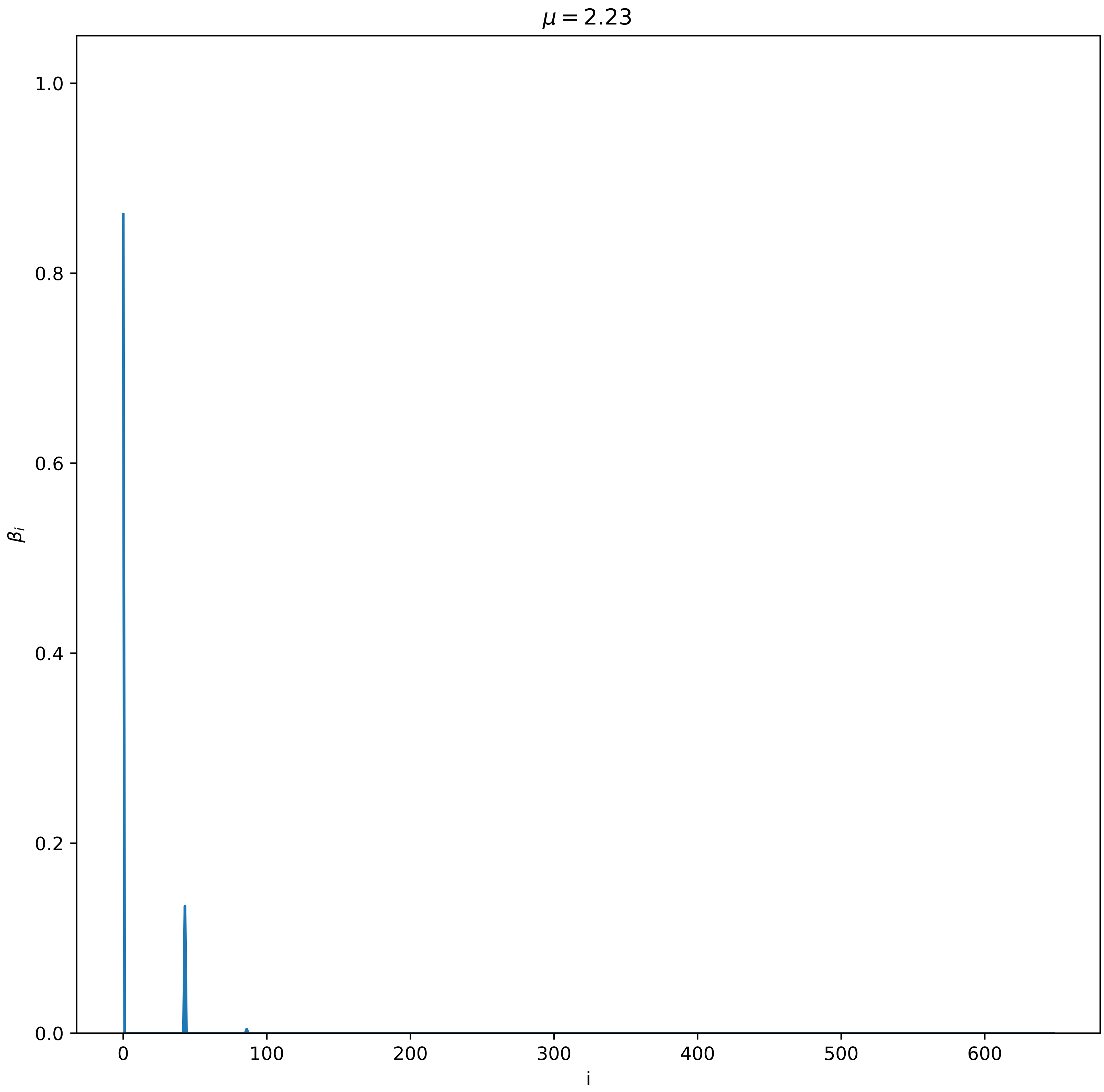

$\mu = 2.20$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



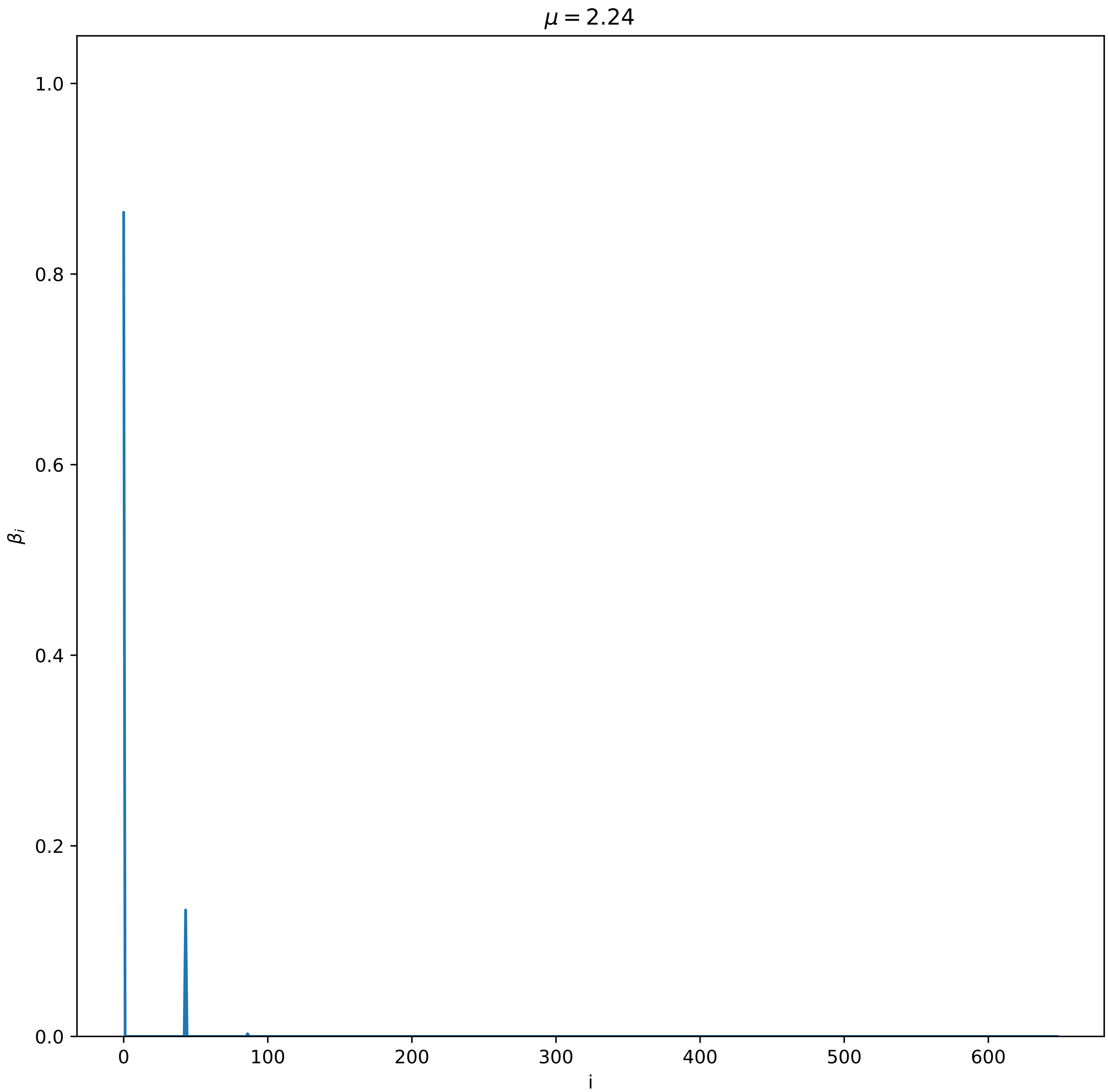
$\mu = 2.21$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



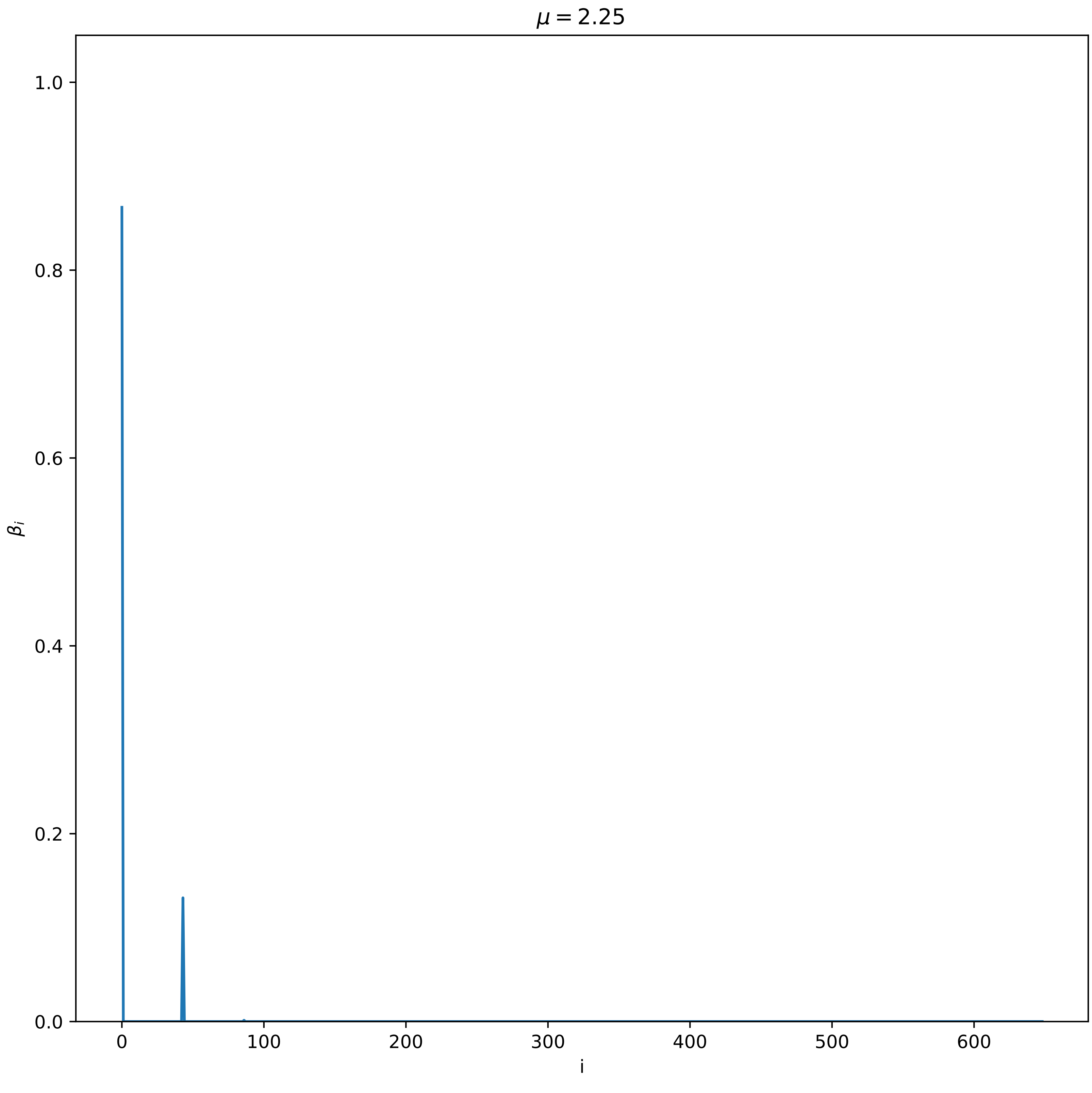
$\mu = 2.22$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



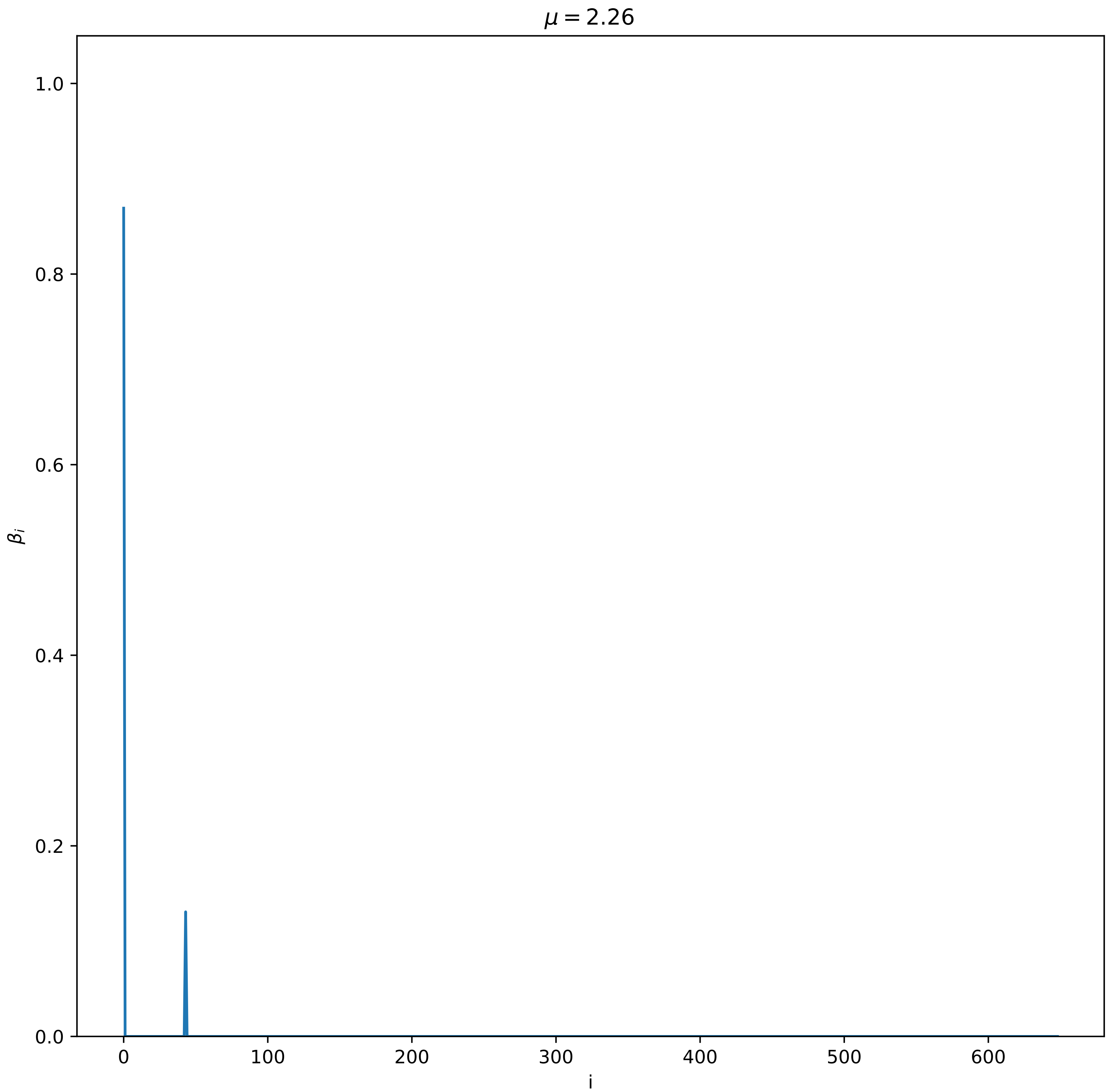
$\mu = 2.23$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



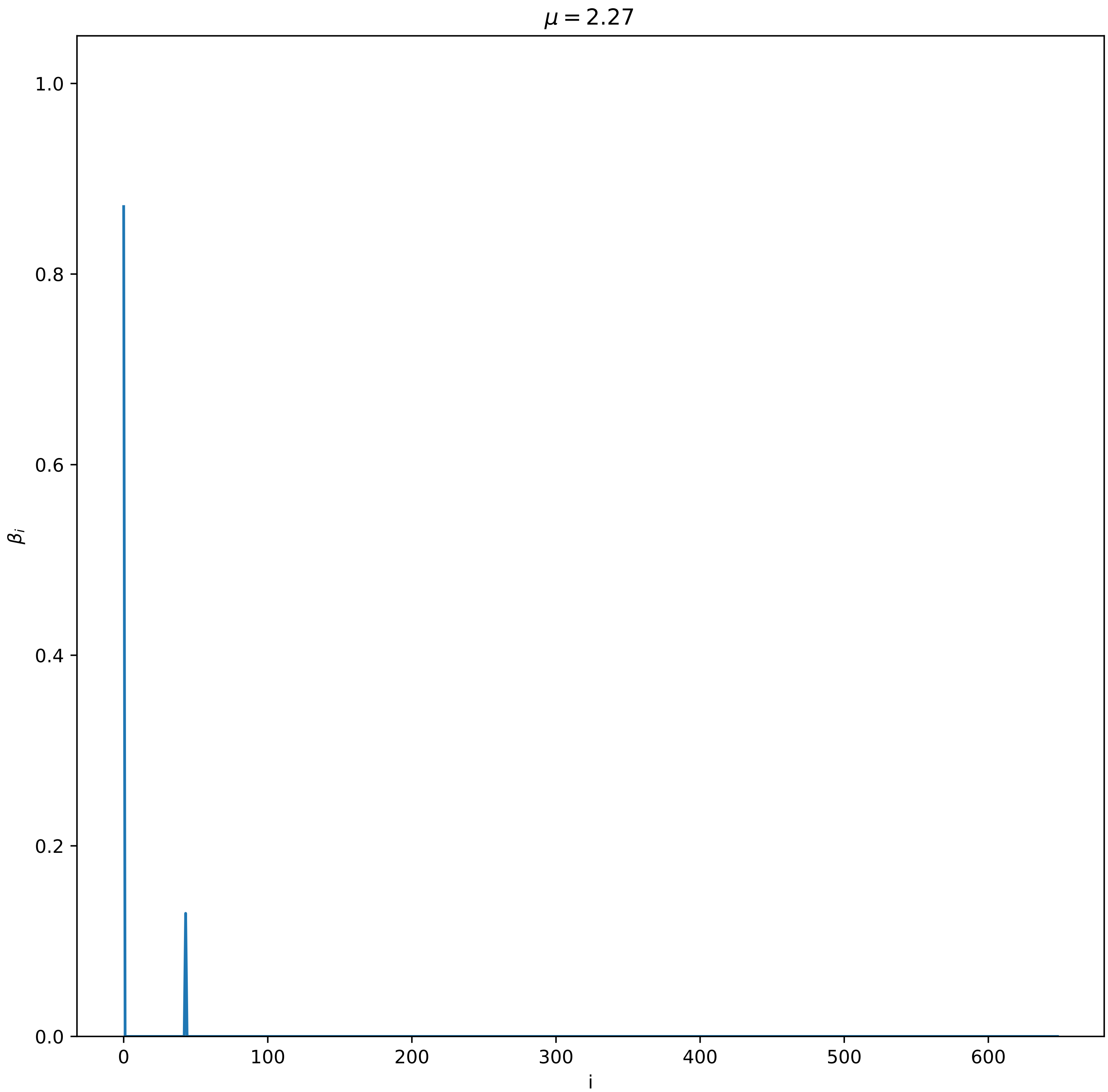
$\mu = 2.24$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



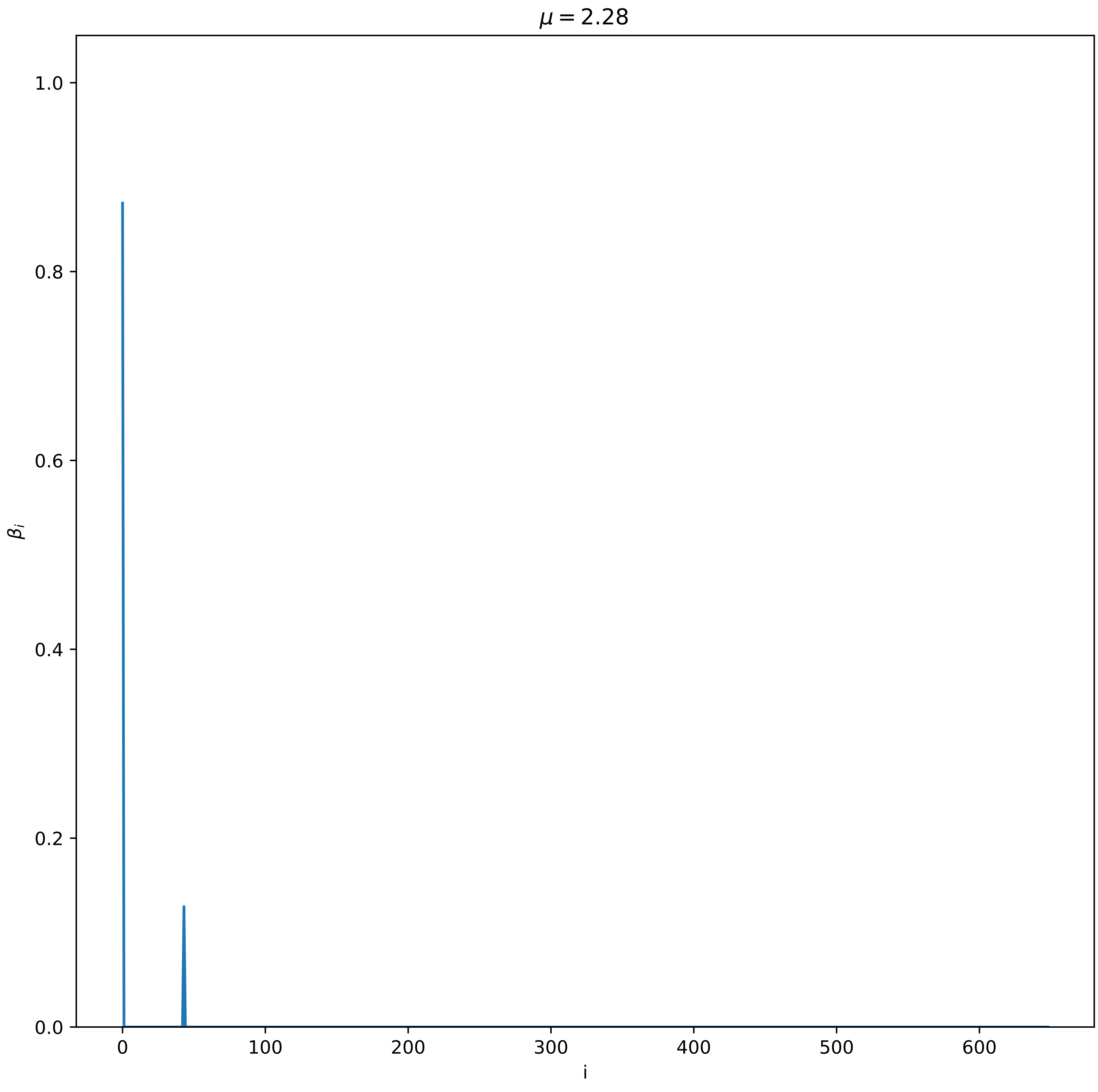
$\mu = 2.25$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



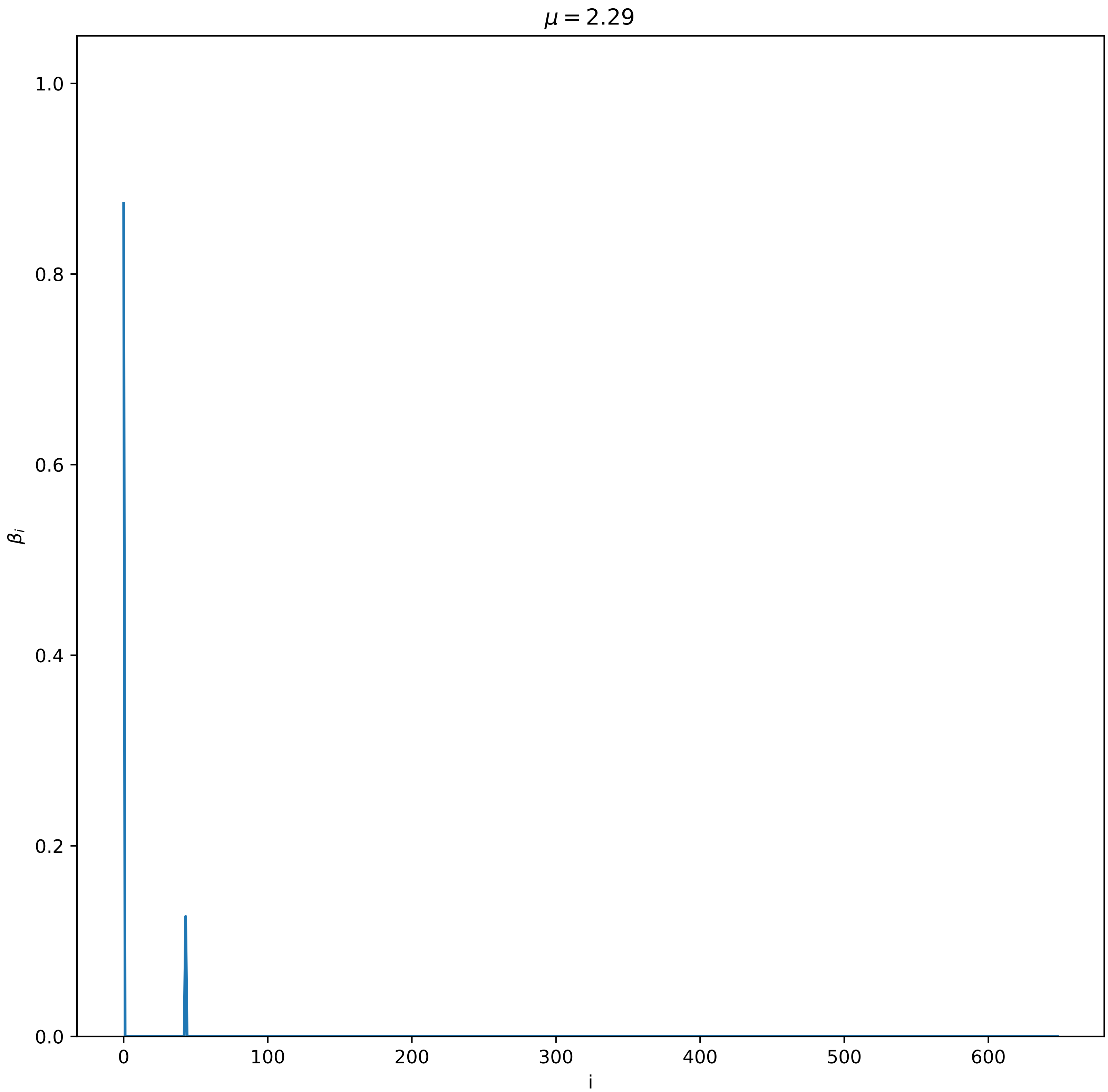
$\mu = 2.26$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



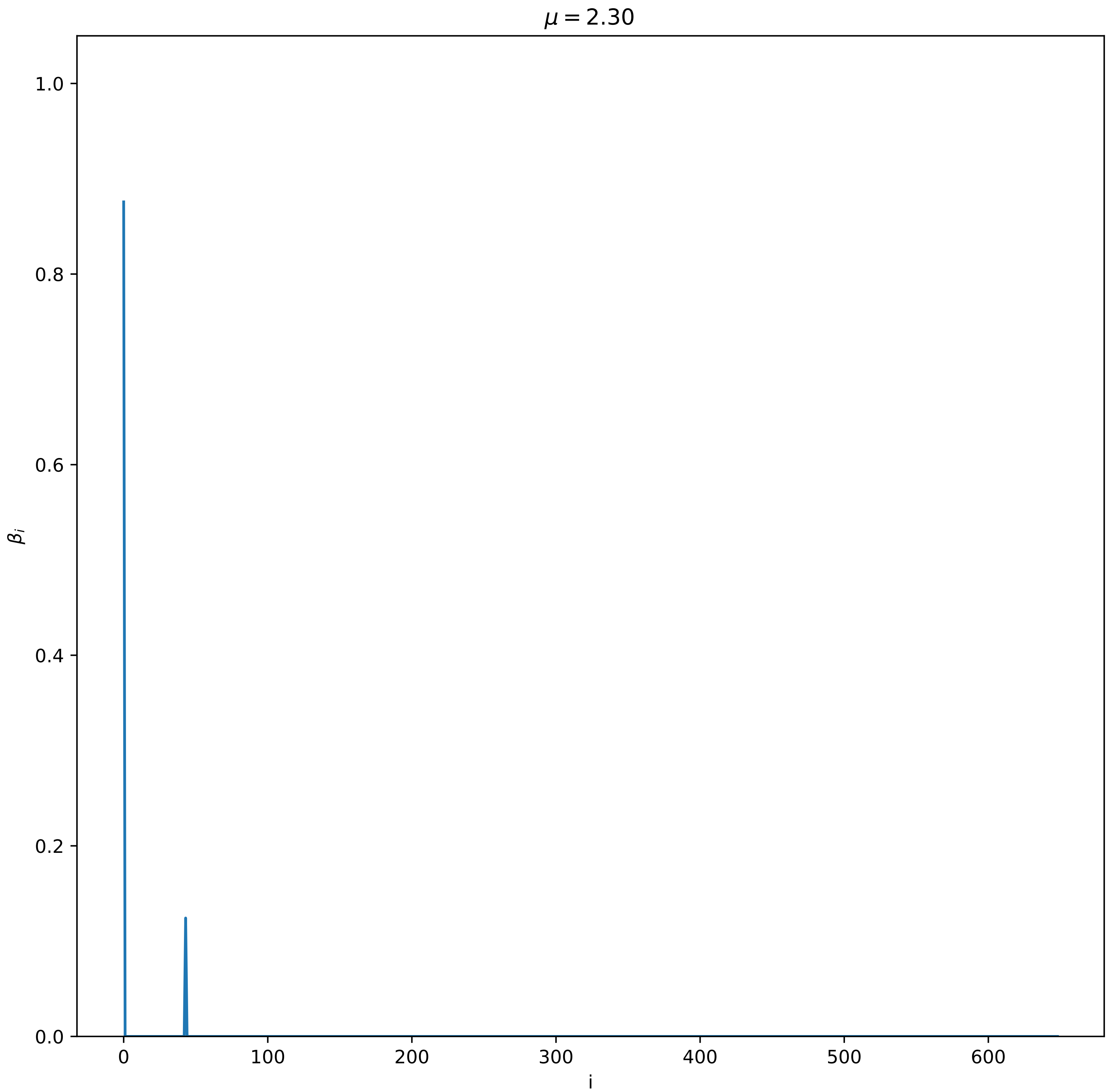
$\mu = 2.27$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



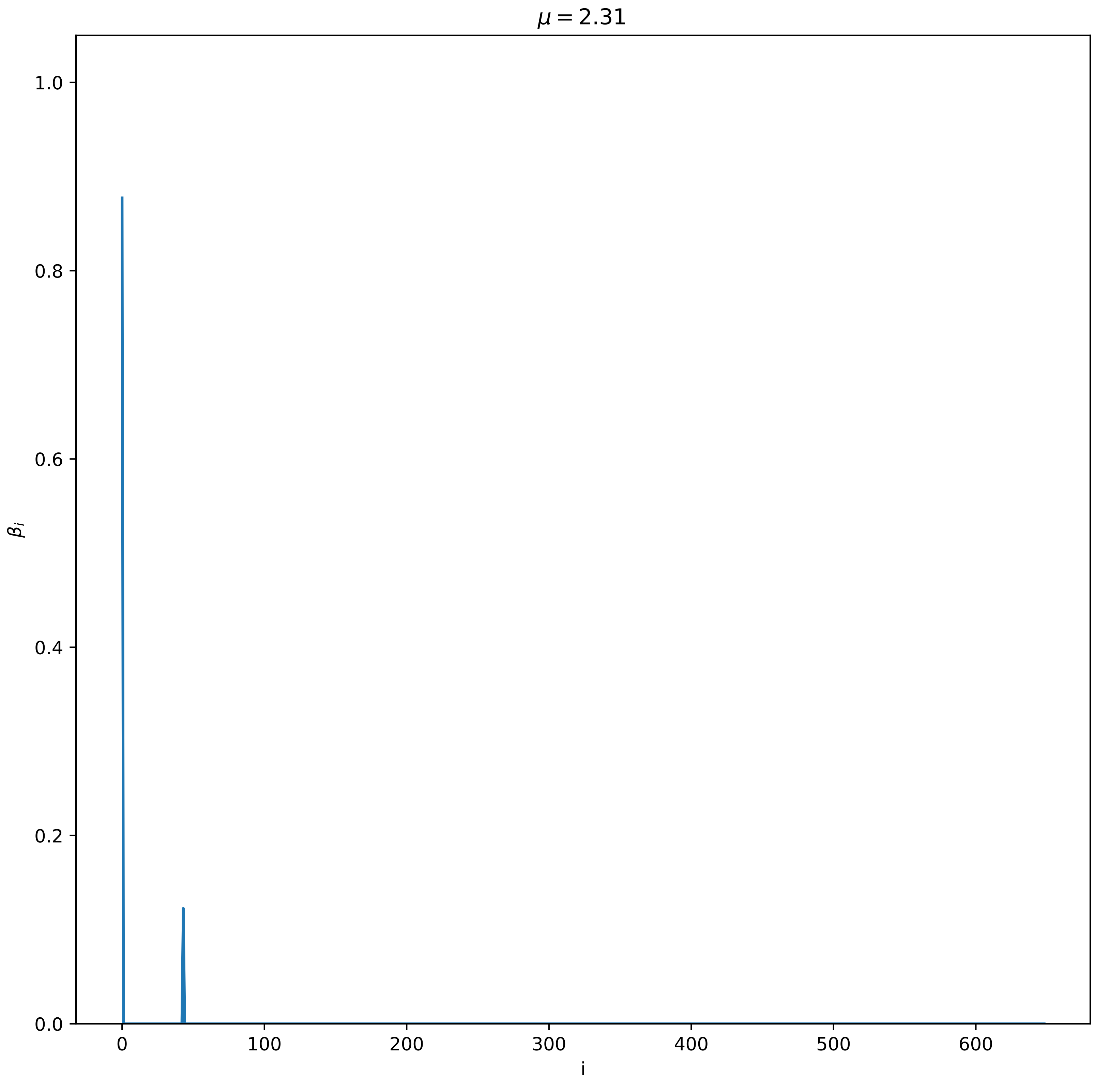
$\mu = 2.28$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

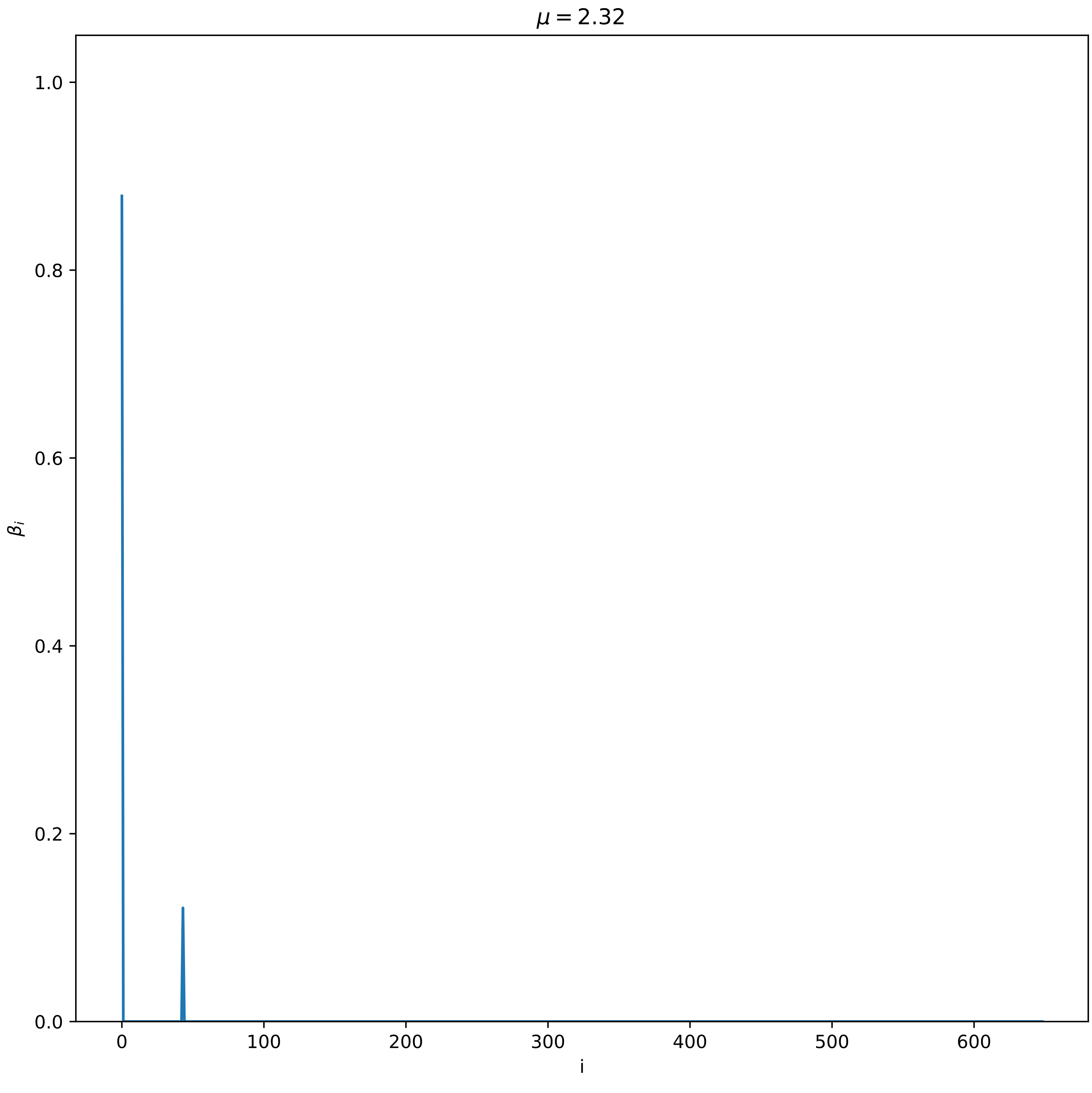


$\mu = 2.29$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

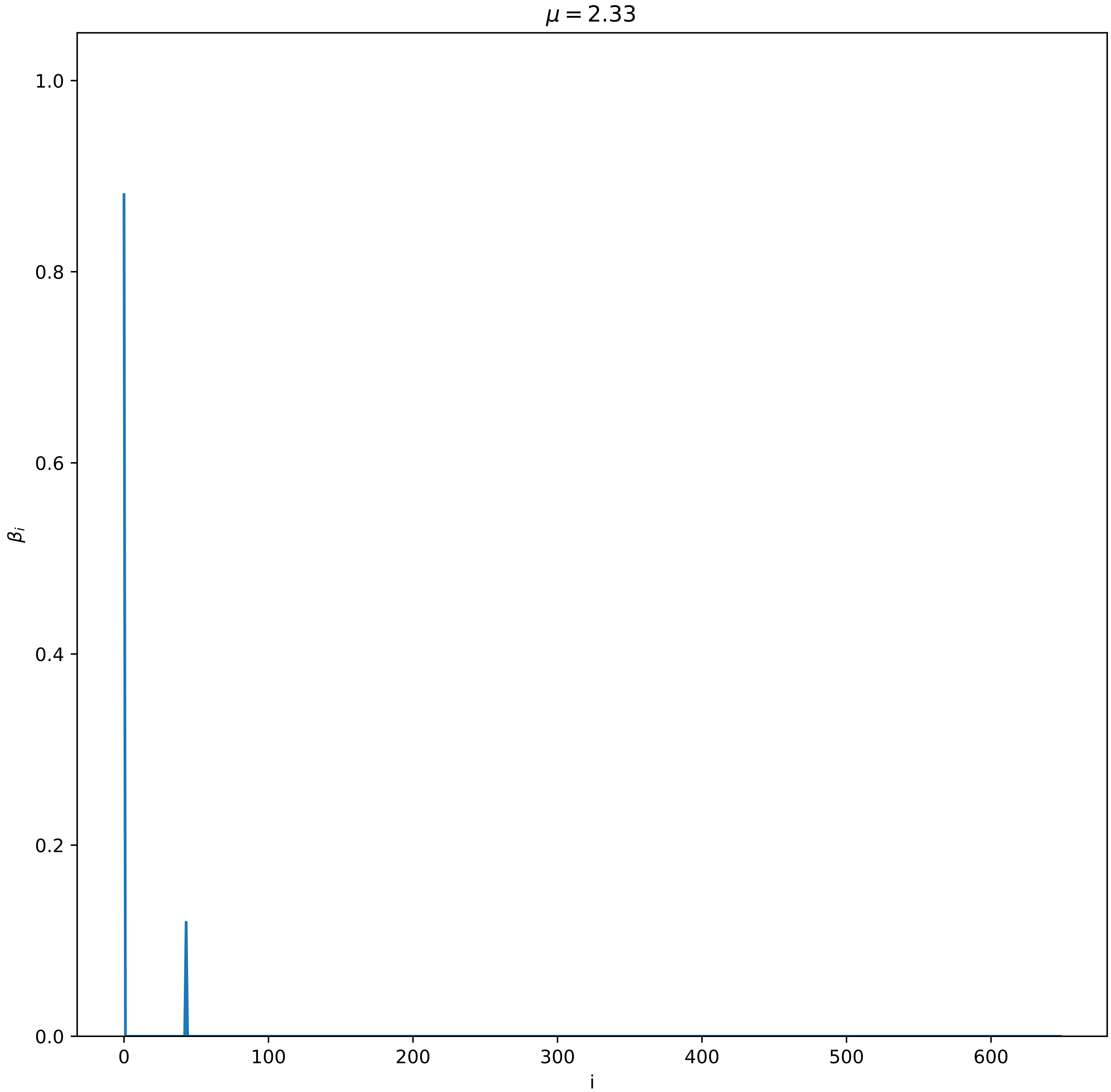


$\mu = 2.30$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

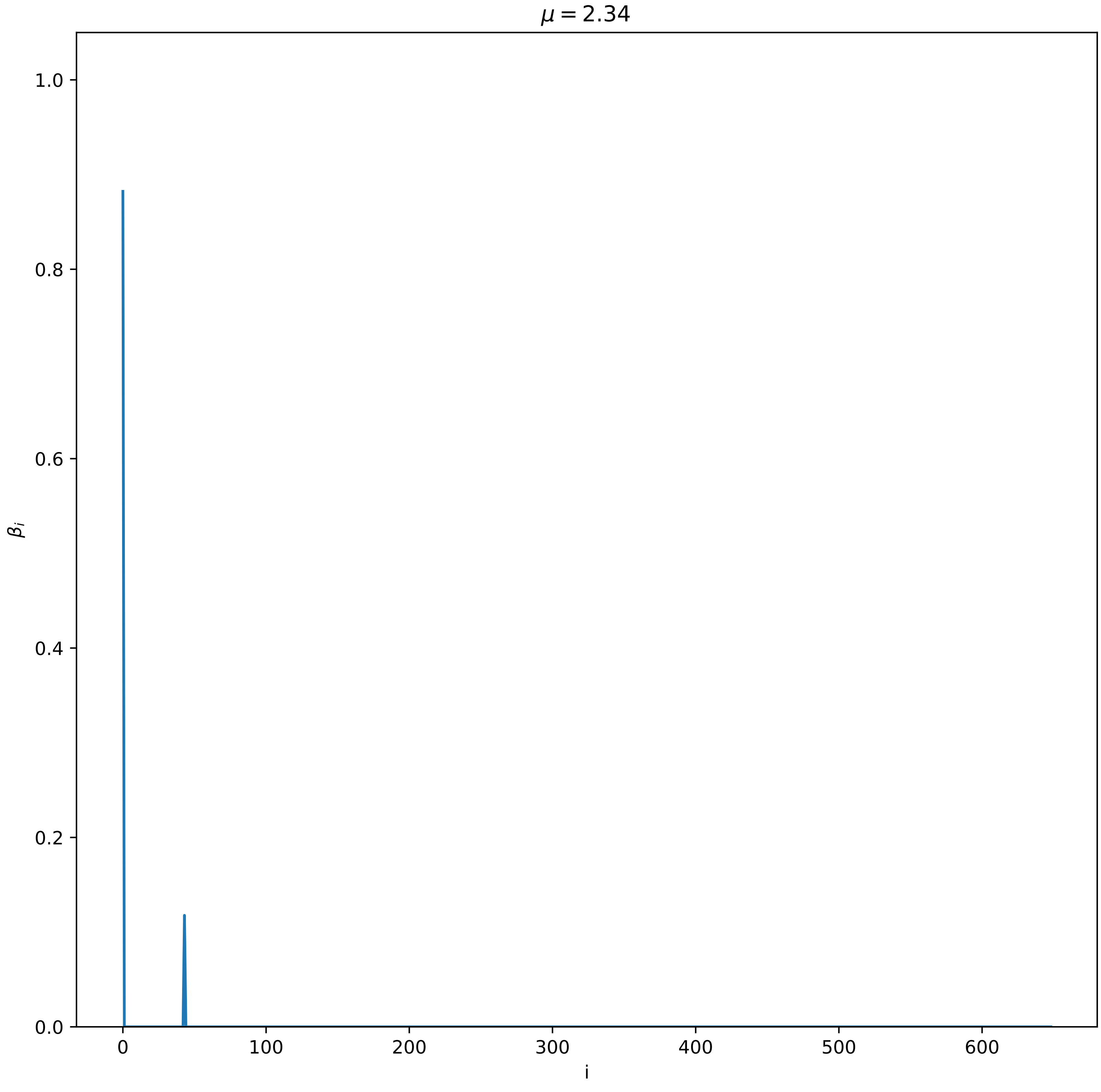

$\mu = 2.31$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



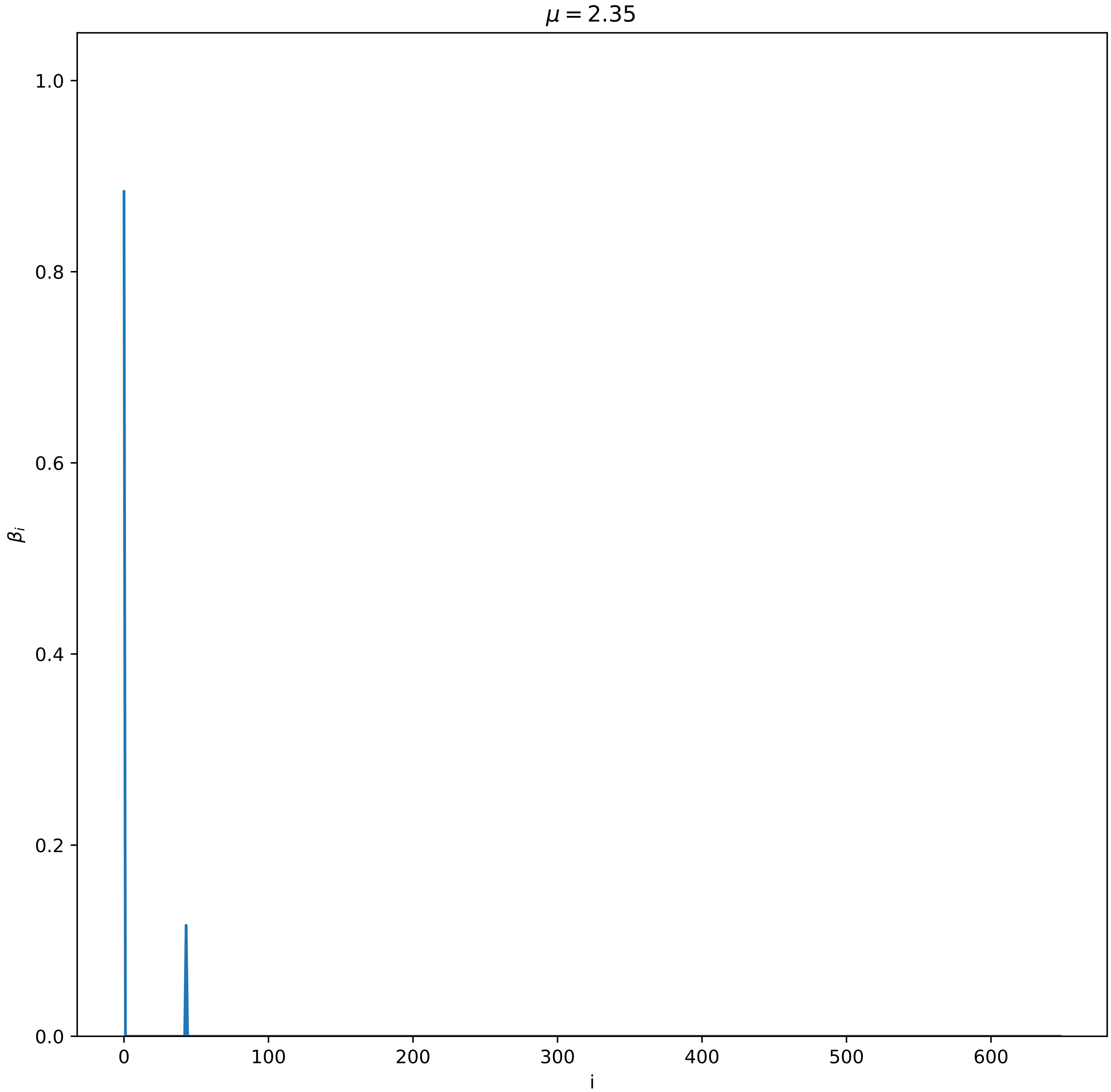
$\mu = 2.32$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



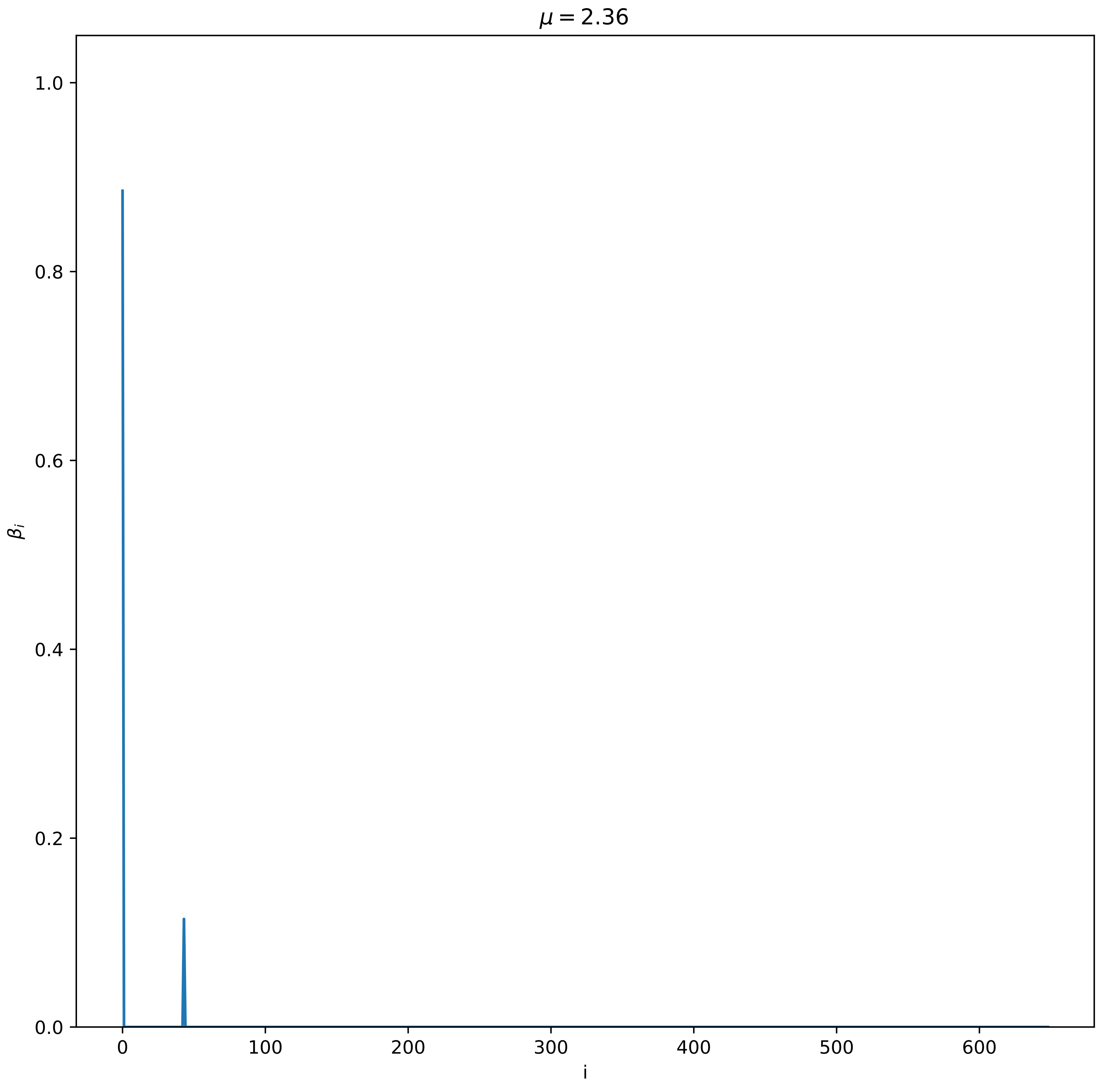
$\mu = 2.33$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



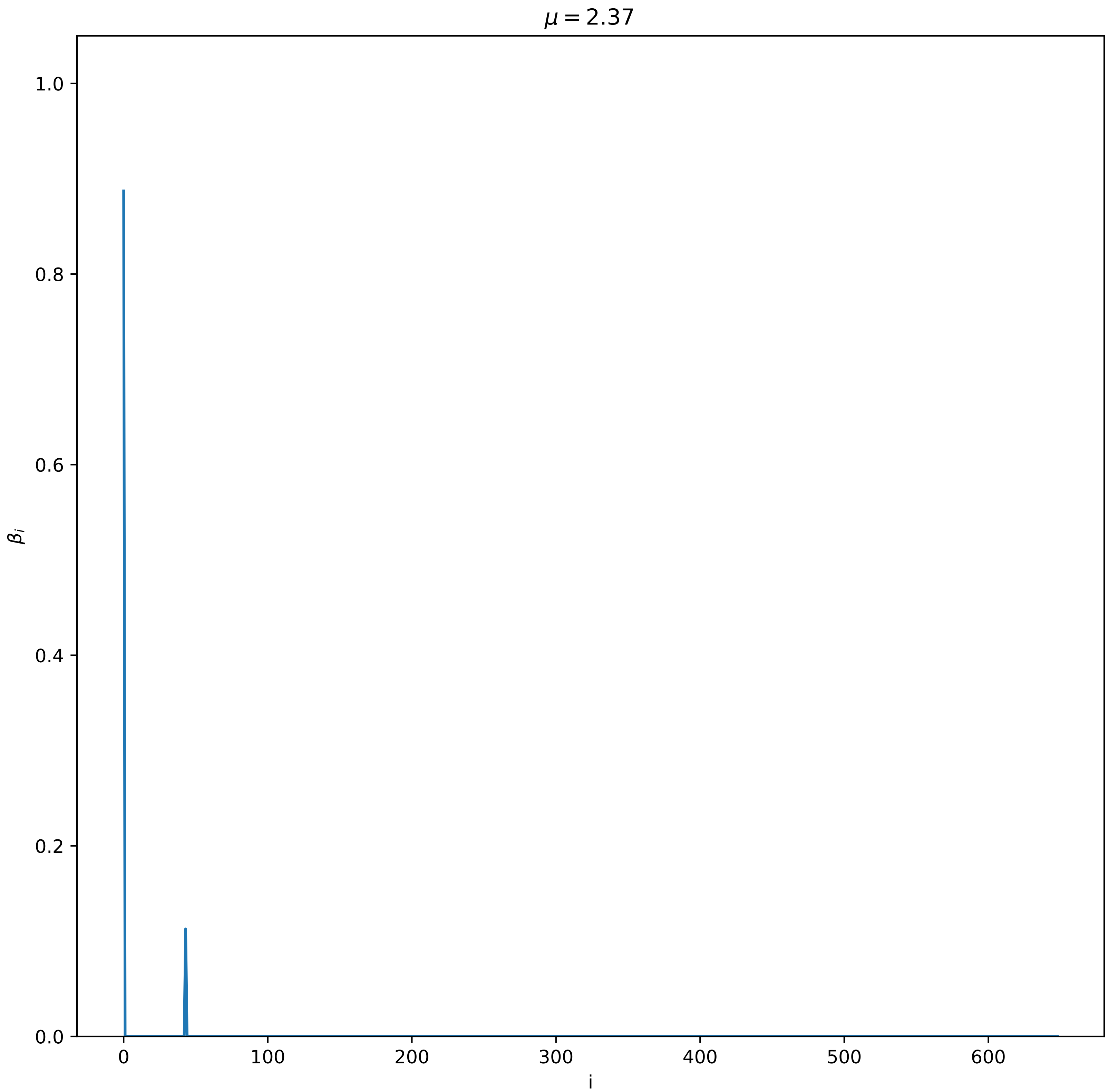
$\mu = 2.34$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



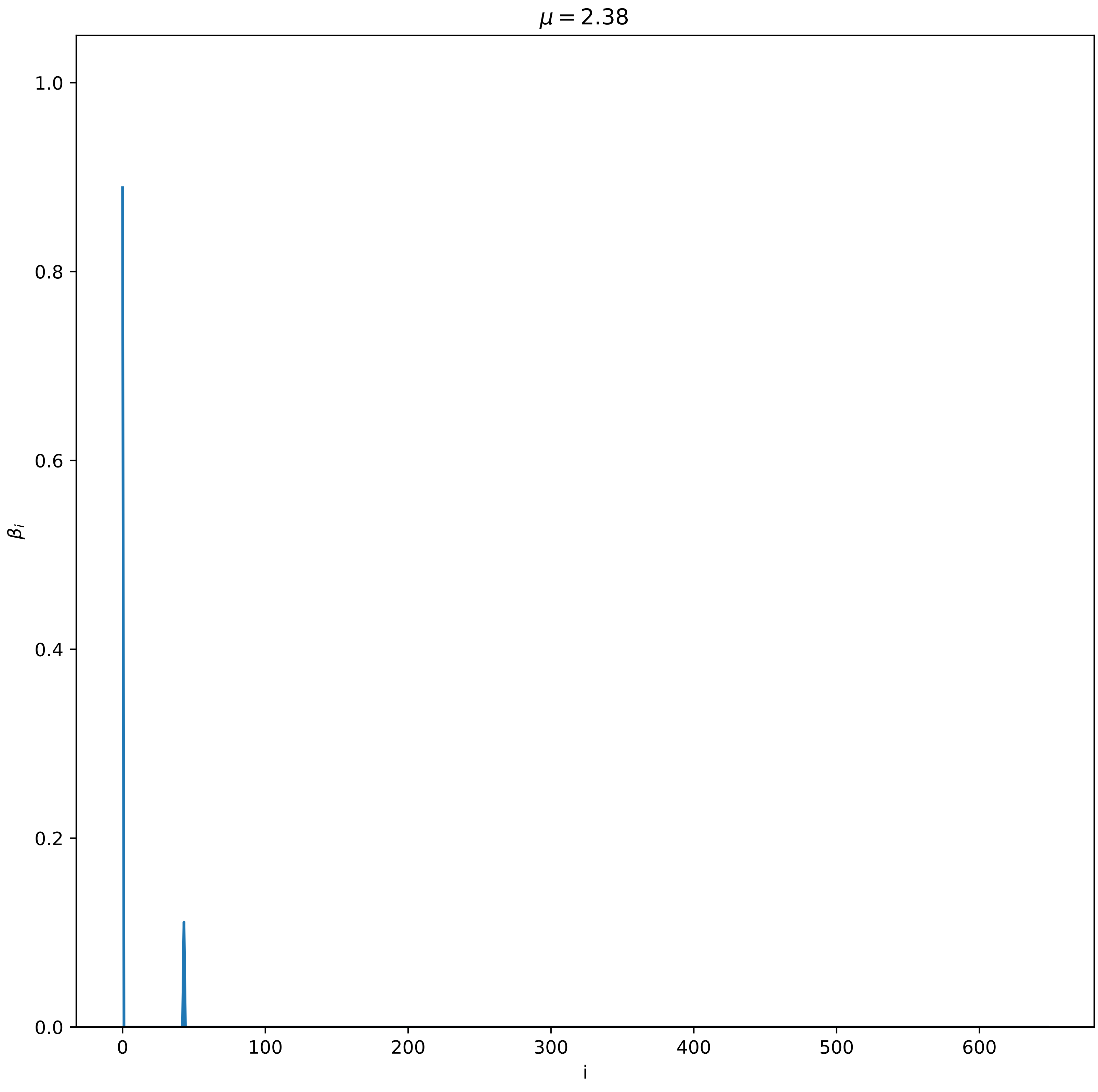
$\mu = 2.35$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



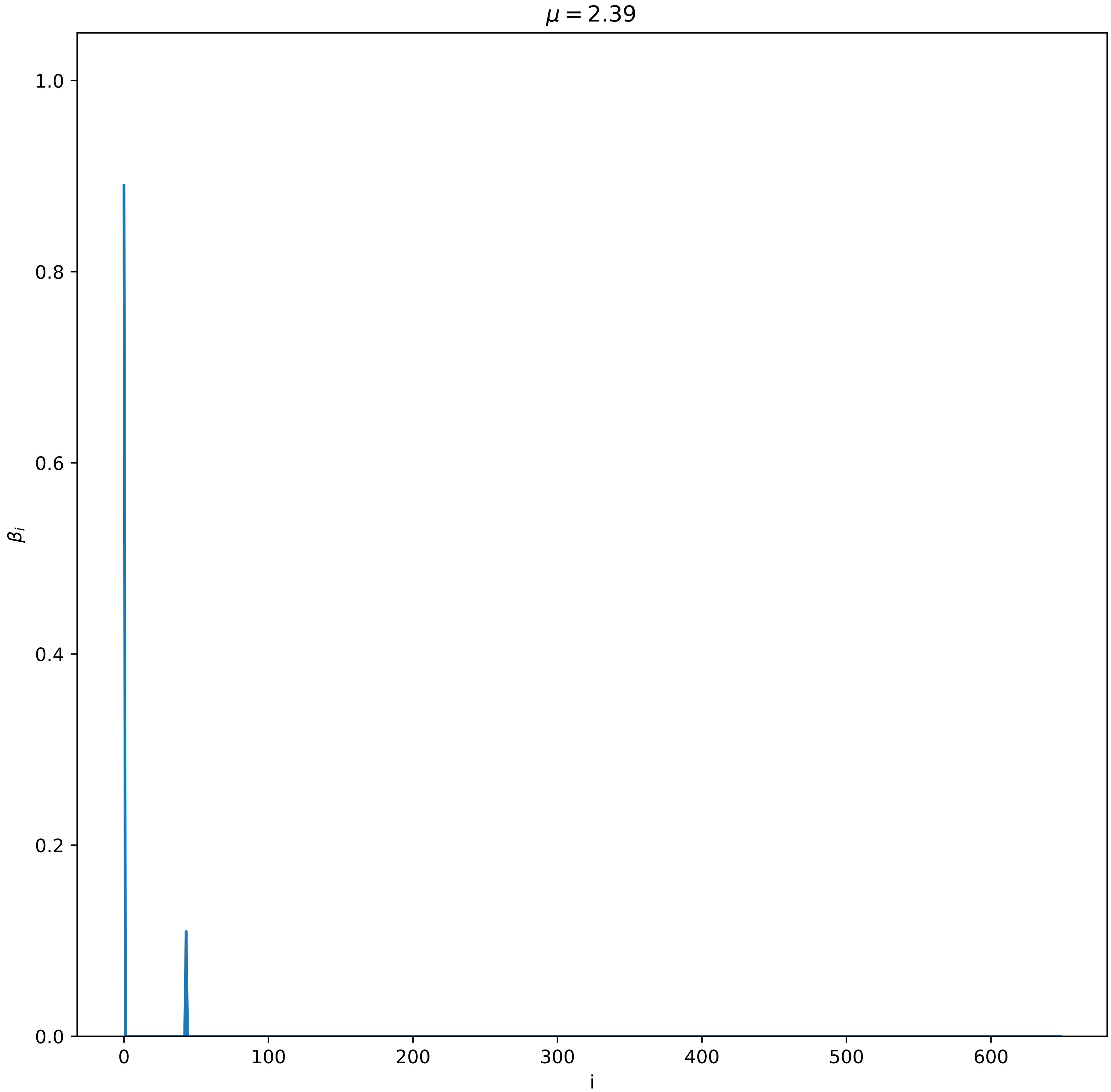
$\mu = 2.36$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

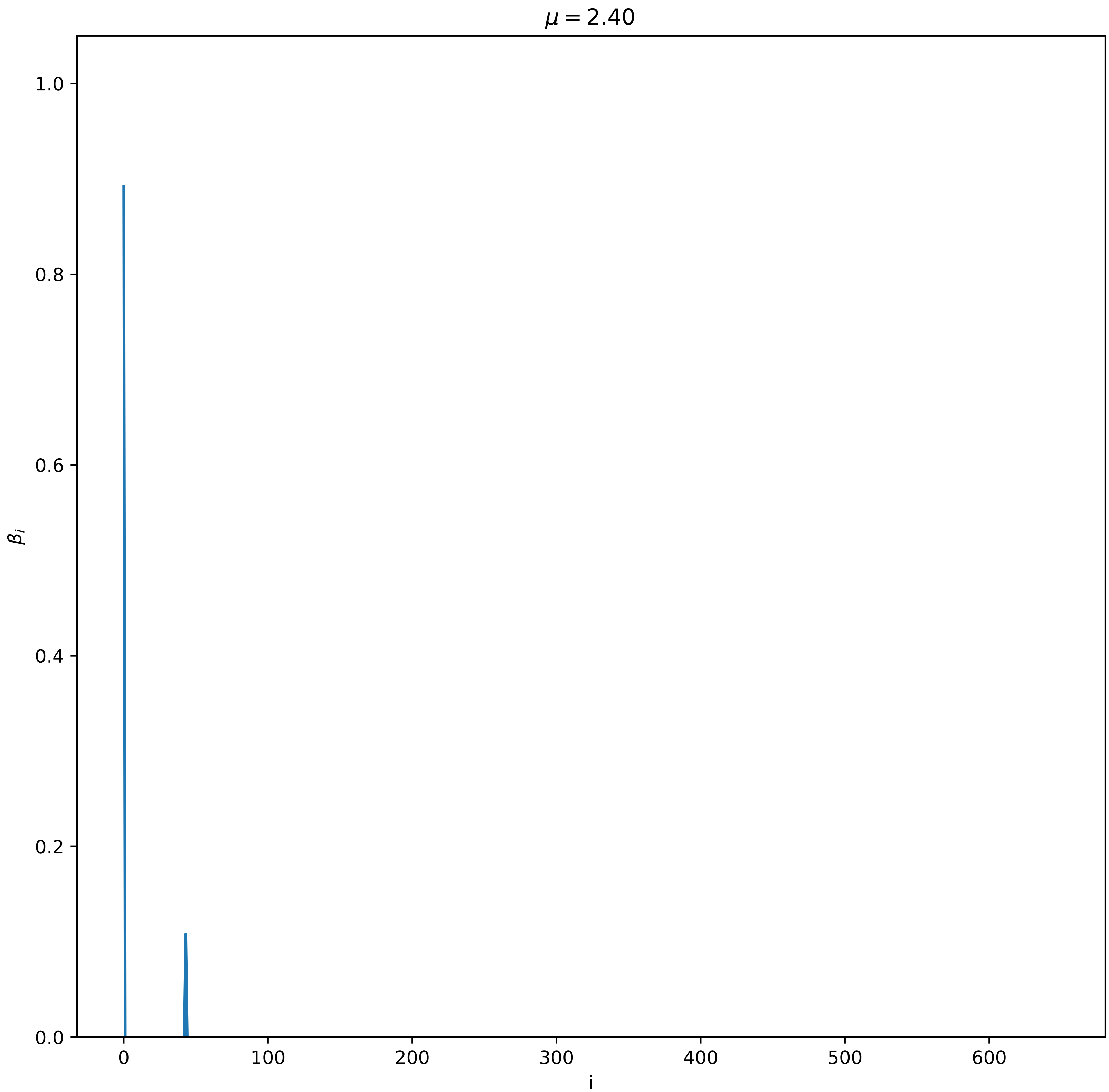


$\mu = 2.37$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

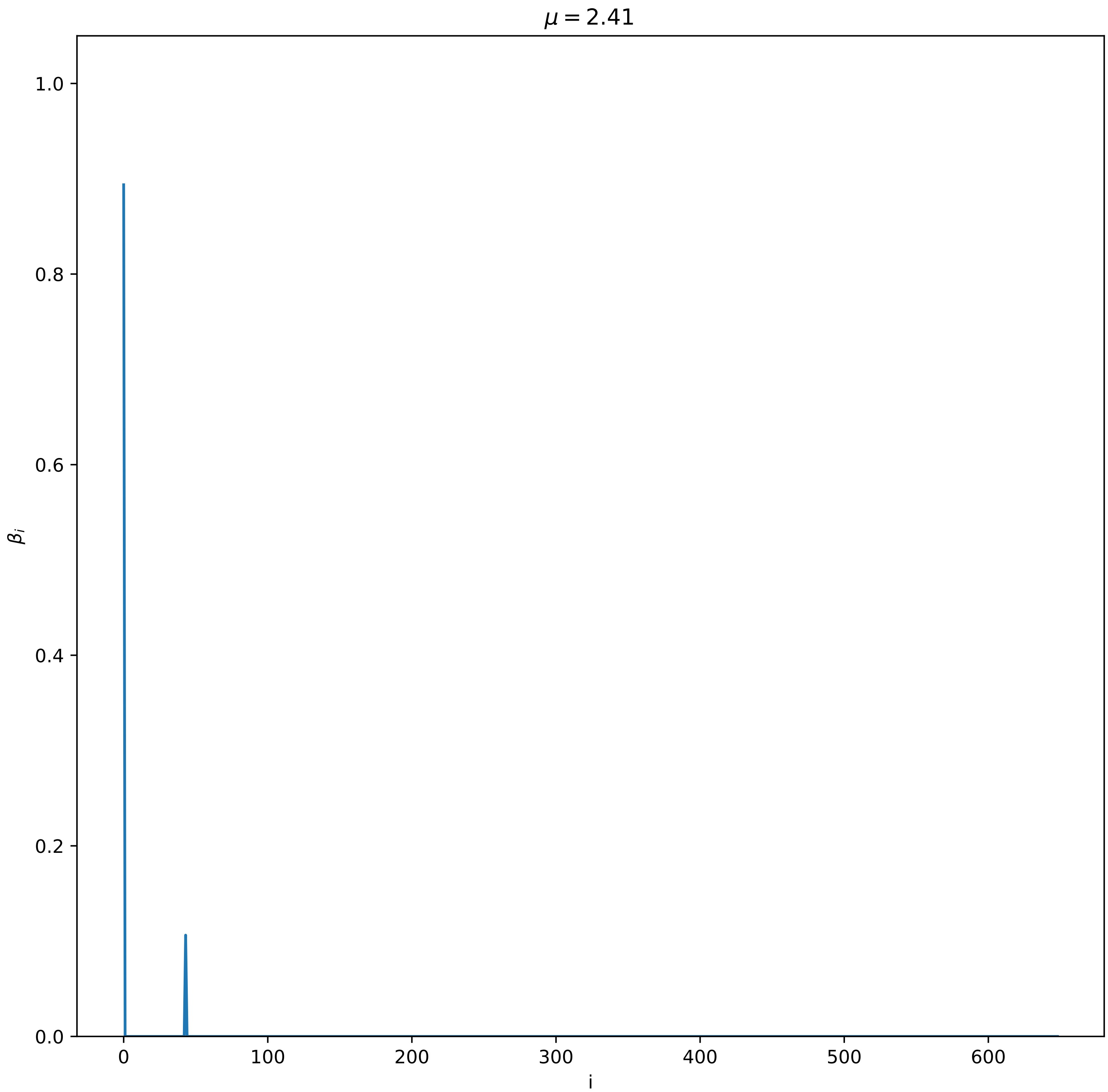


$\mu = 2.38$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

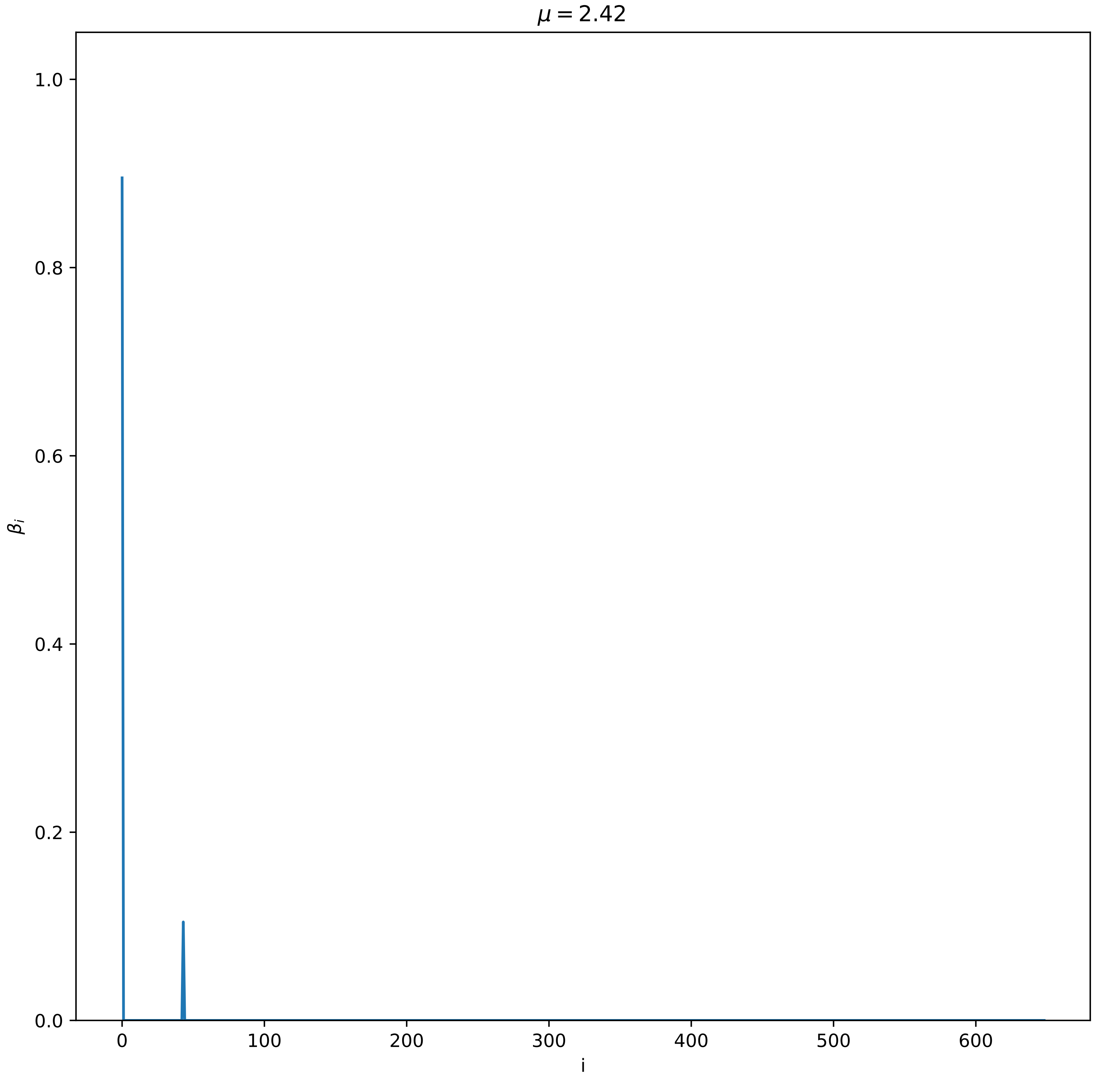

$\mu = 2.39$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



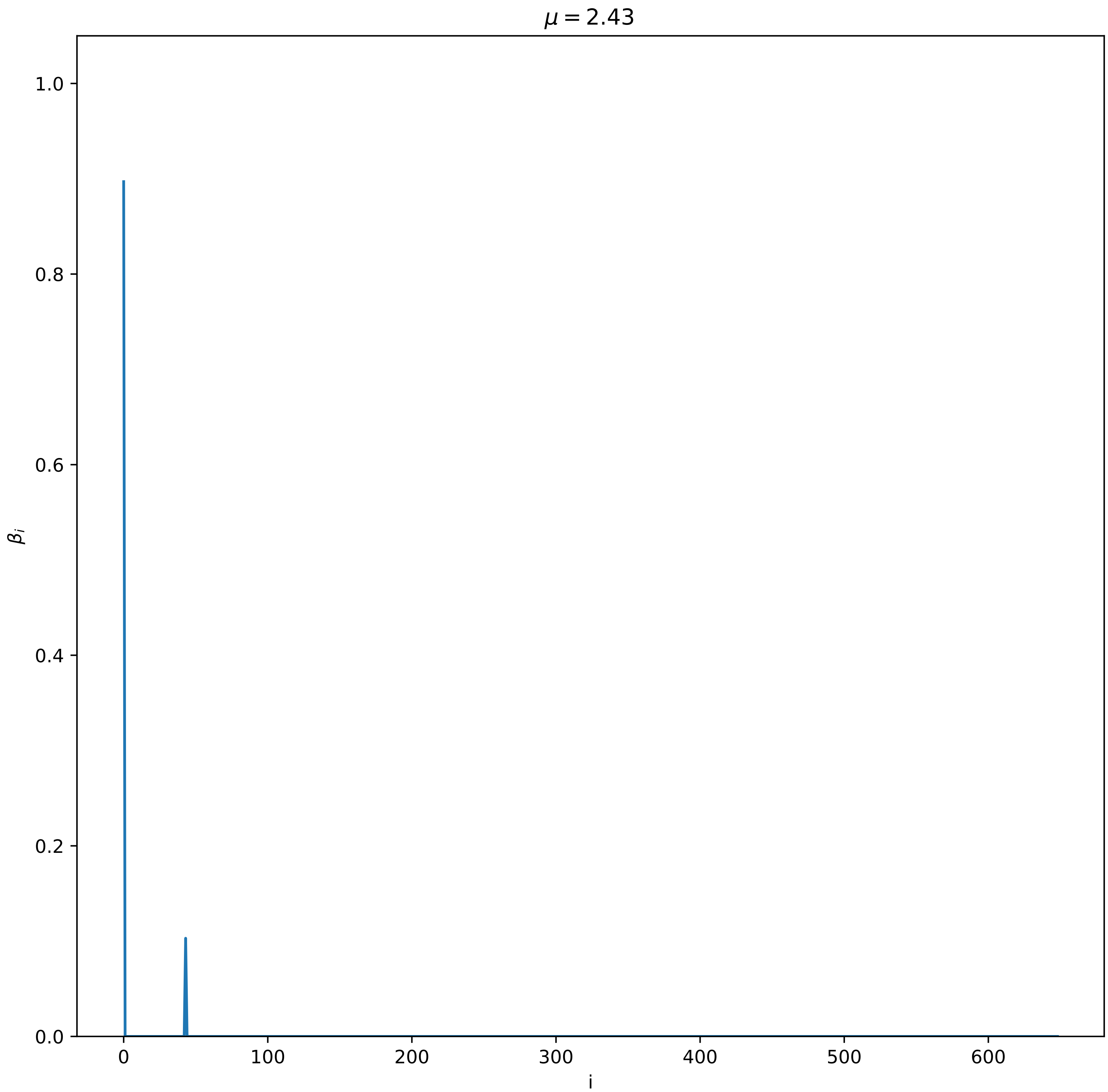
$\mu = 2.40$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

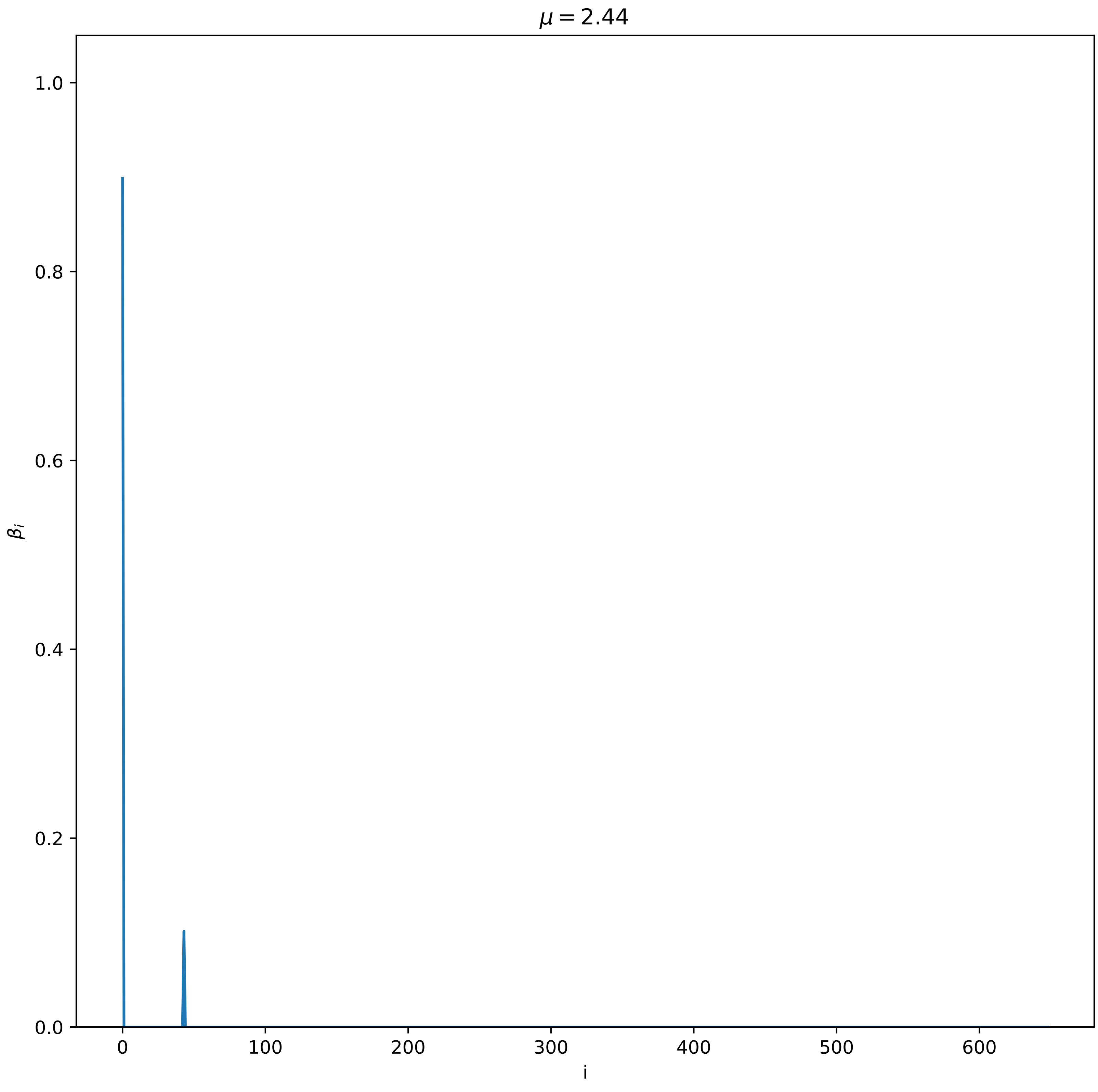


$\mu = 2.41$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

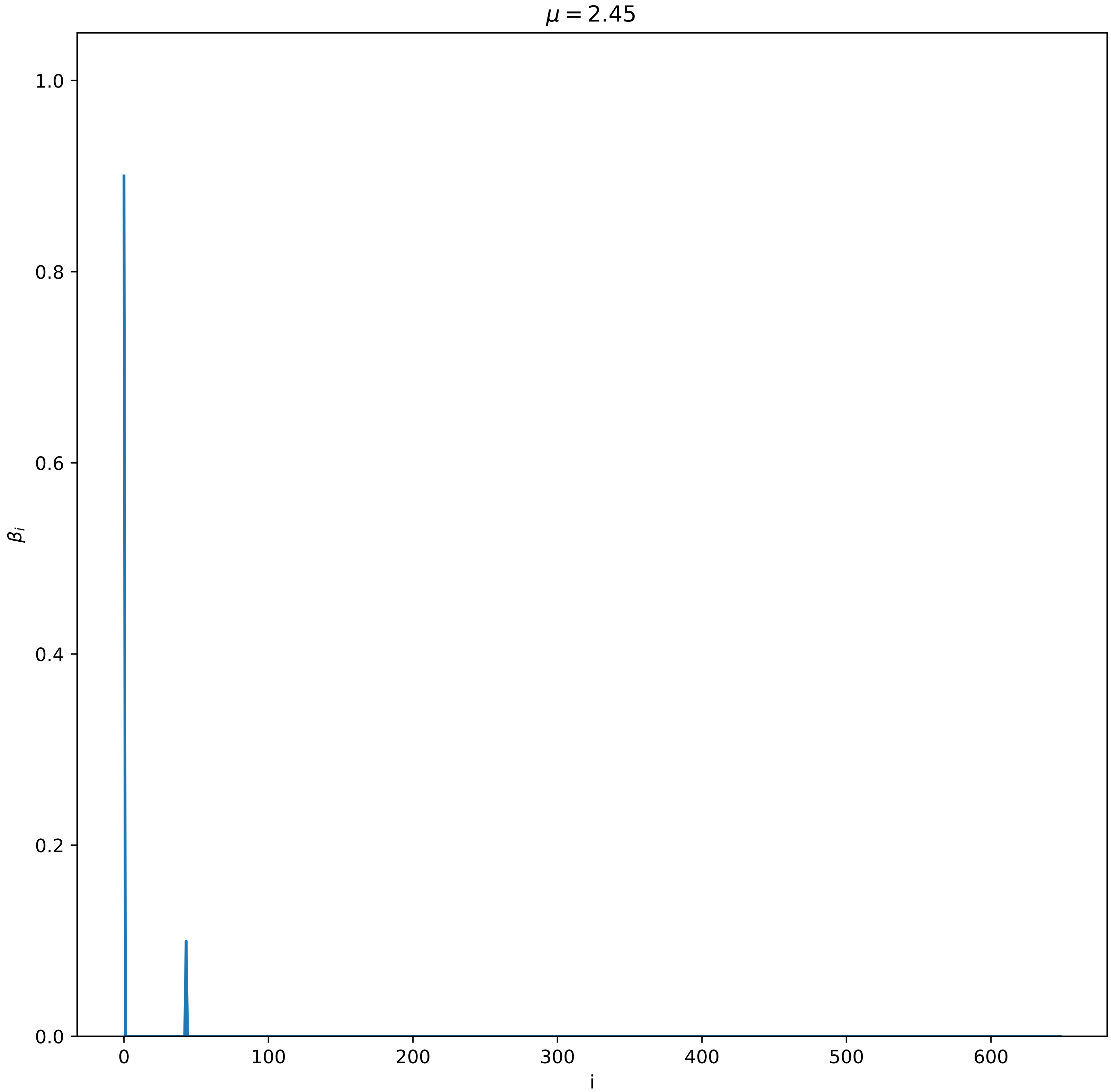


$\mu = 2.42$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

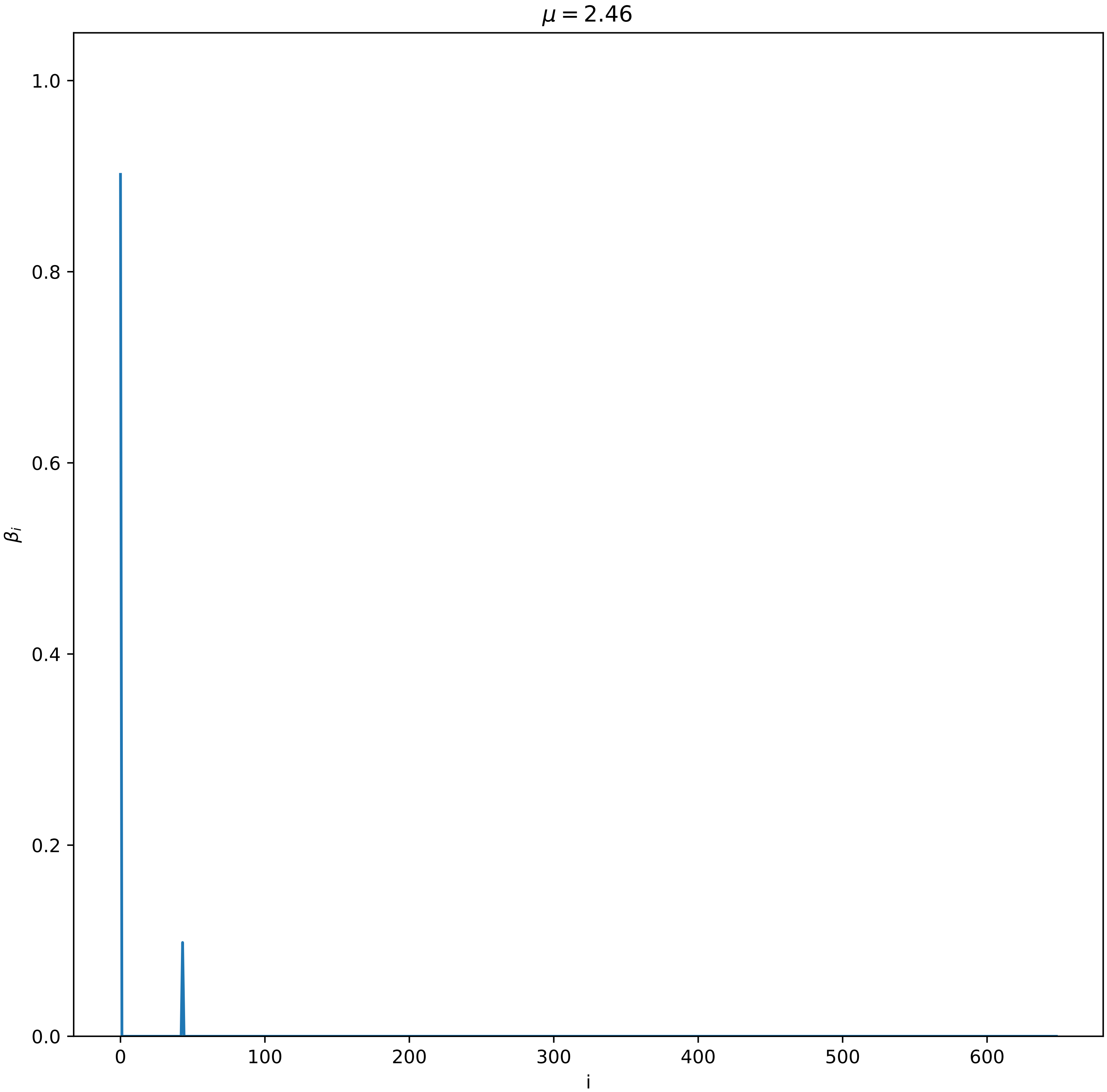

$\mu = 2.43$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



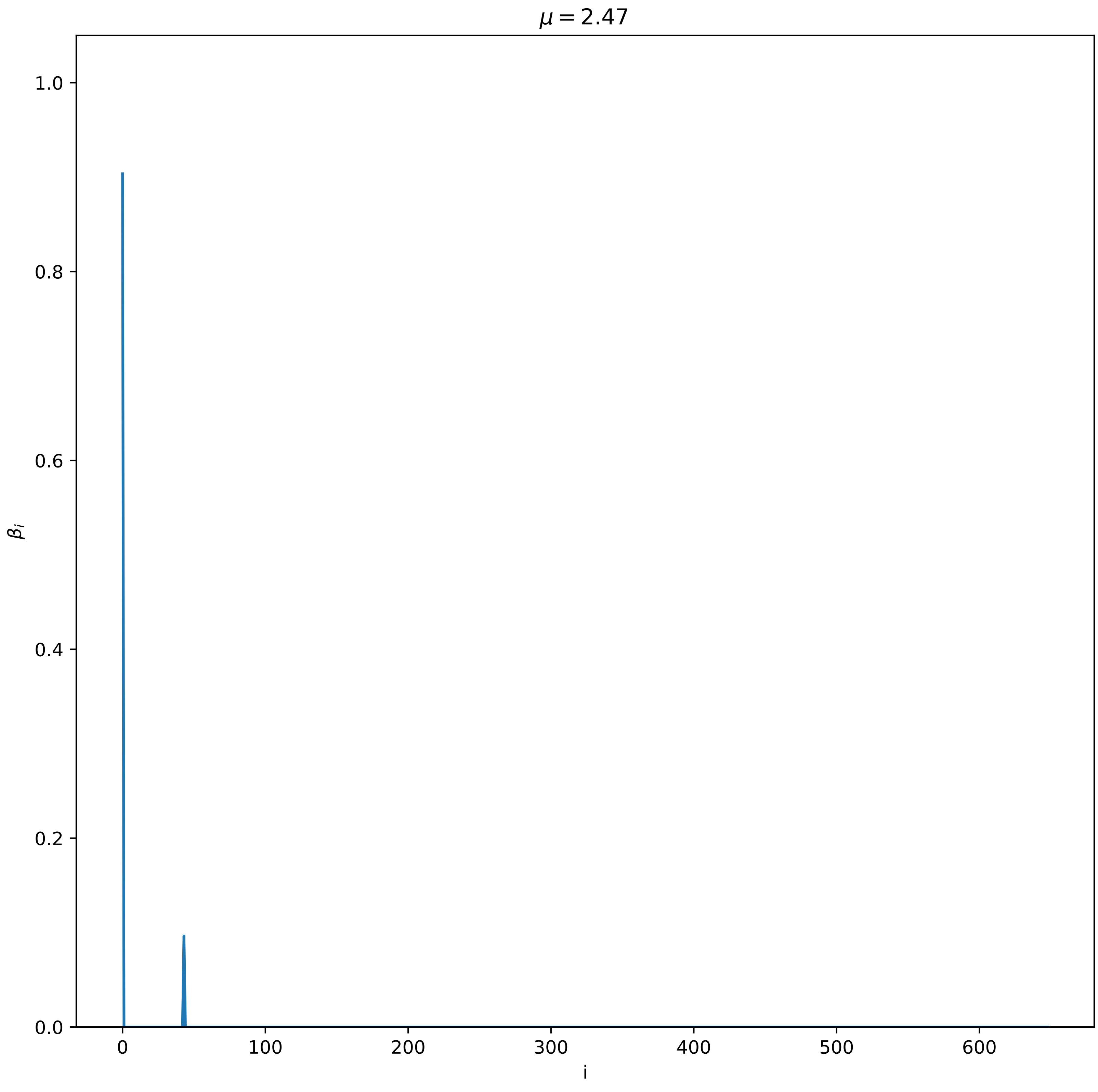
$\mu = 2.44$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



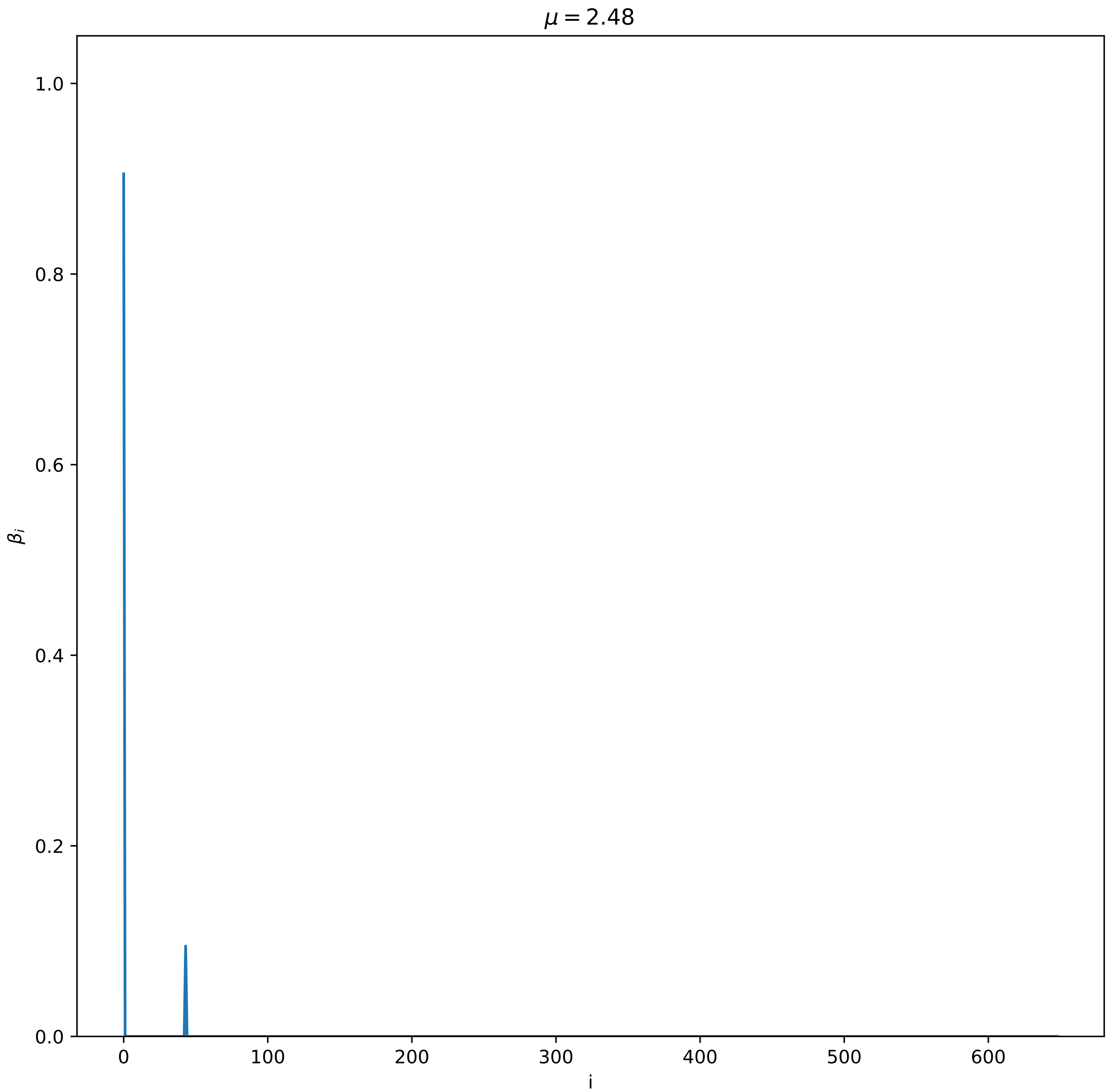
$\mu = 2.45$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



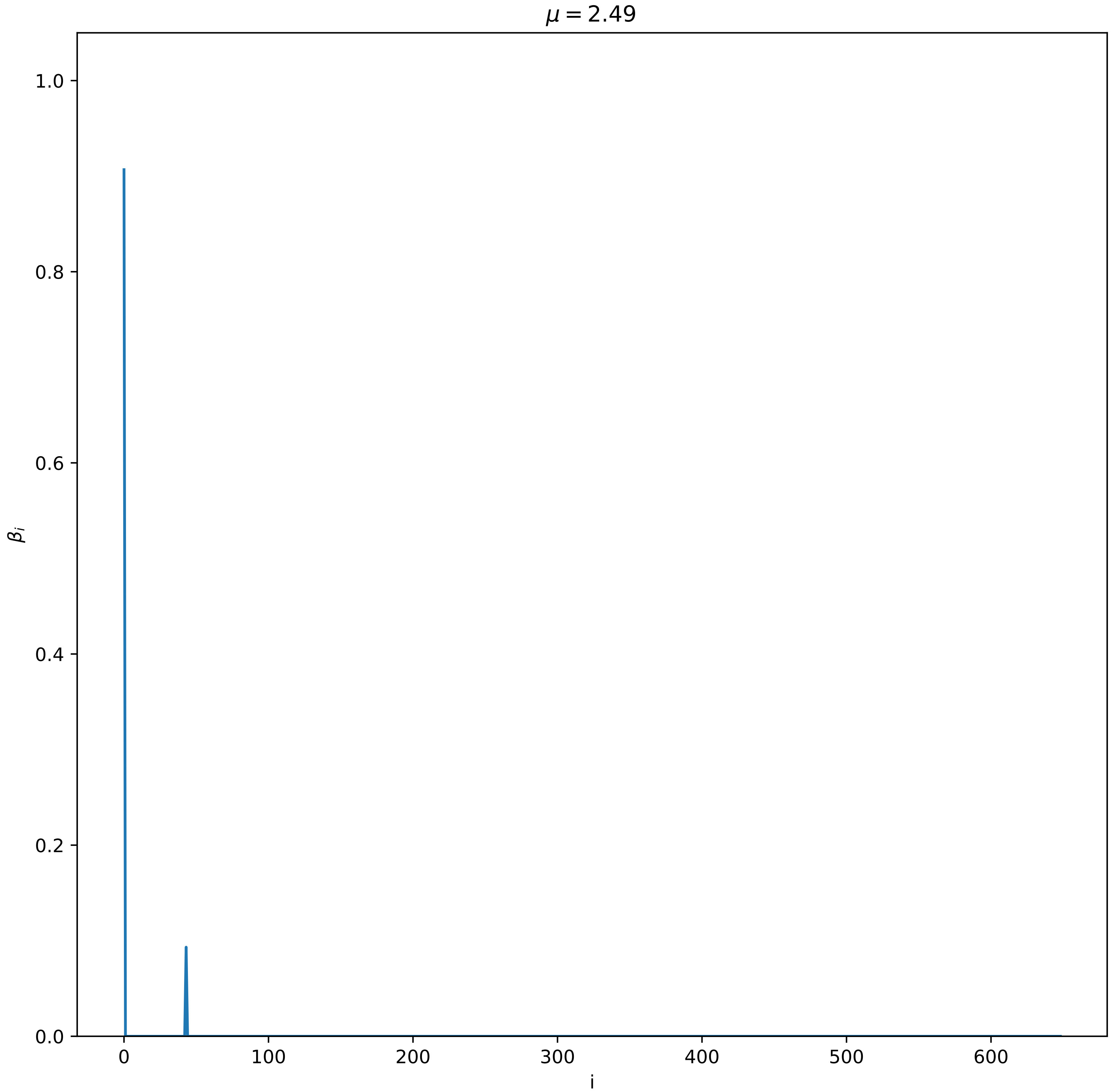
$\mu = 2.46$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

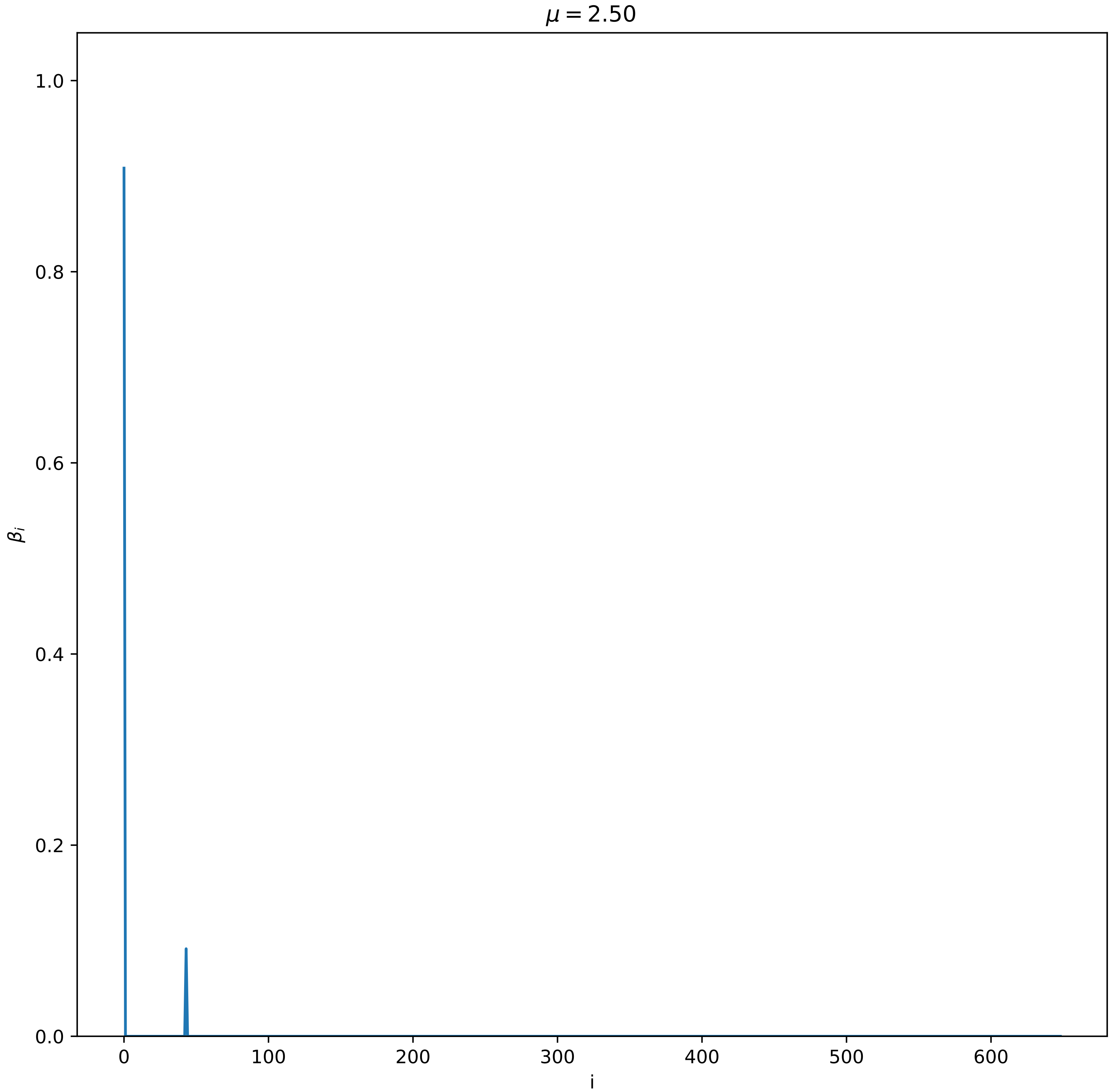


$\mu = 2.47$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

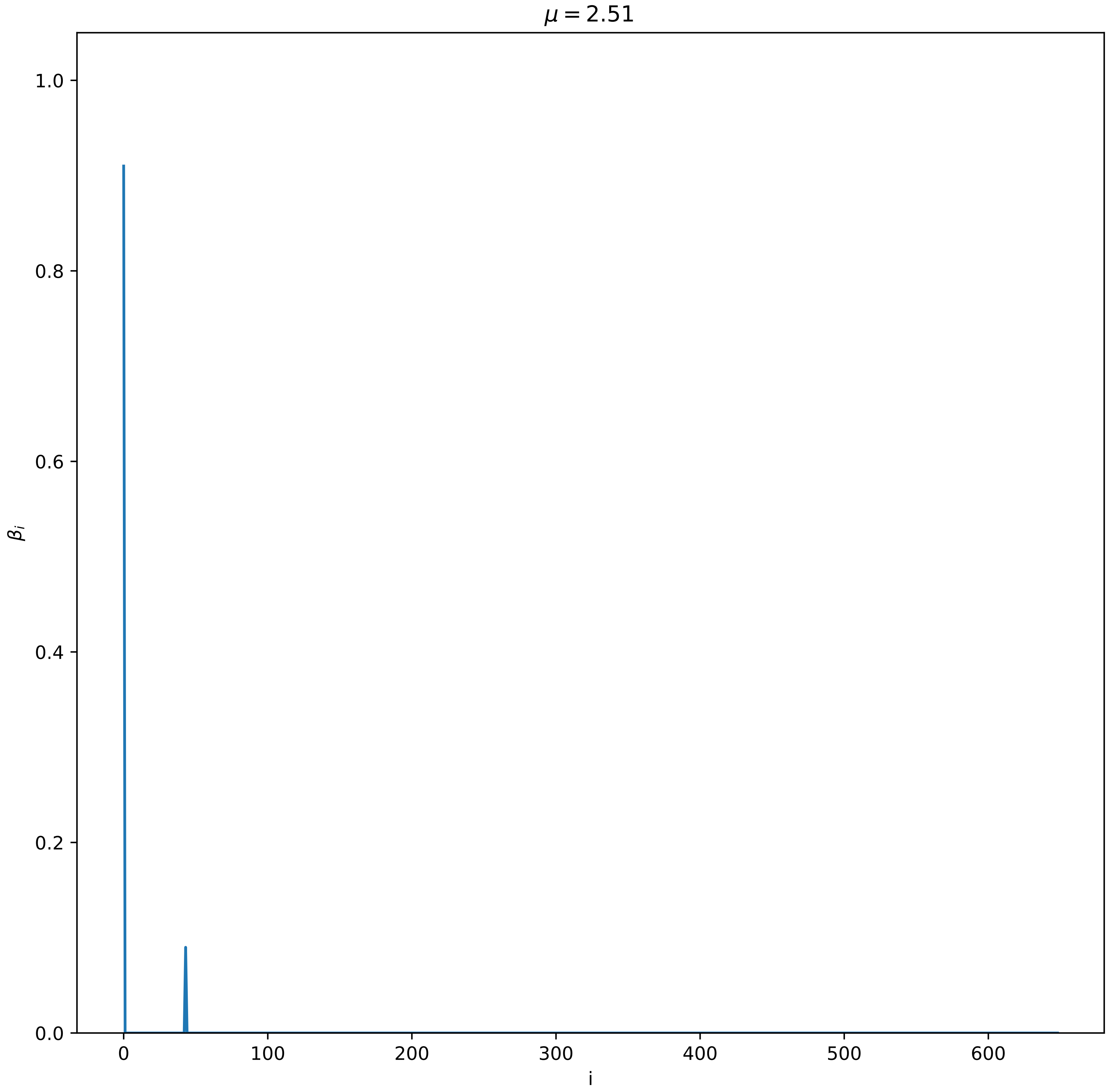


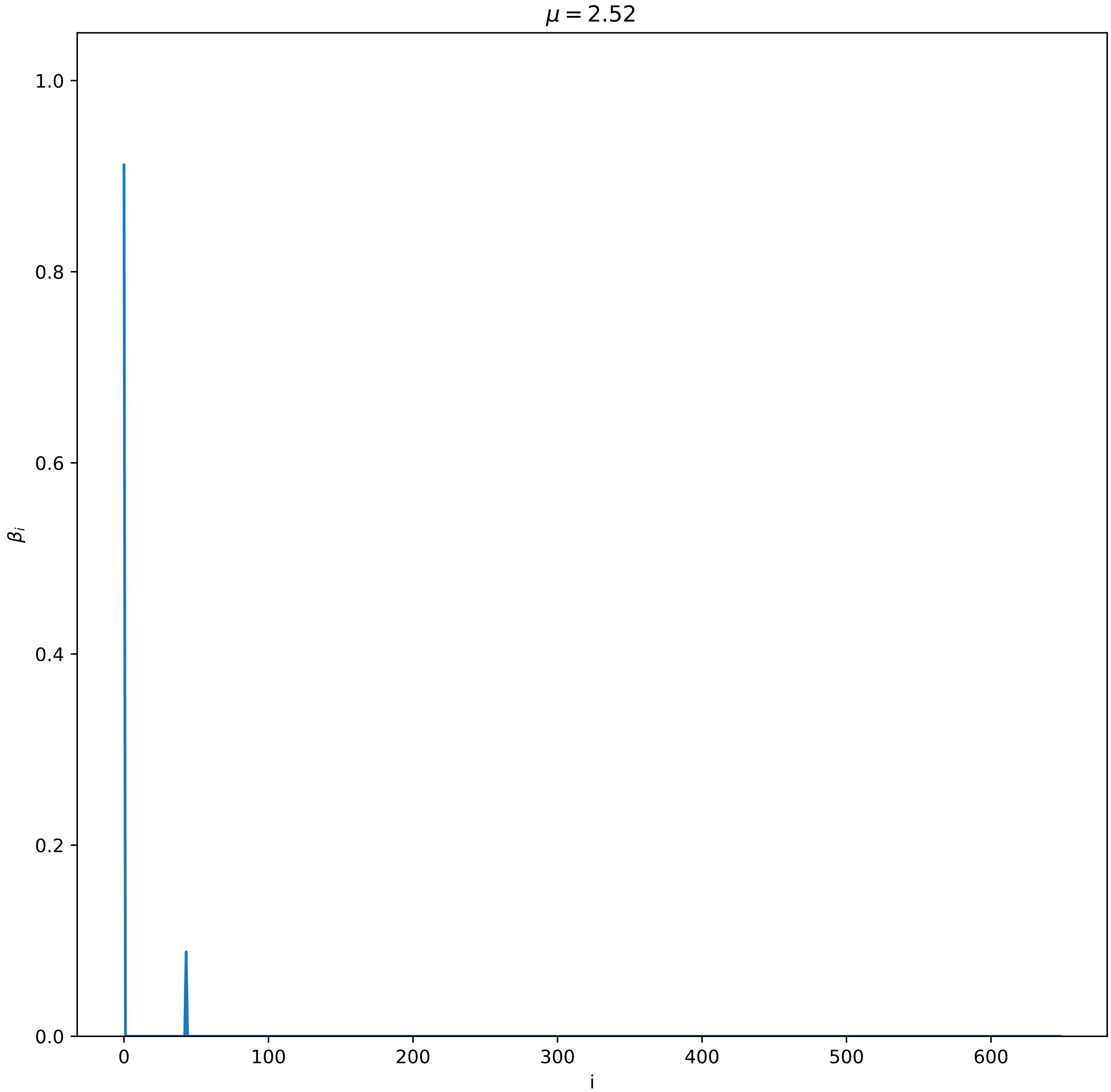
$\mu = 2.48$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

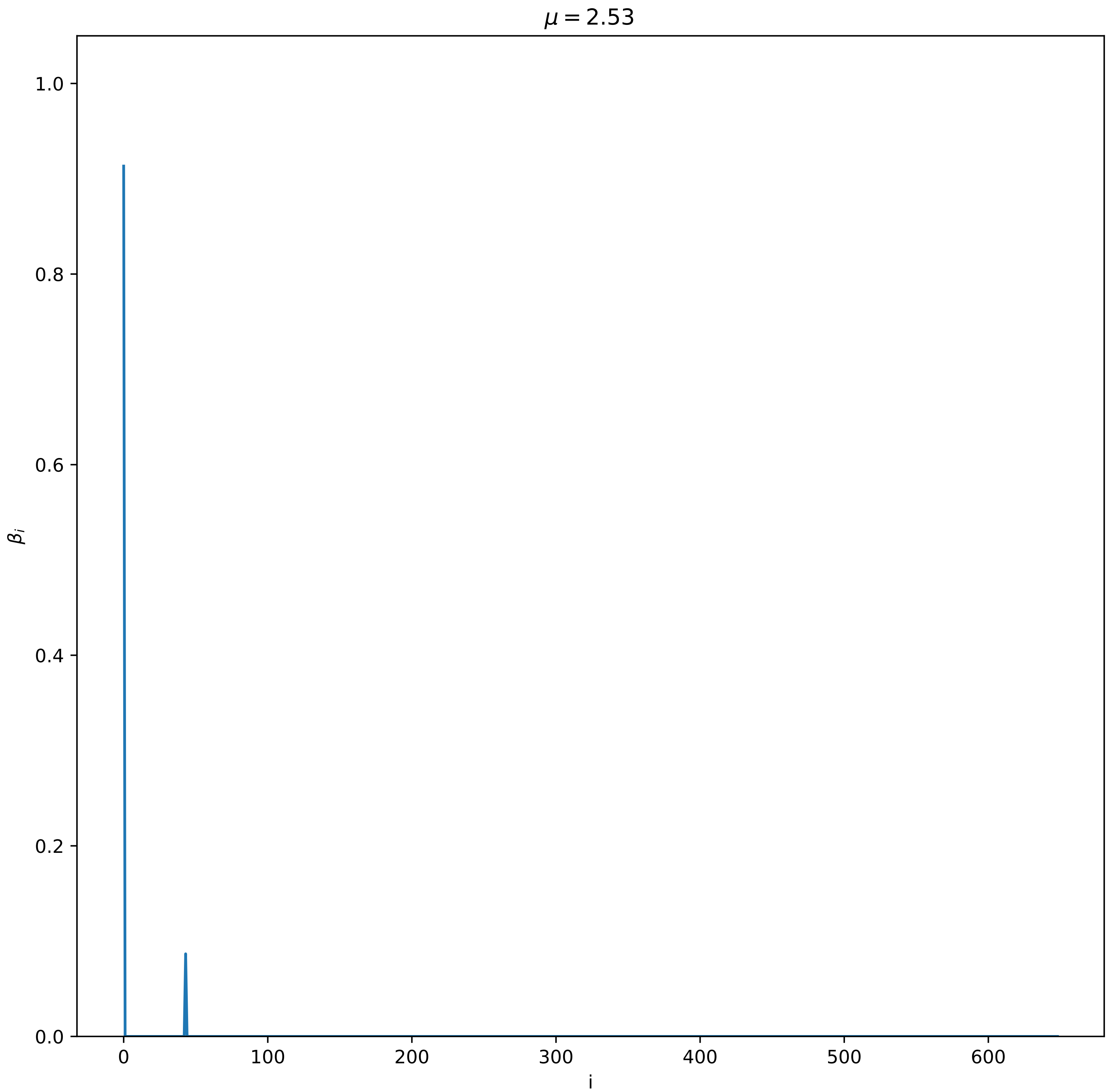

$\mu = 2.49$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

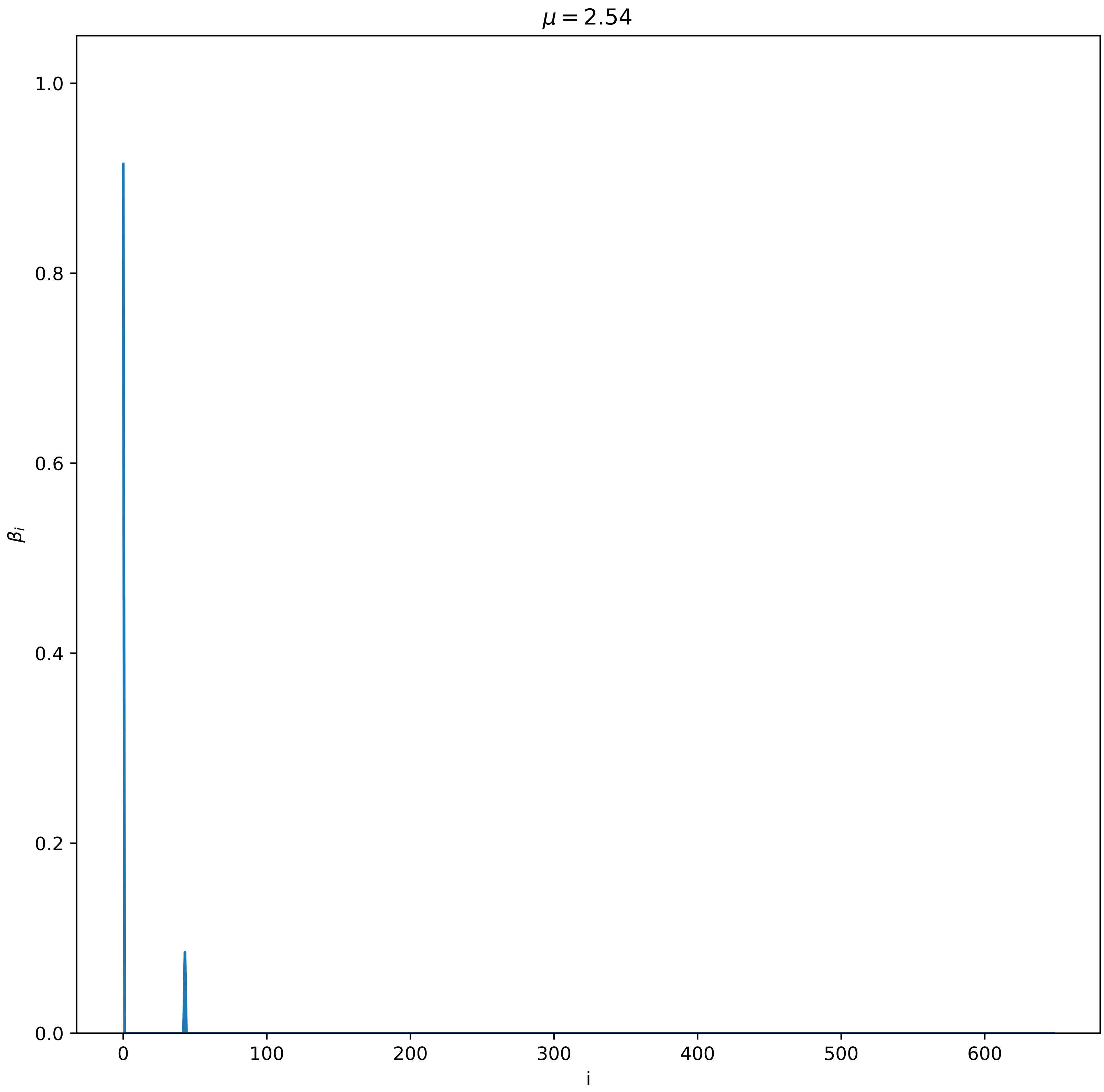


$\mu = 2.50$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

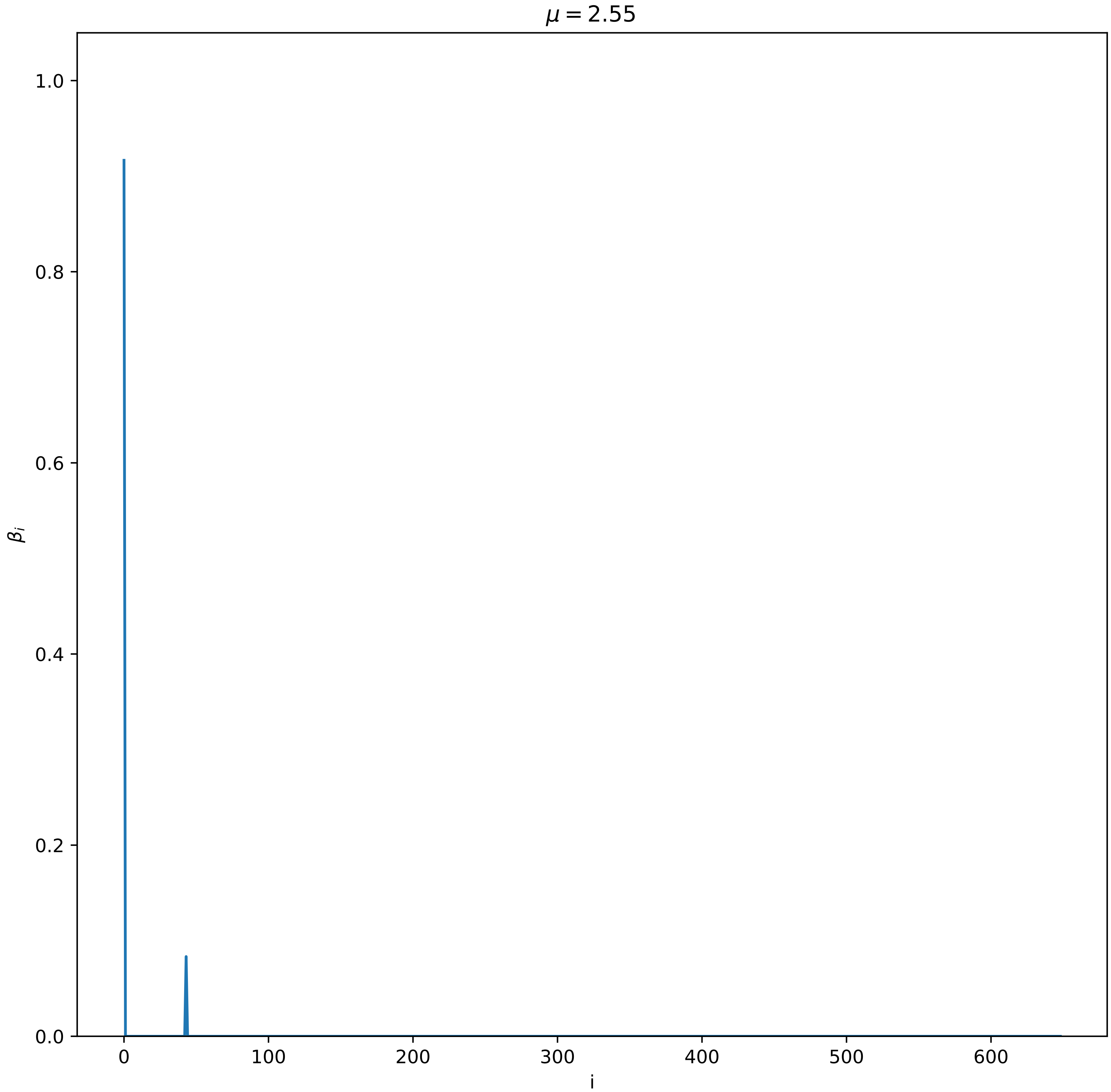

$\mu = 2.51$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 2.52$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

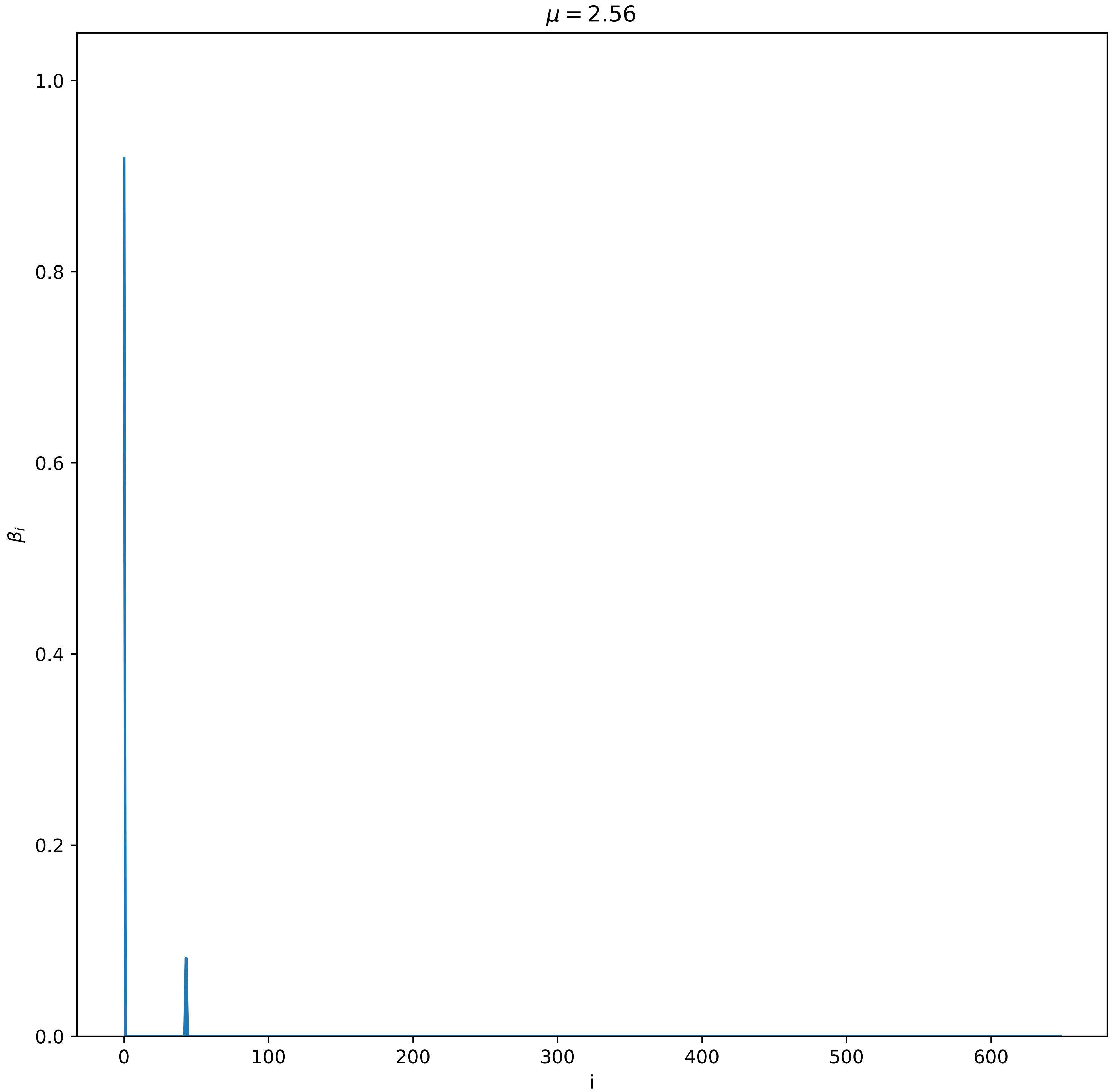

$\mu = 2.53$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

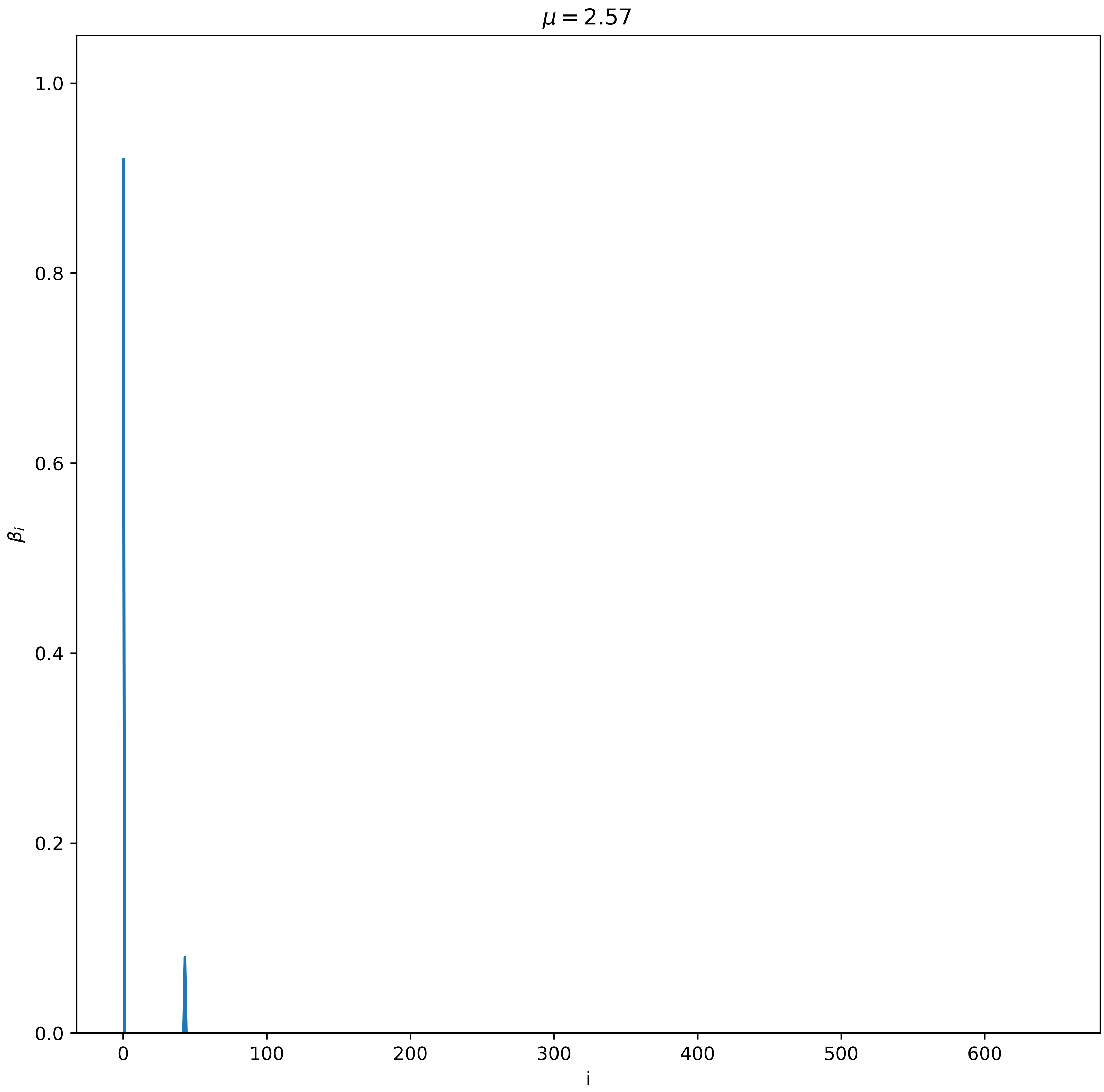


$\mu = 2.54$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

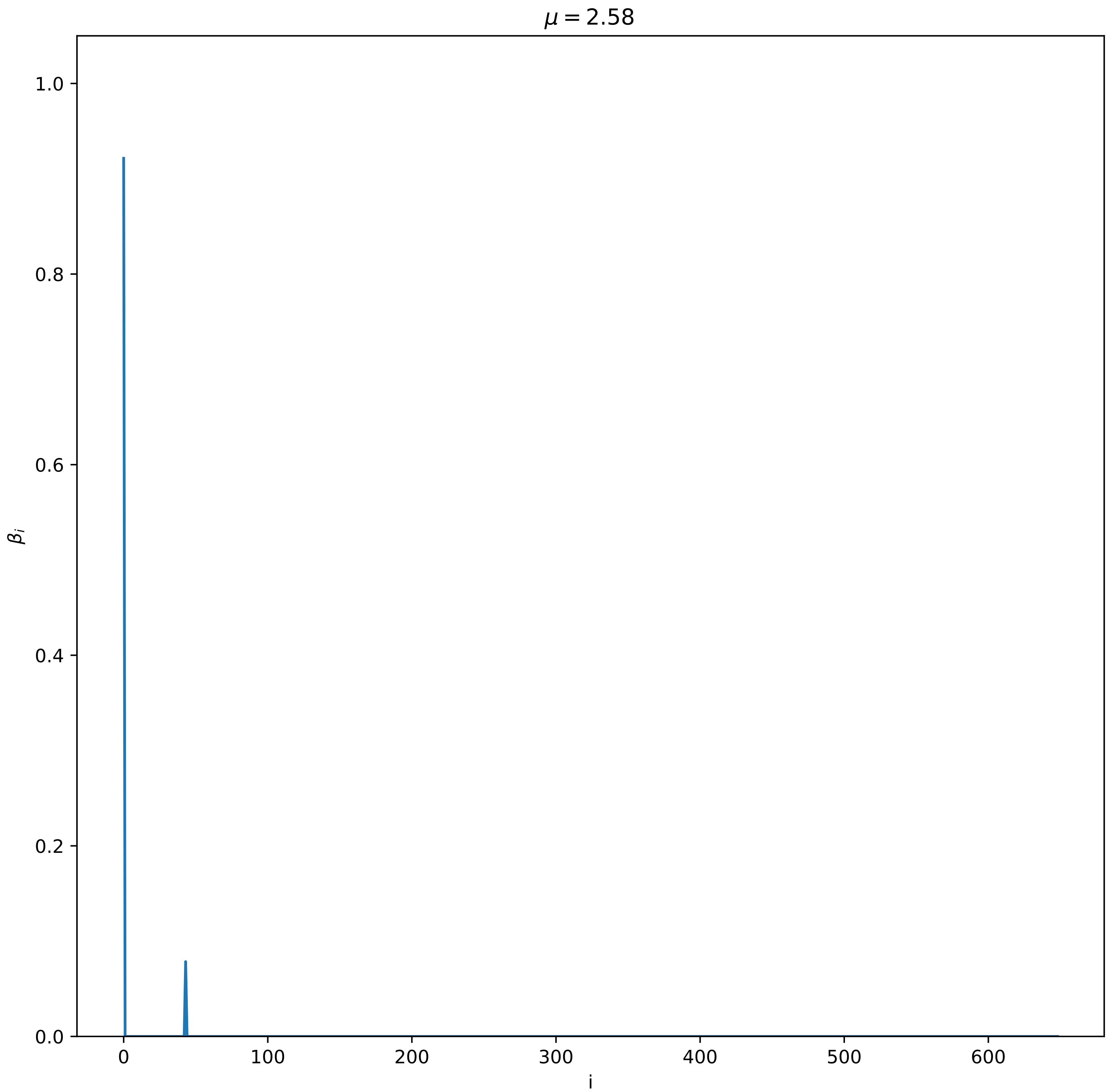


$\mu = 2.55$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

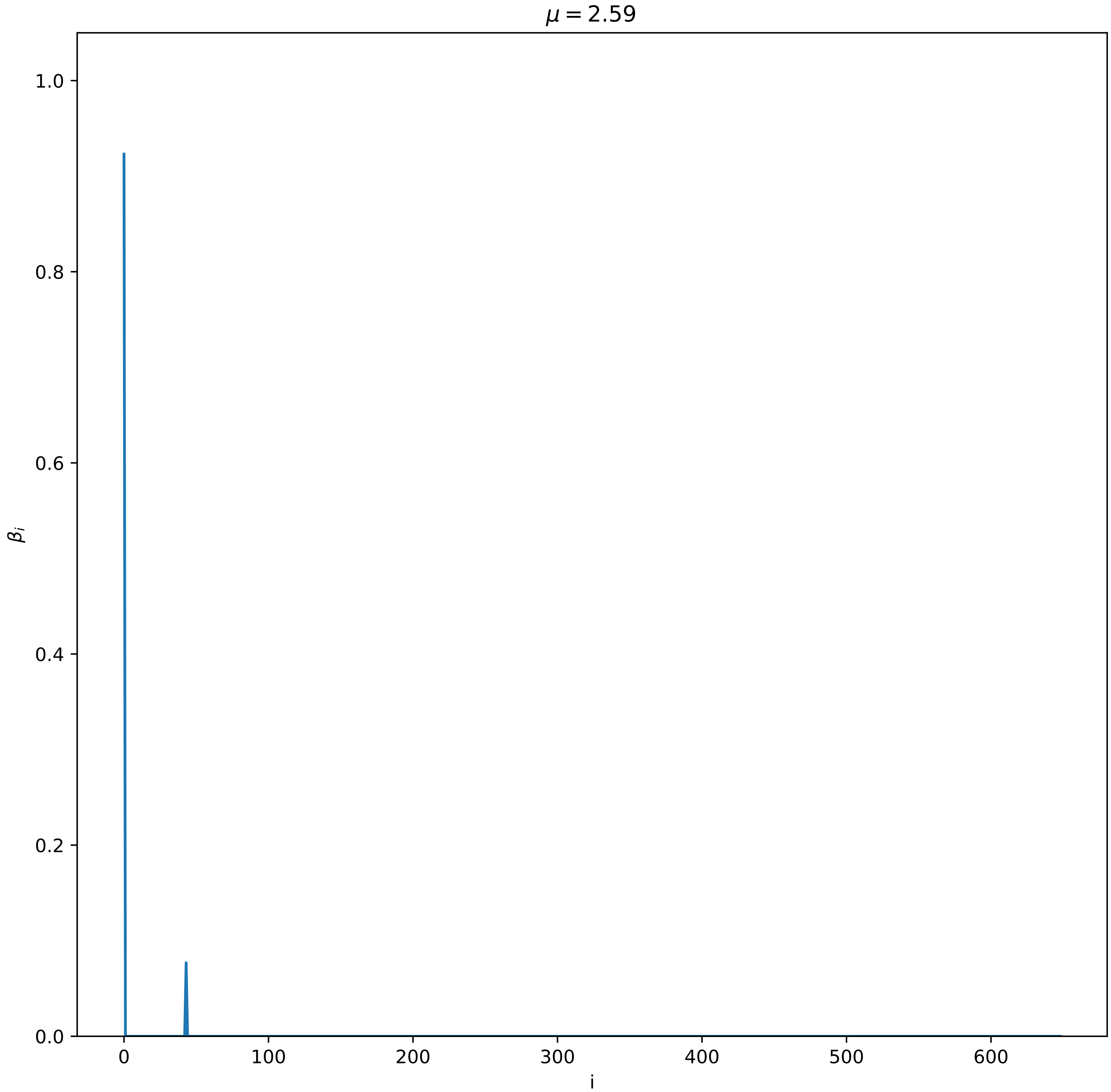

$\mu = 2.56$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



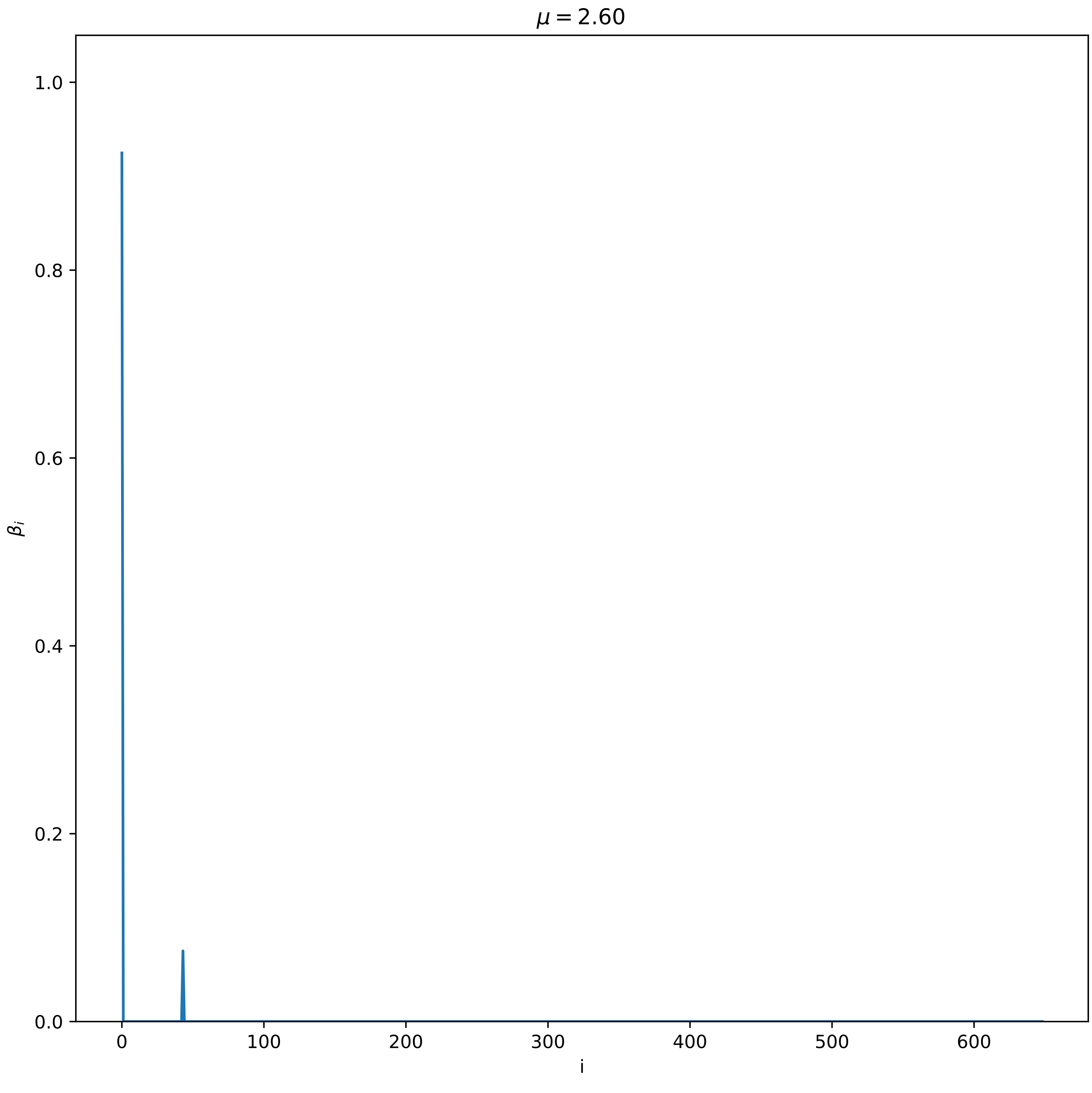
$\mu = 2.57$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



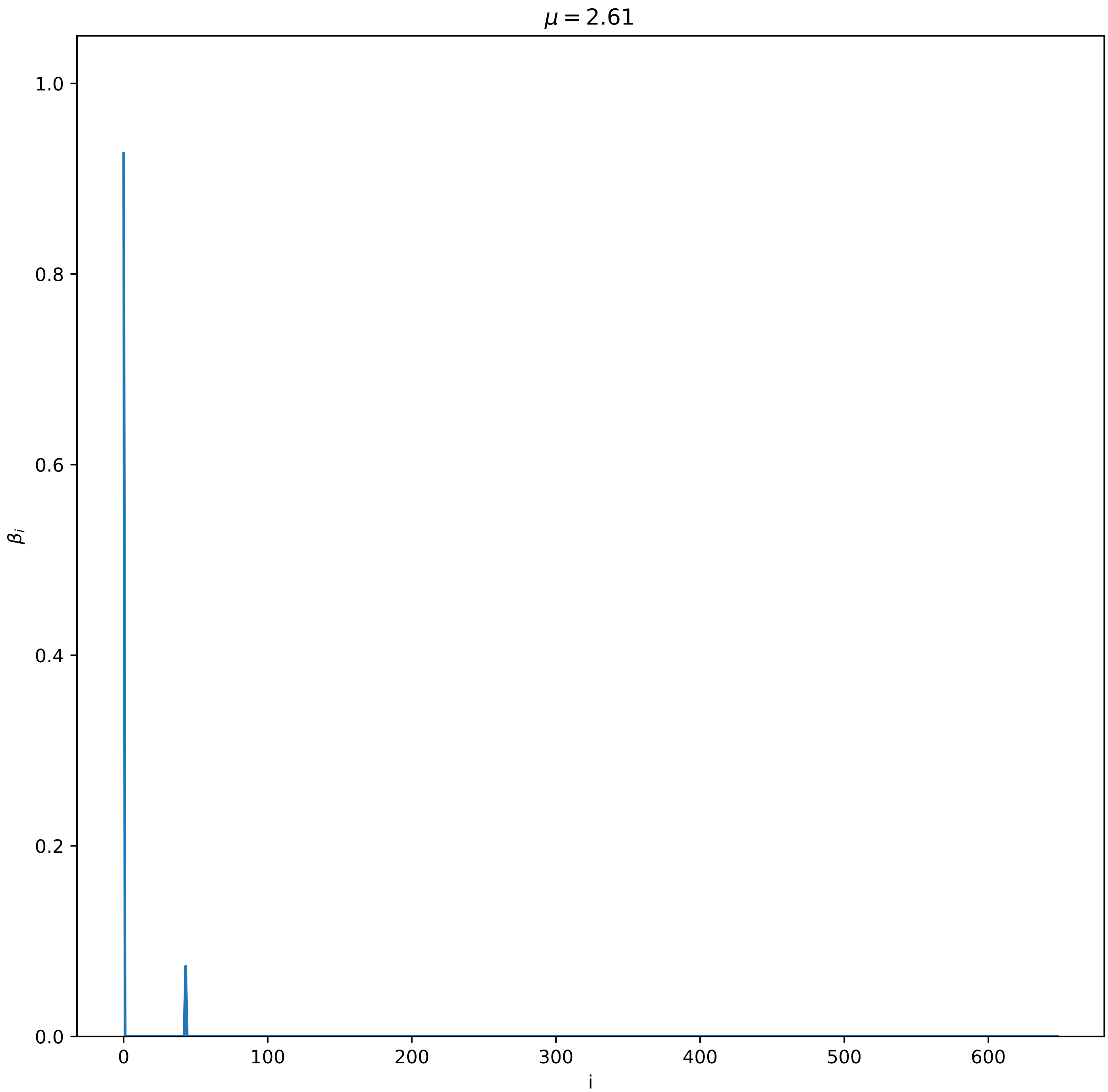
$\mu = 2.58$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

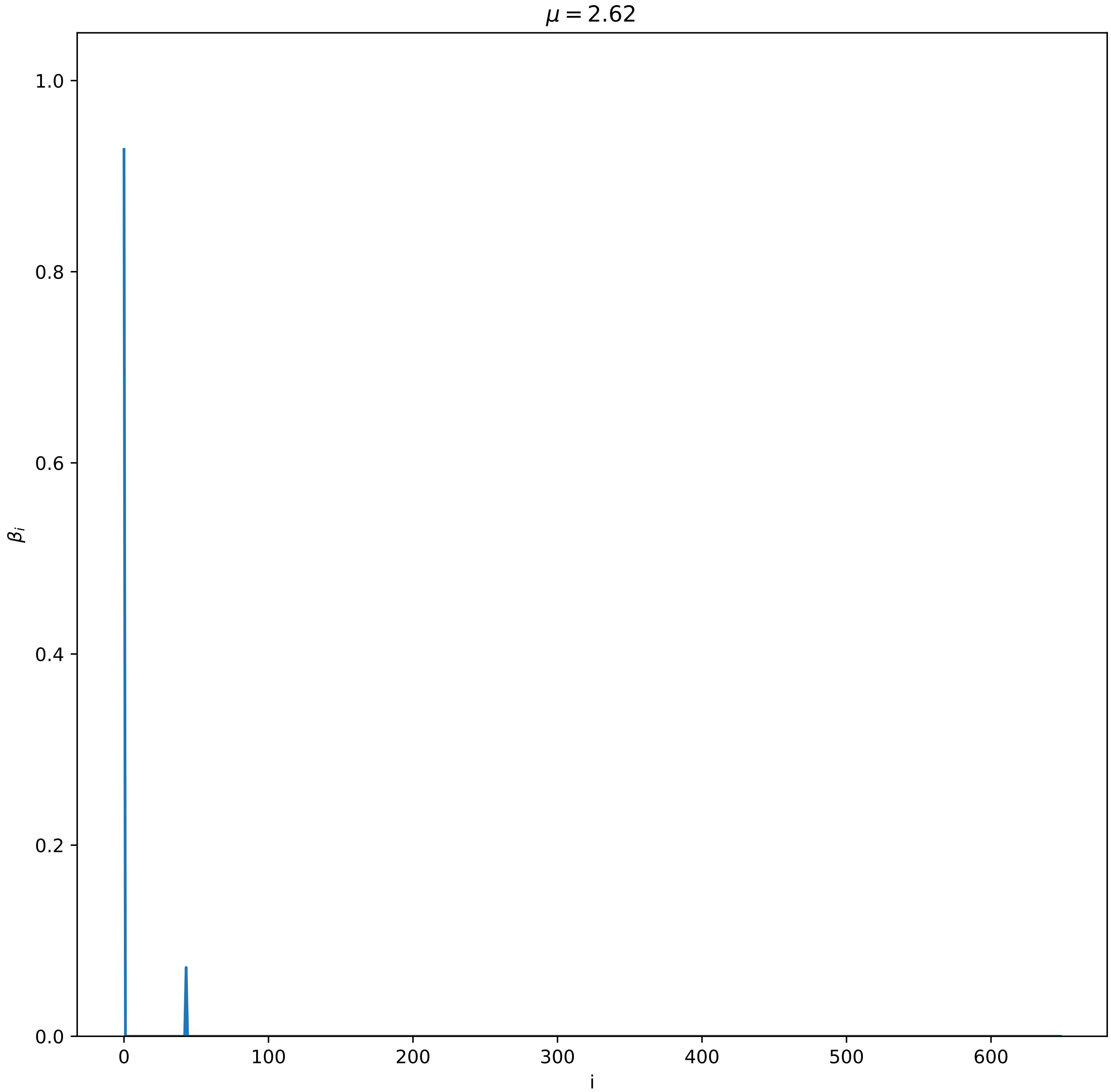


$\mu = 2.59$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

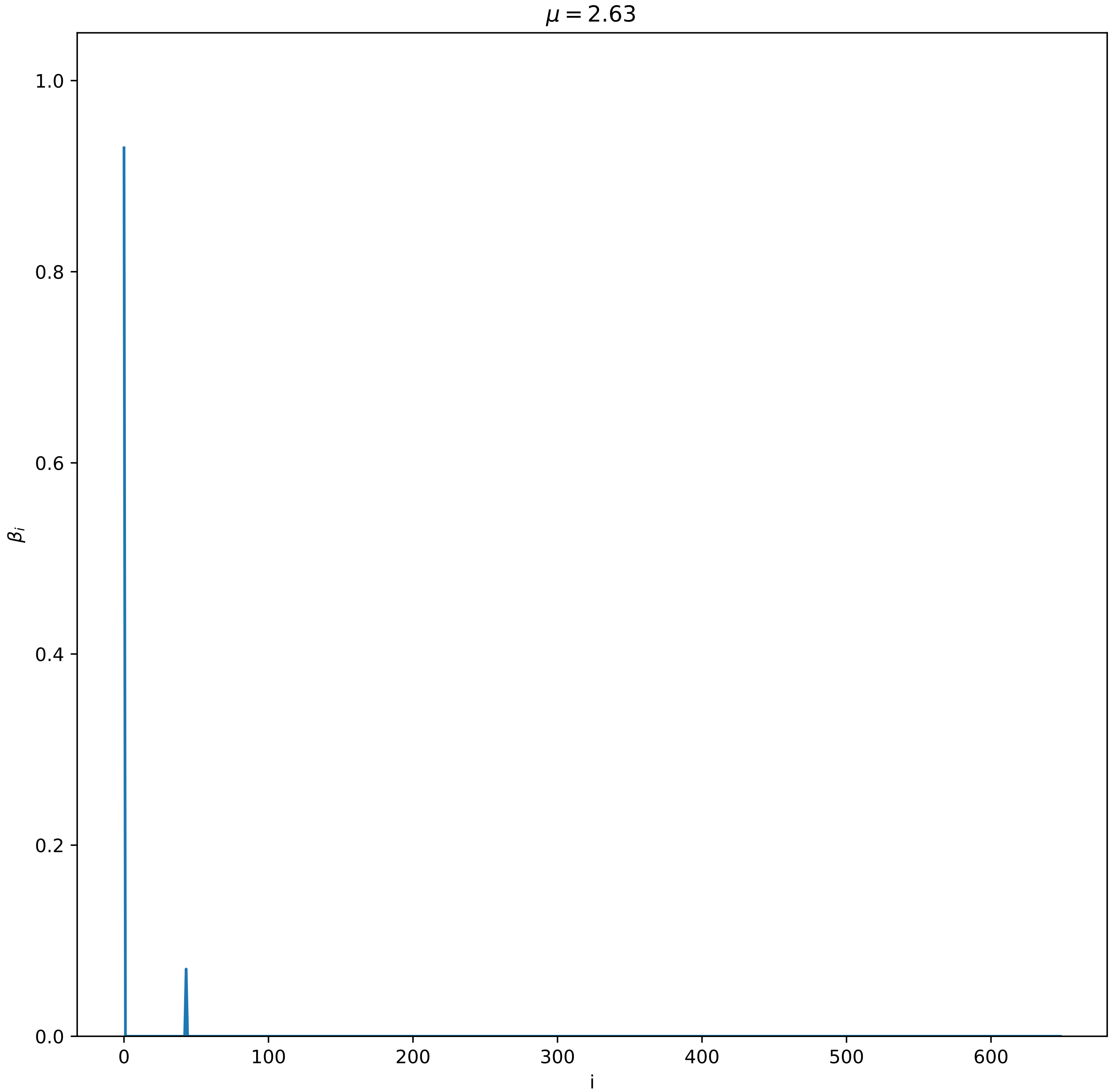


$\mu = 2.60$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

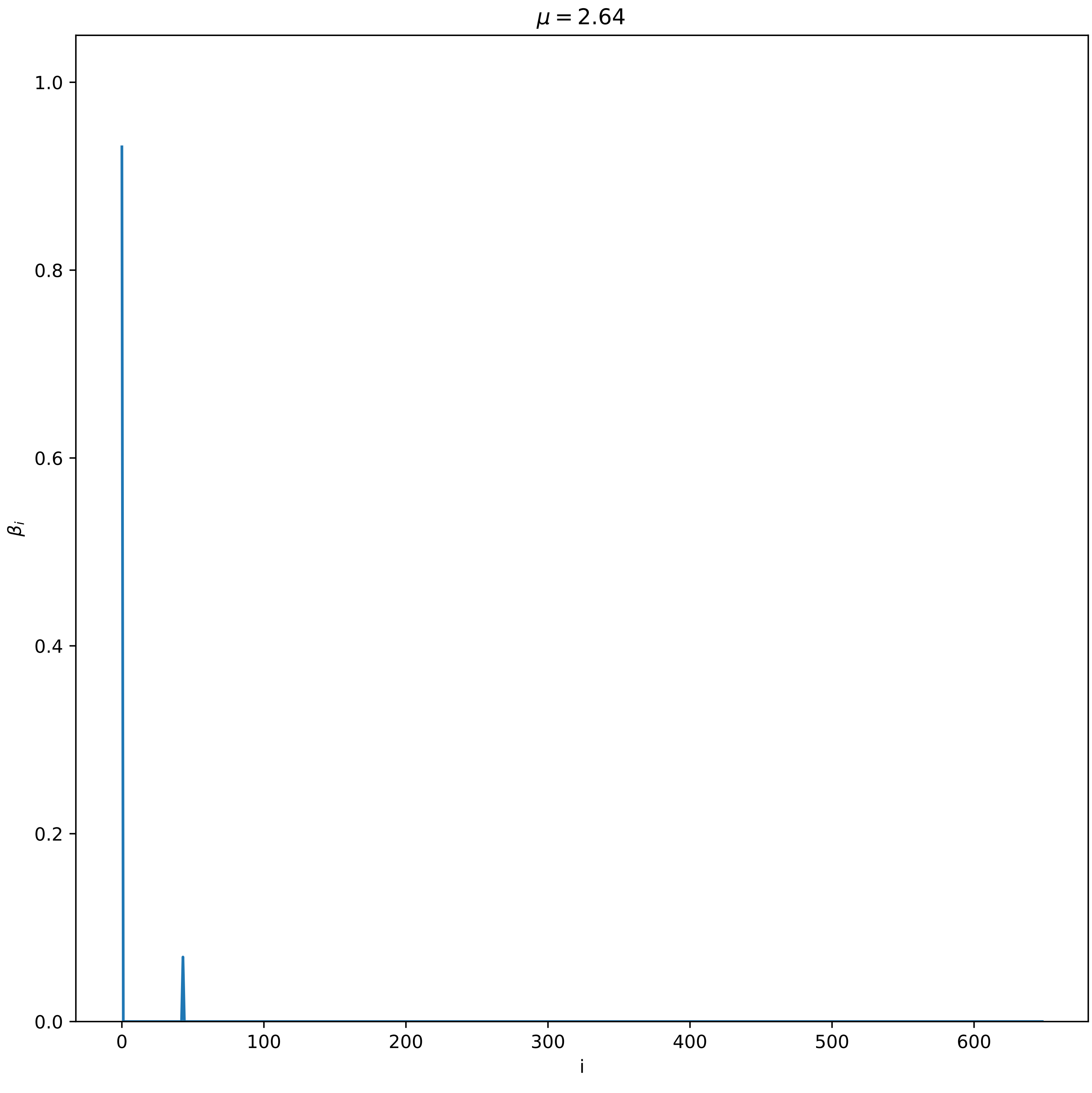

$\mu = 2.61$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

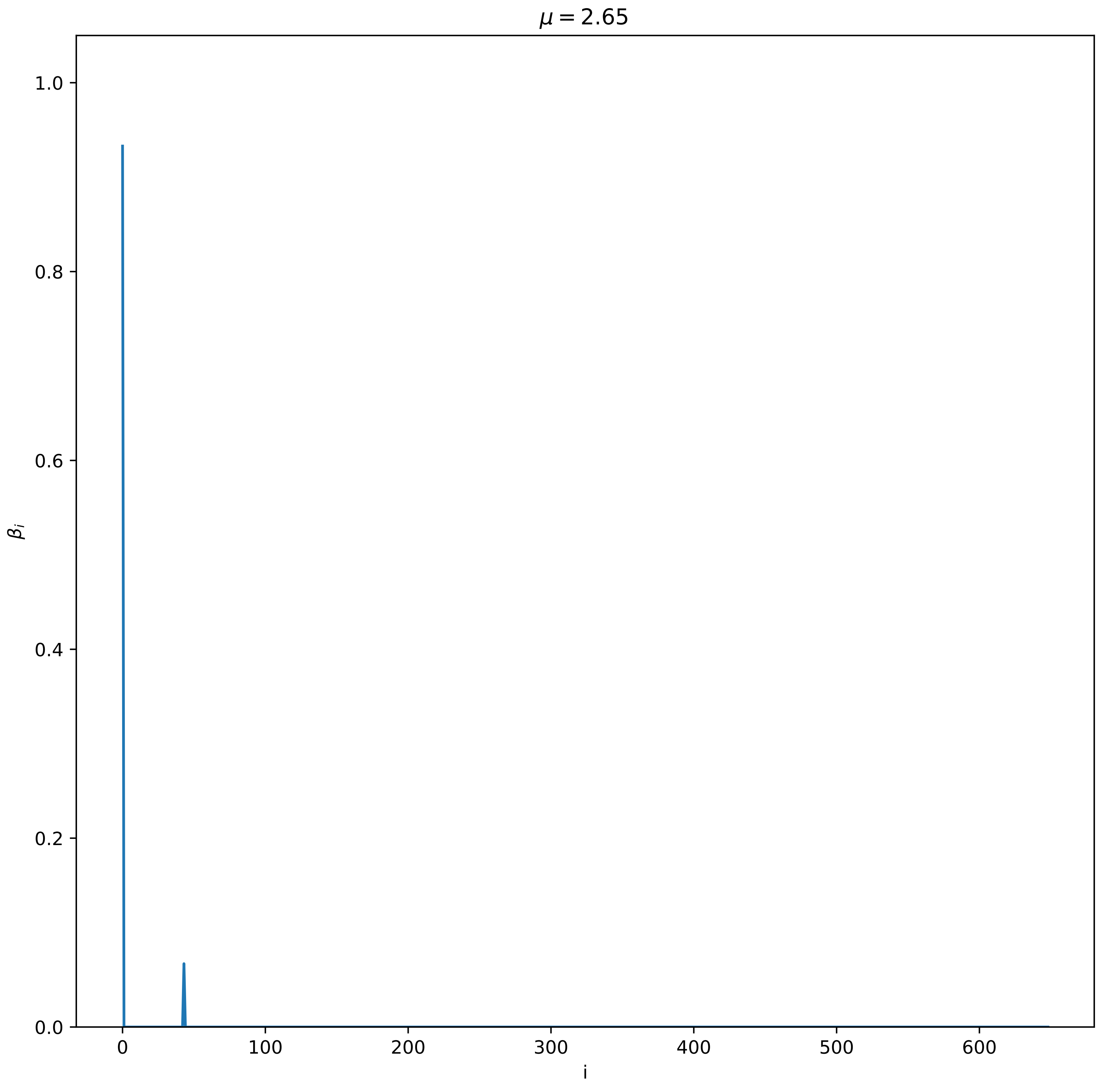


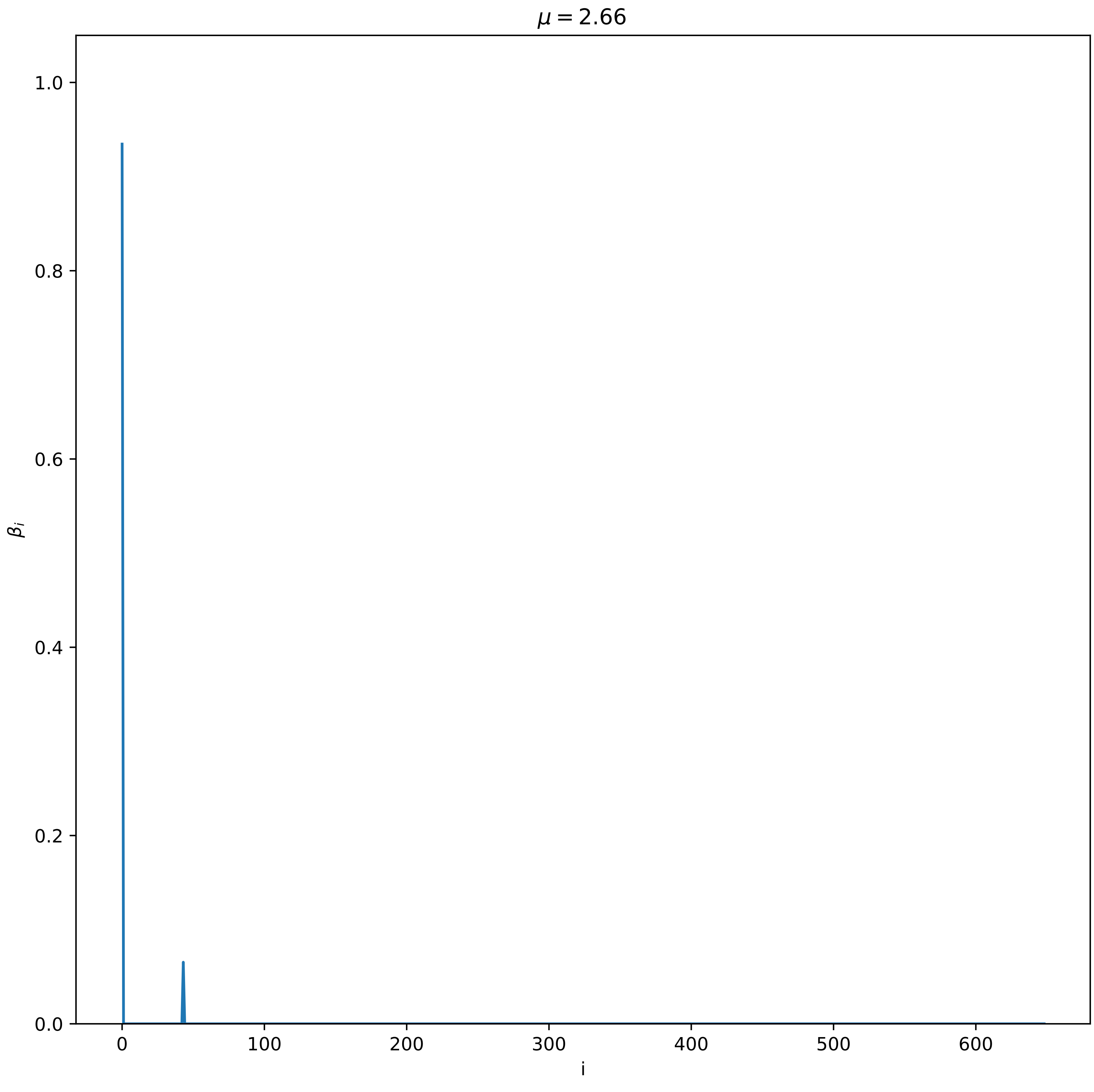
$\mu = 2.62$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



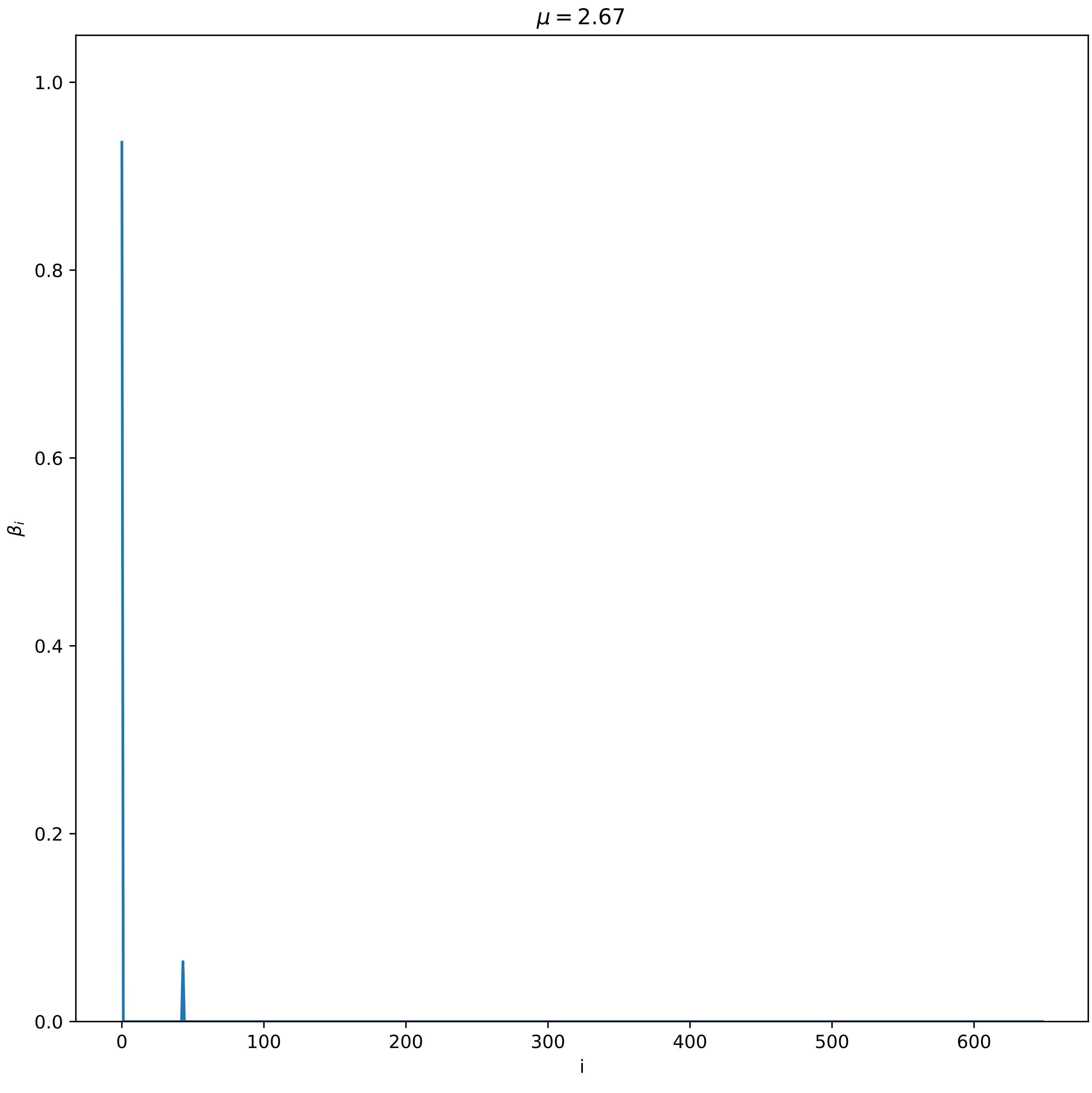
$\mu = 2.63$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

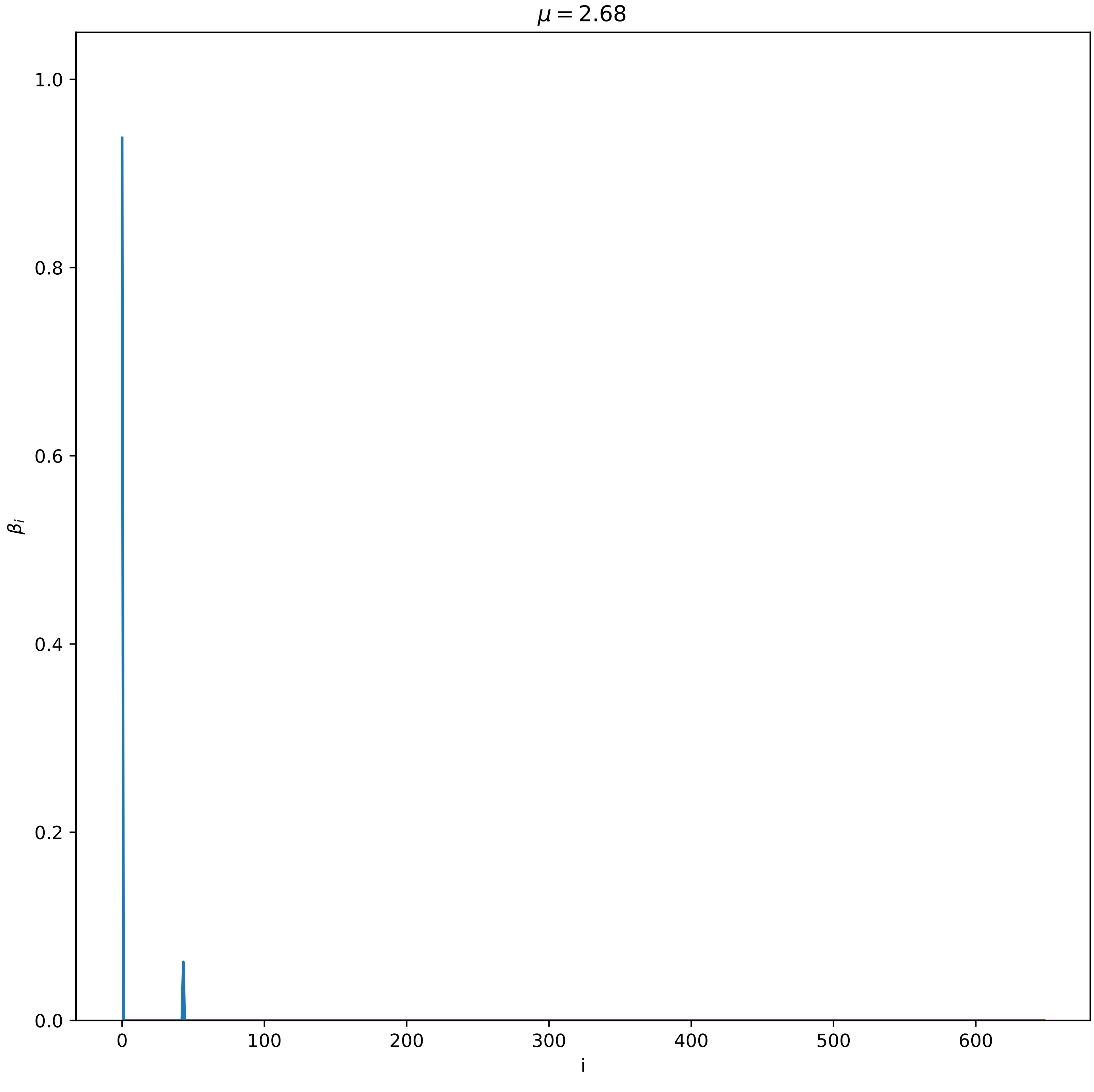

$\mu = 2.64$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

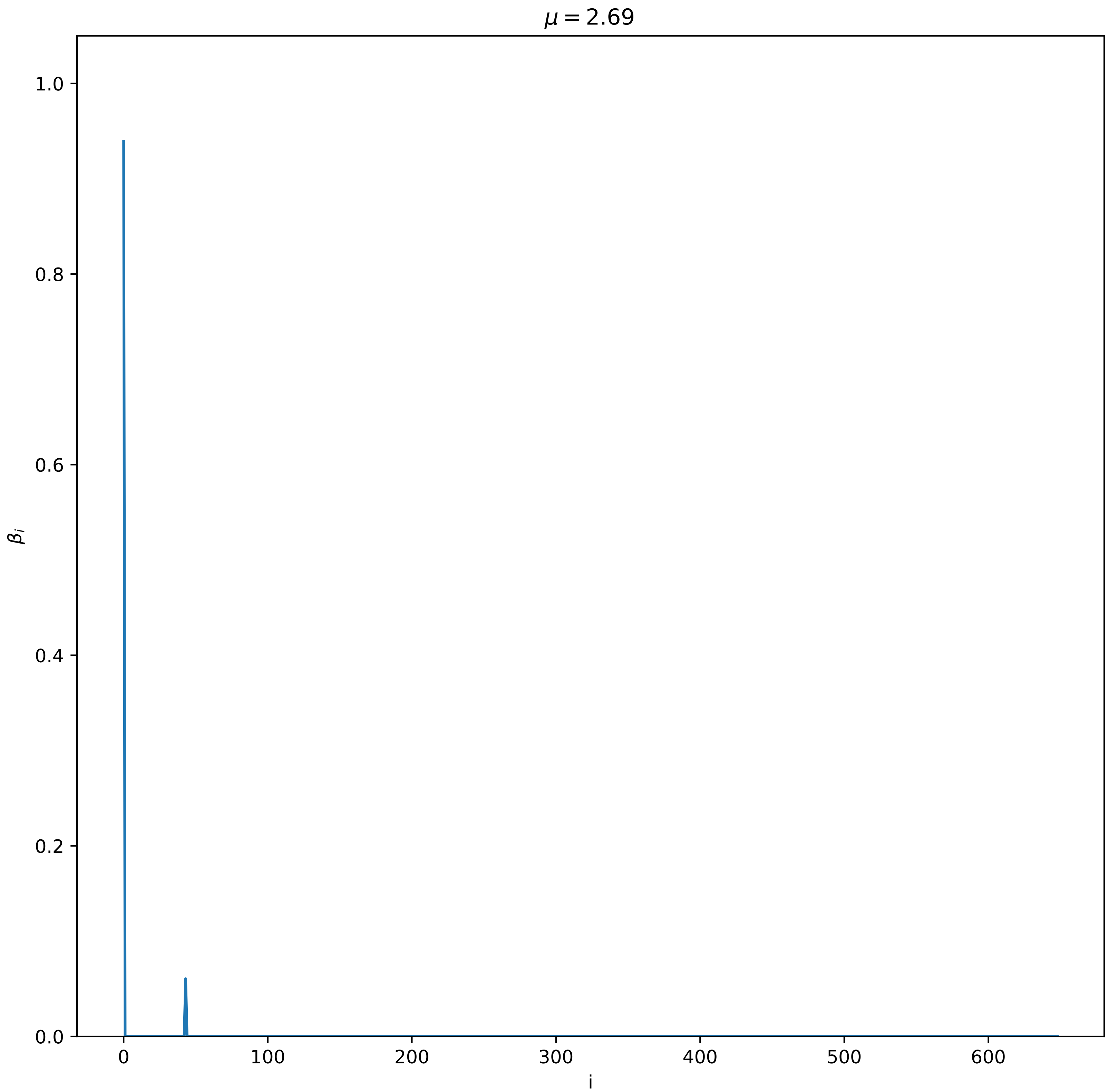

$\mu = 2.65$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



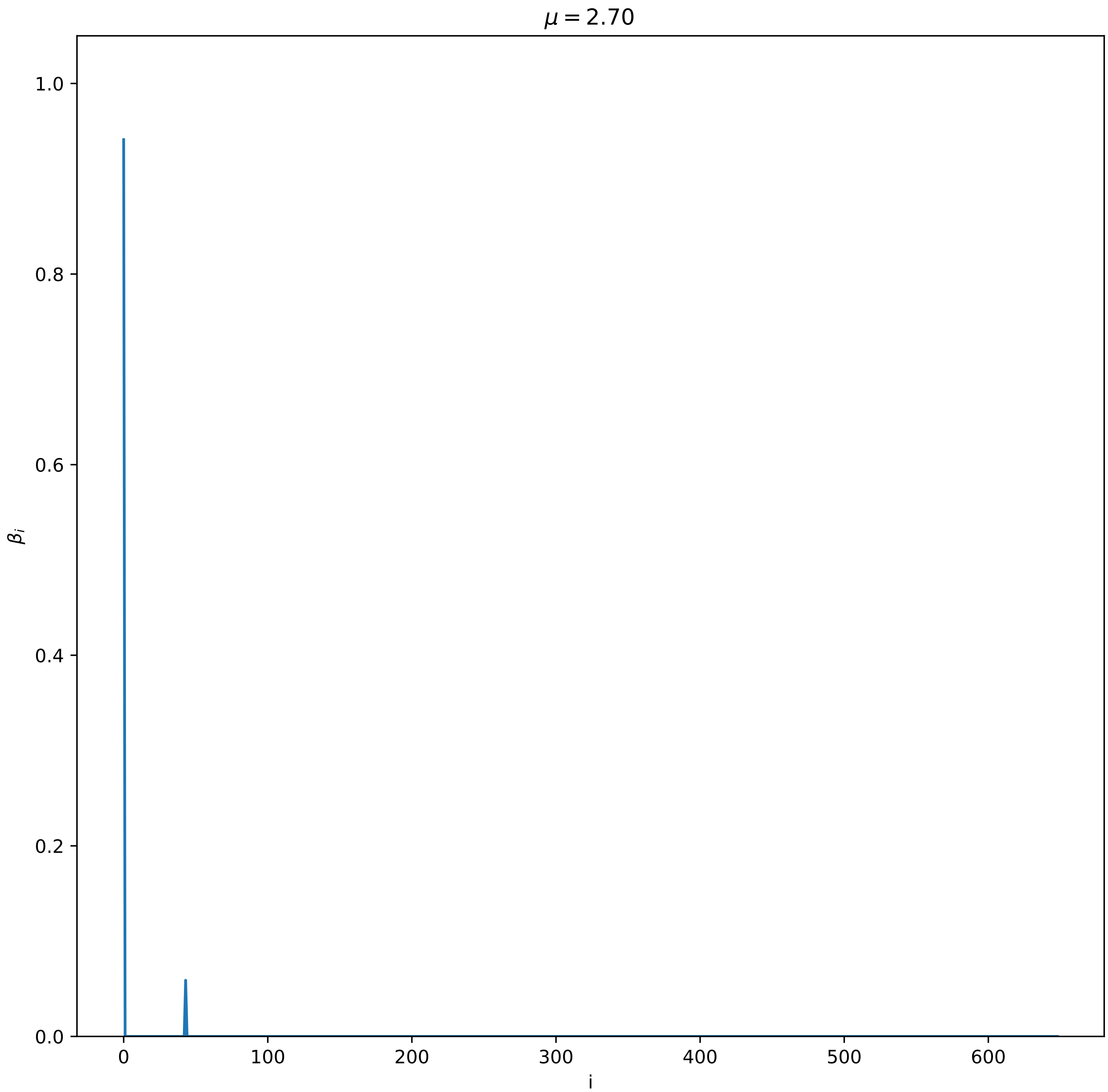
$\mu = 2.66$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

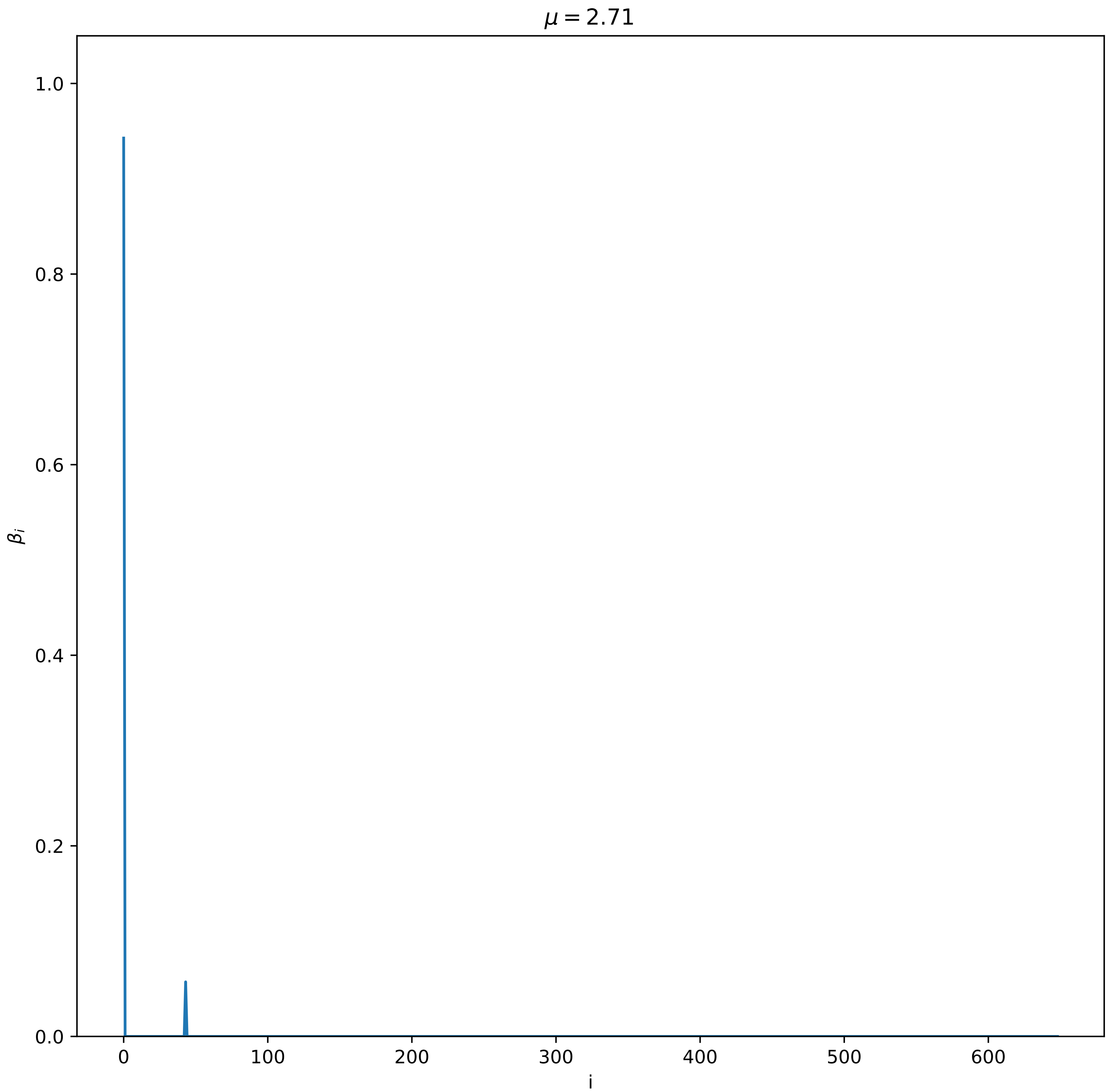

$\mu = 2.67$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 2.68$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

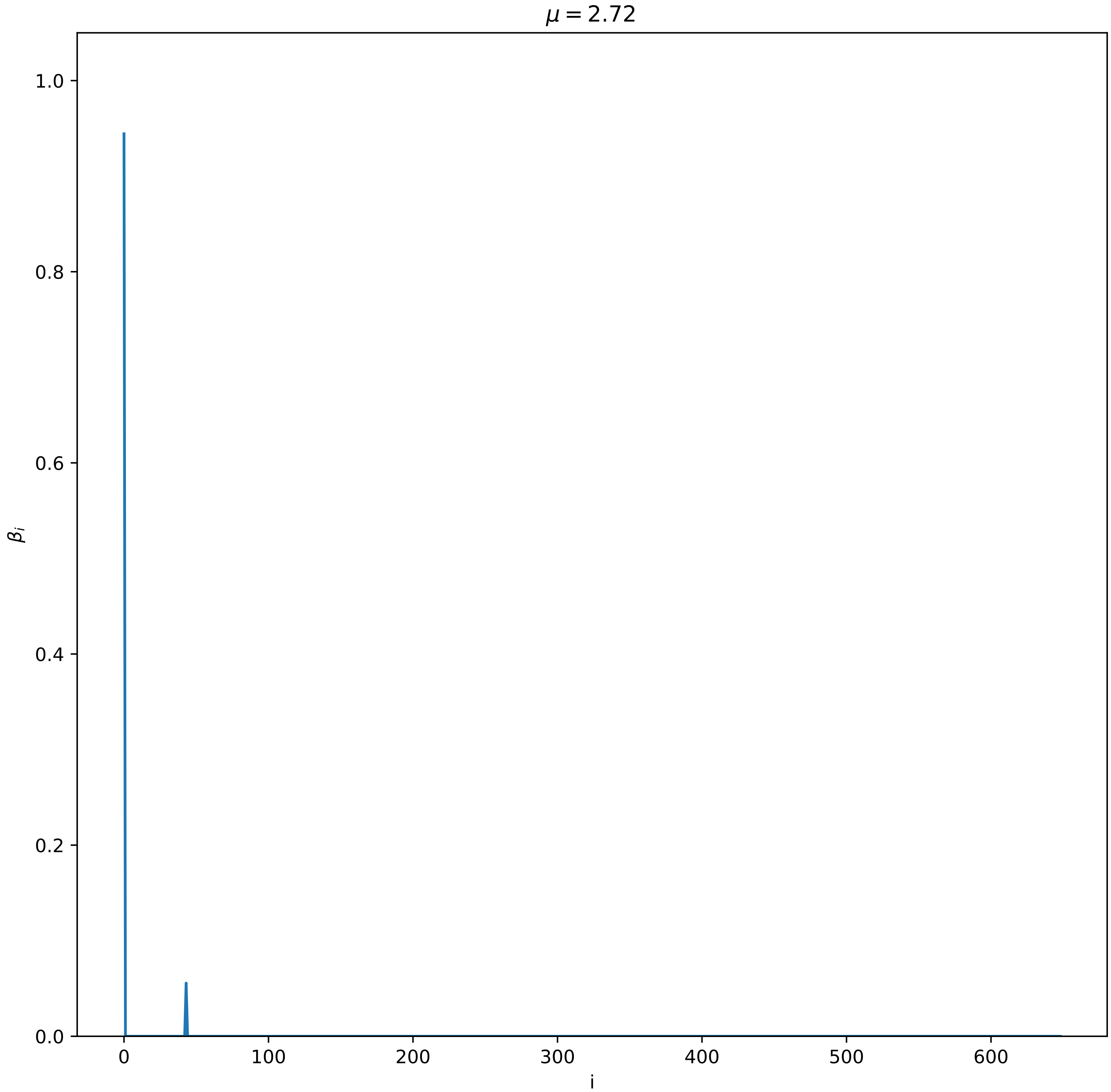


$\mu = 2.69$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

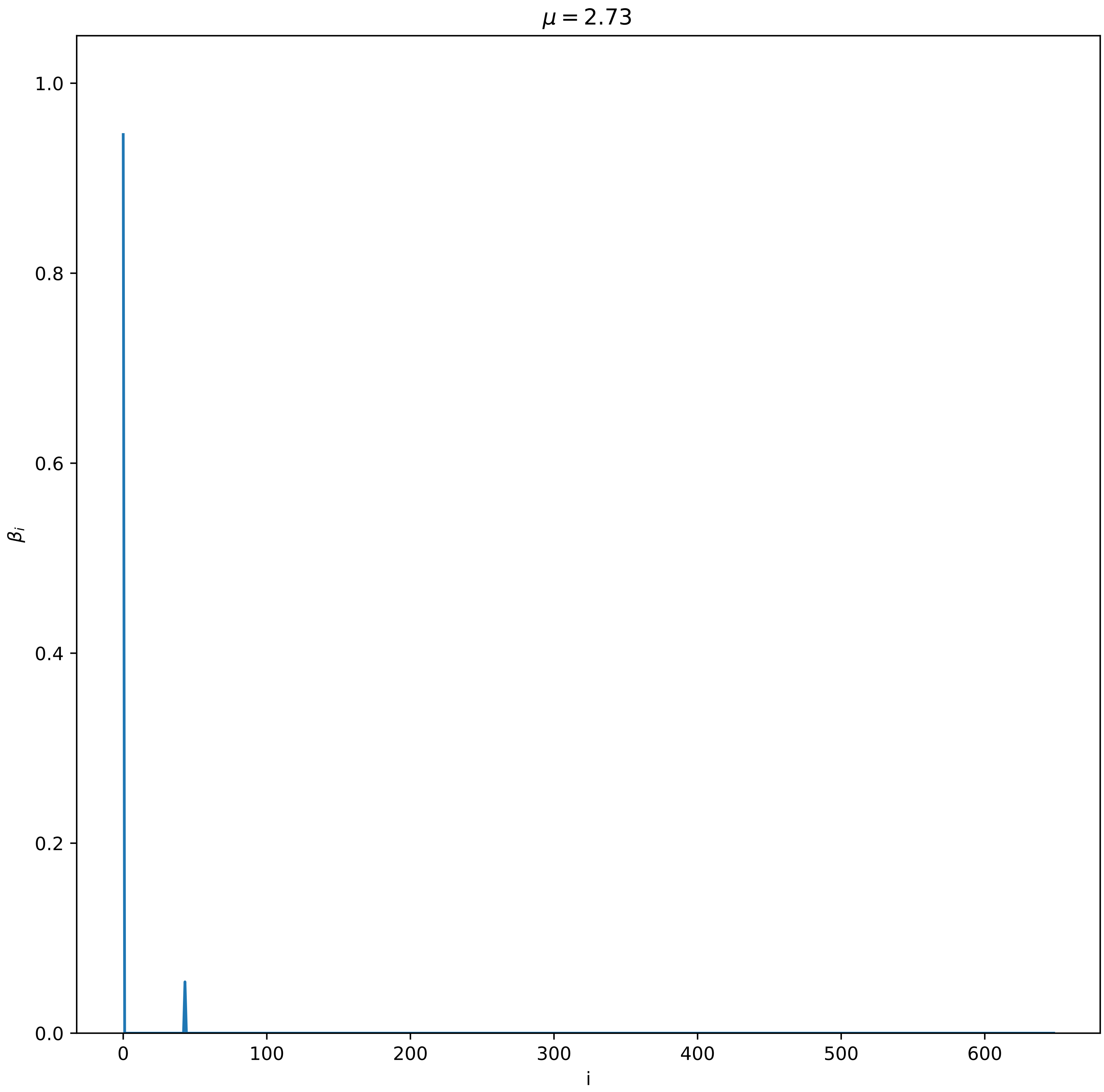

$\mu = 2.70$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



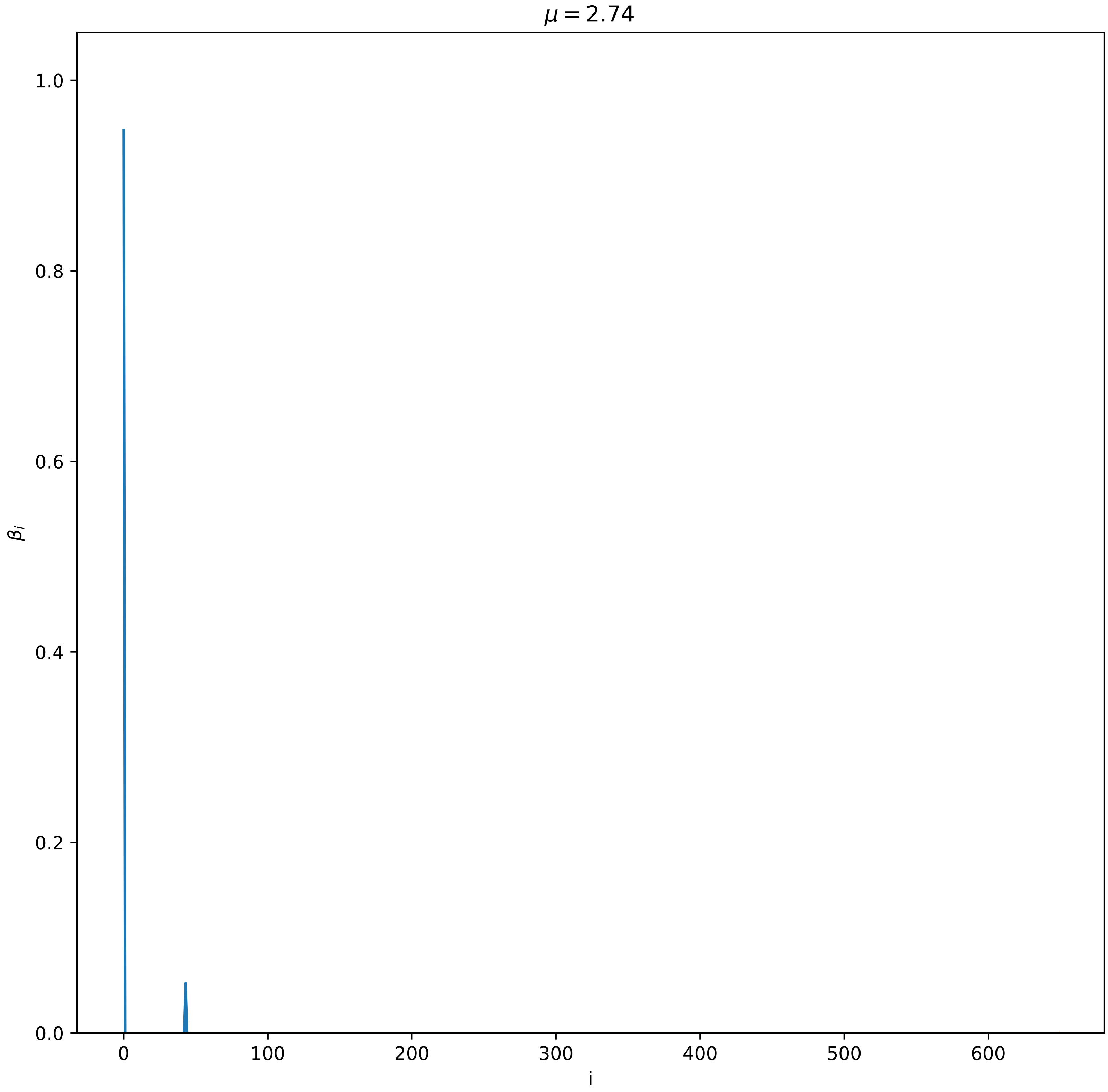
$\mu = 2.71$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

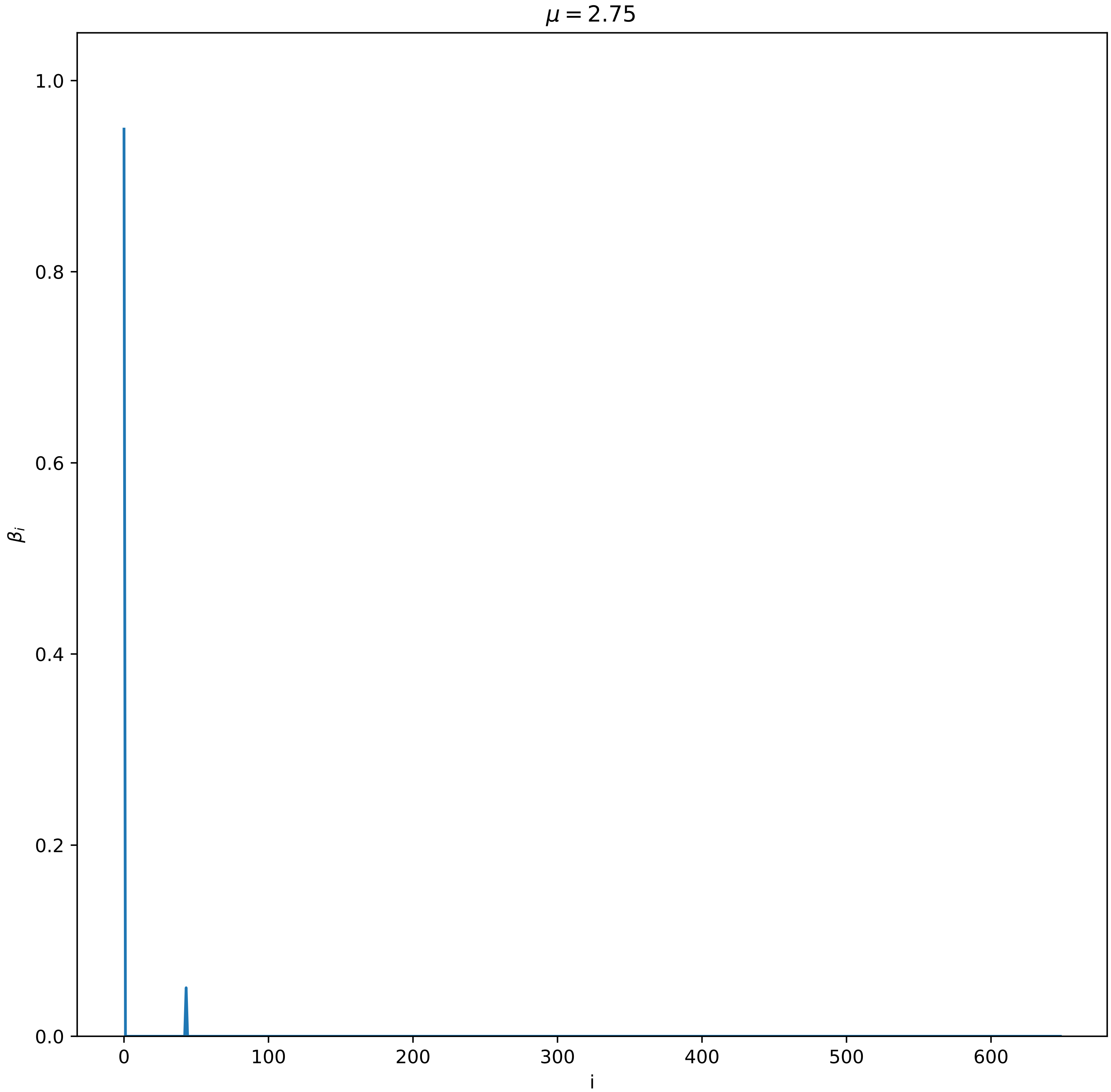


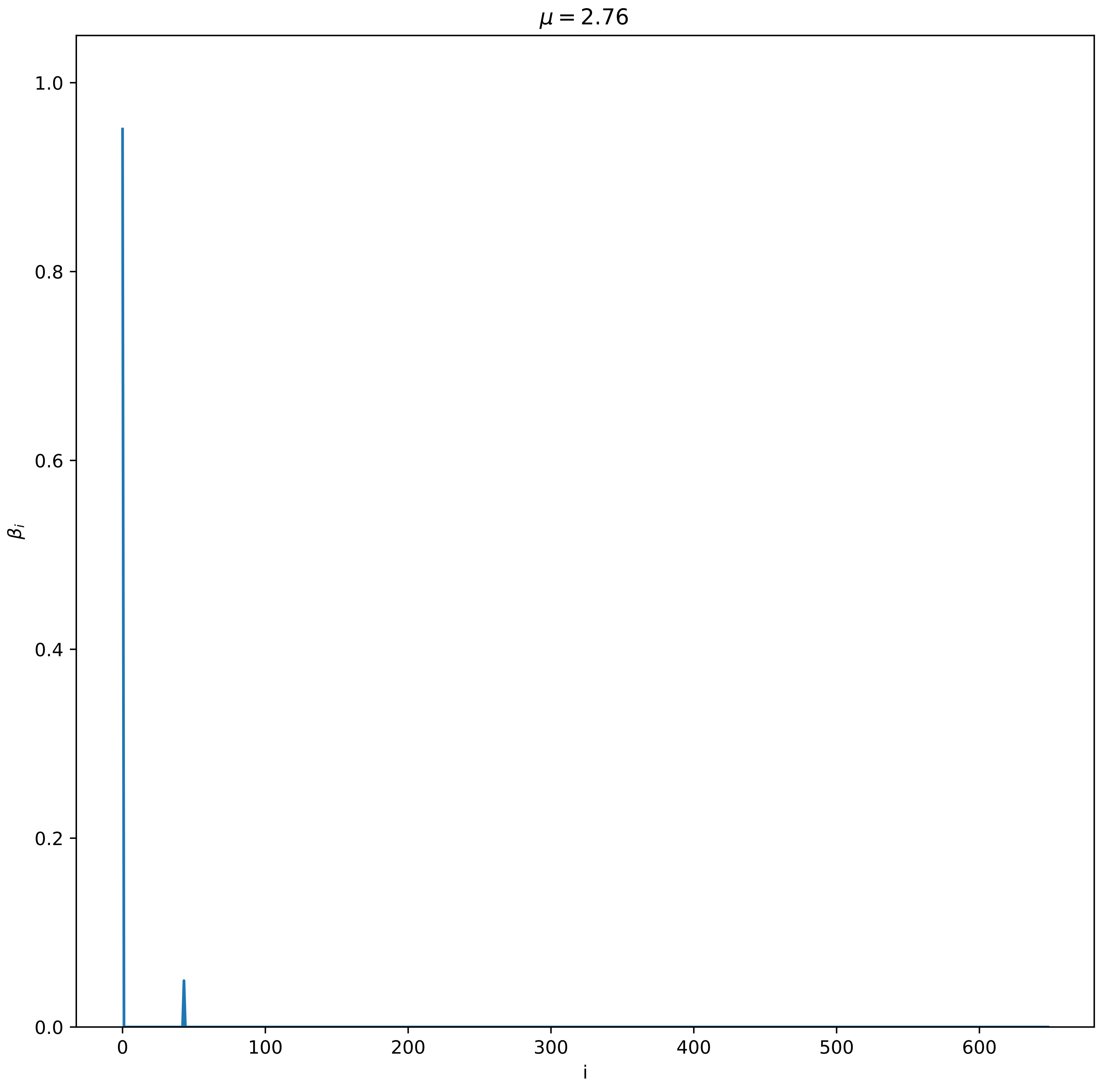
$\mu = 2.72$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



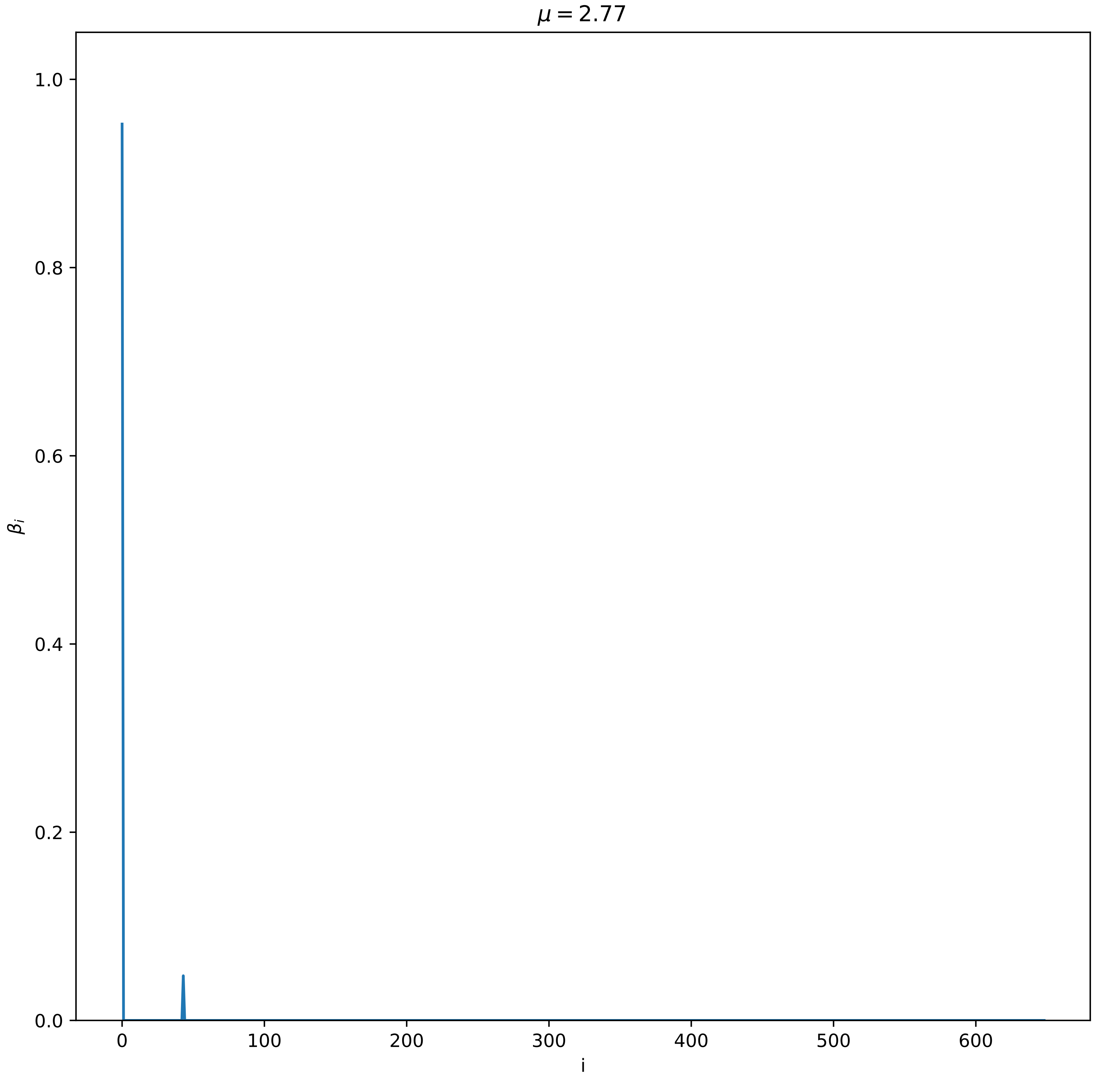
$\mu = 2.73$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

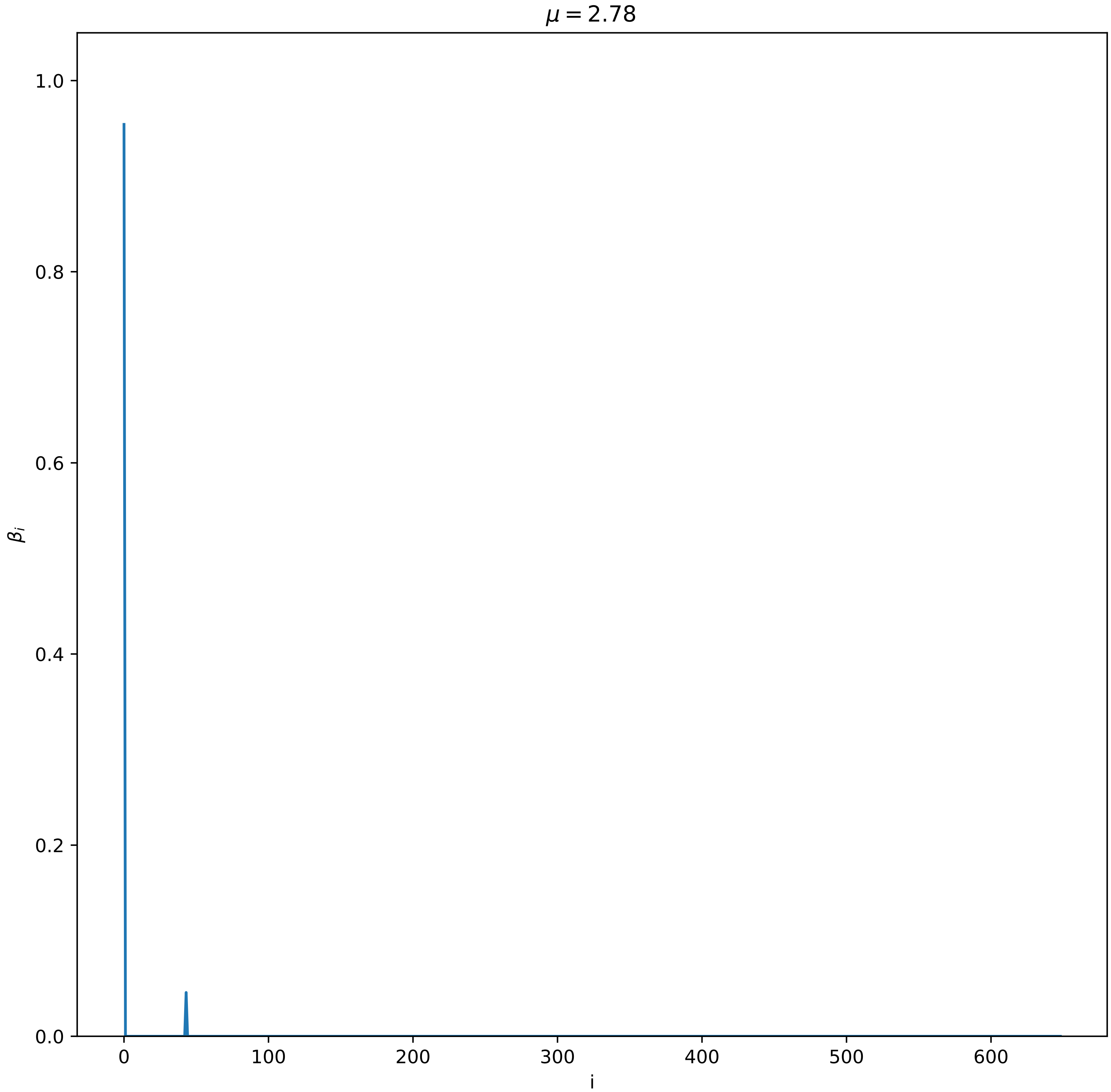

$\mu = 2.74$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 2.75$
β_i
i
0.0
0.2
0.4
0.6
0.8
1.0
0
100
200
300
400
500
600

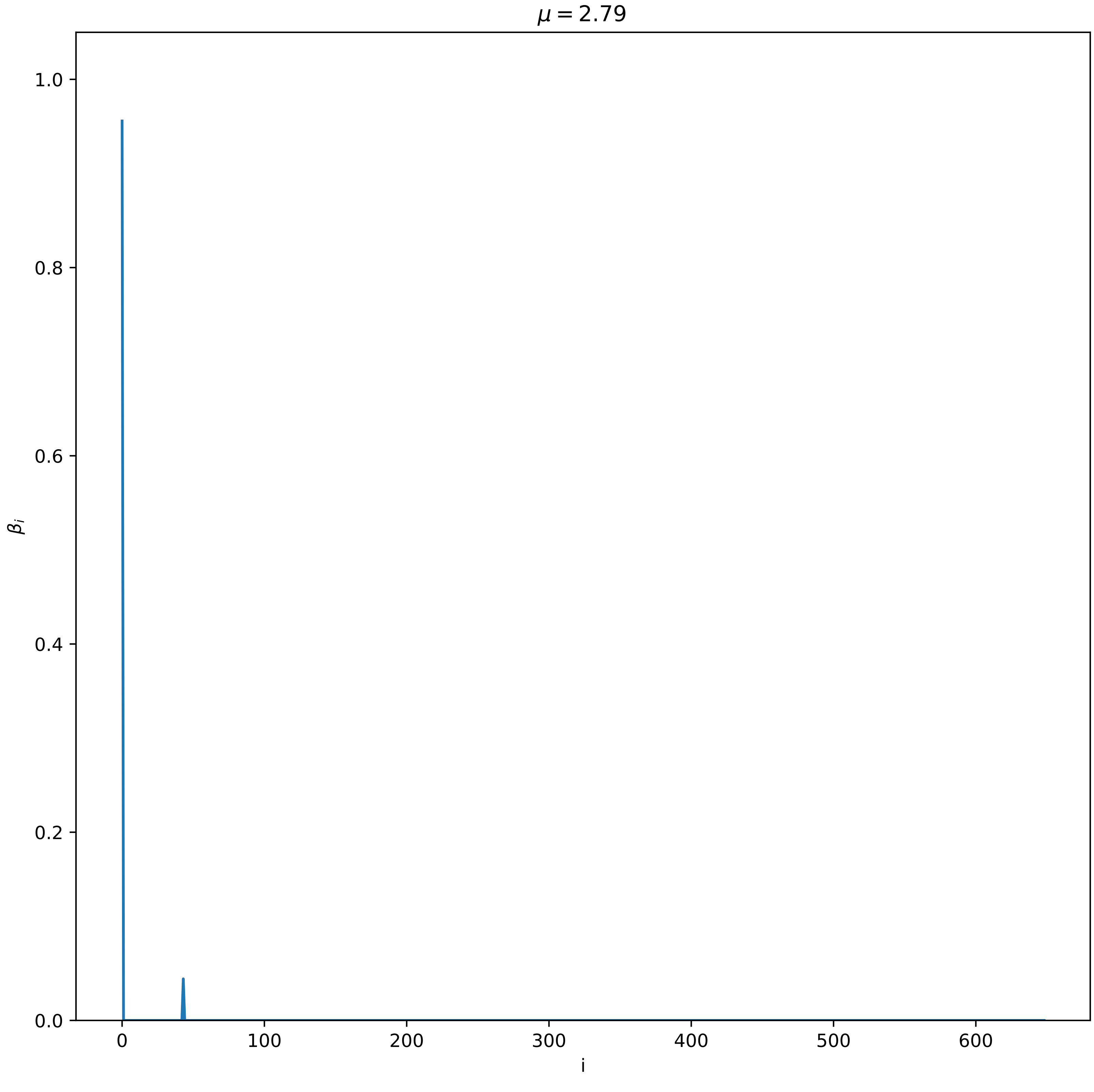


$\mu = 2.76$
β_i
i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600

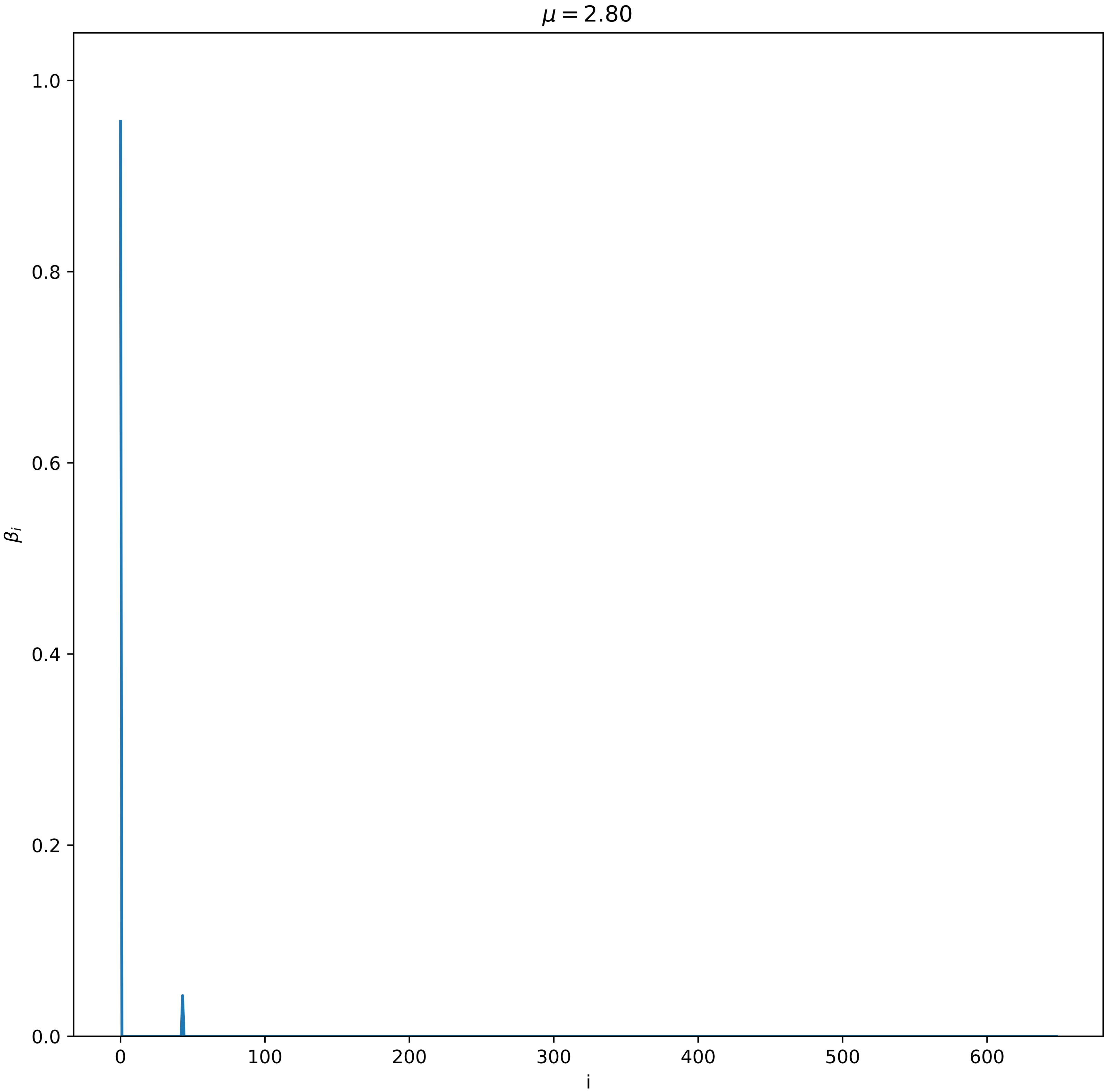

$\mu = 2.77$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



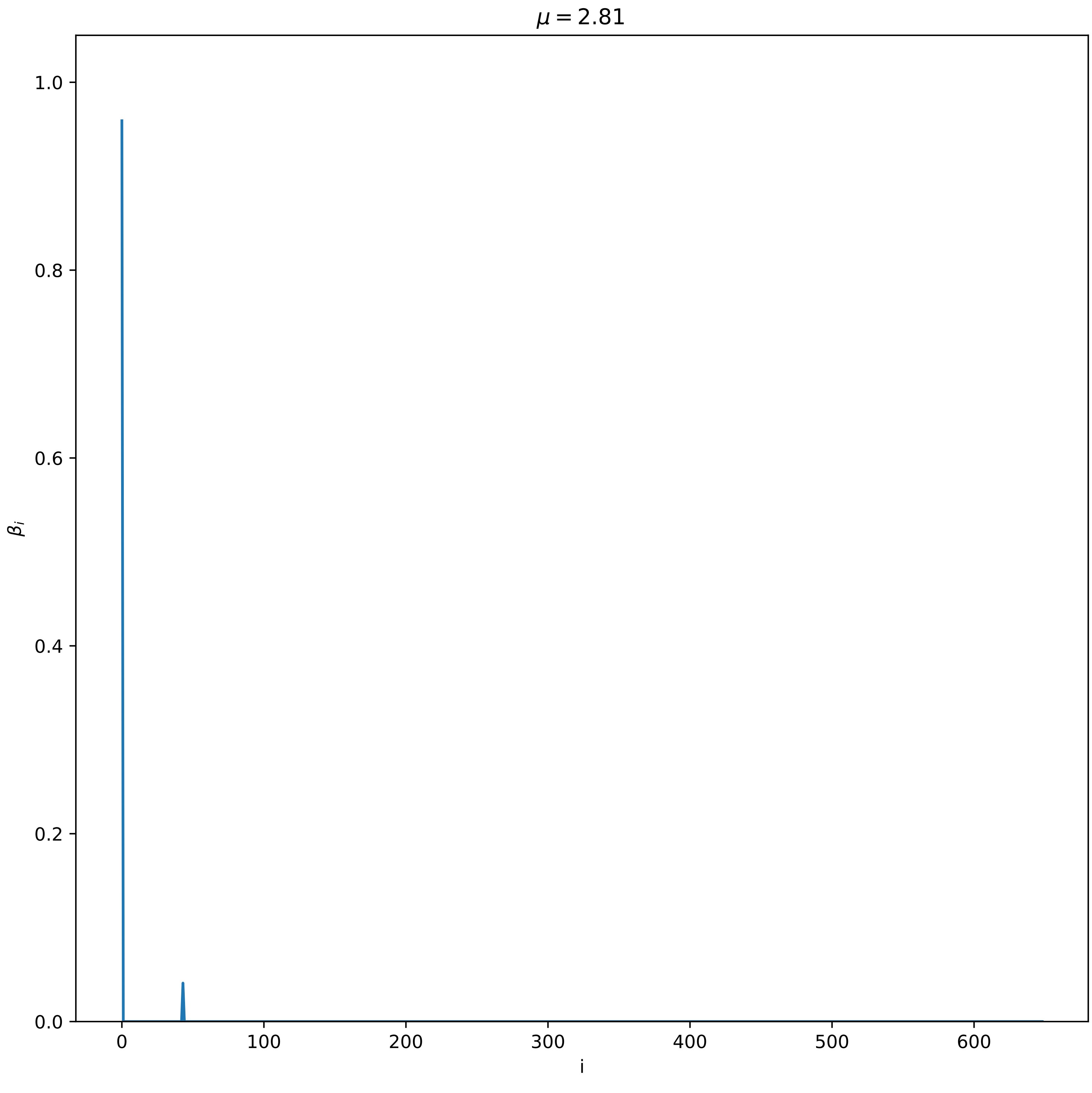
$\mu = 2.78$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



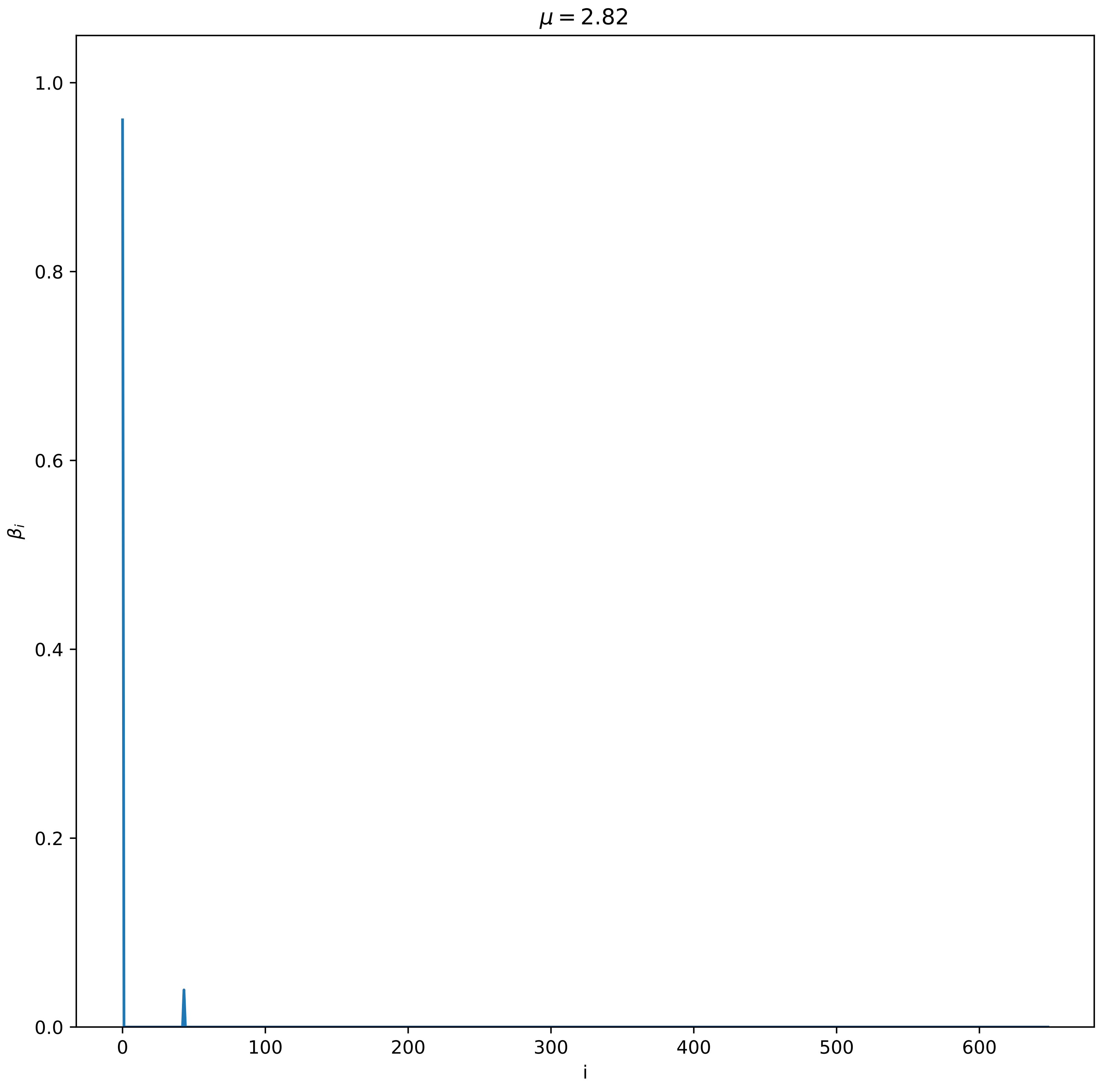
$\mu = 2.79$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i



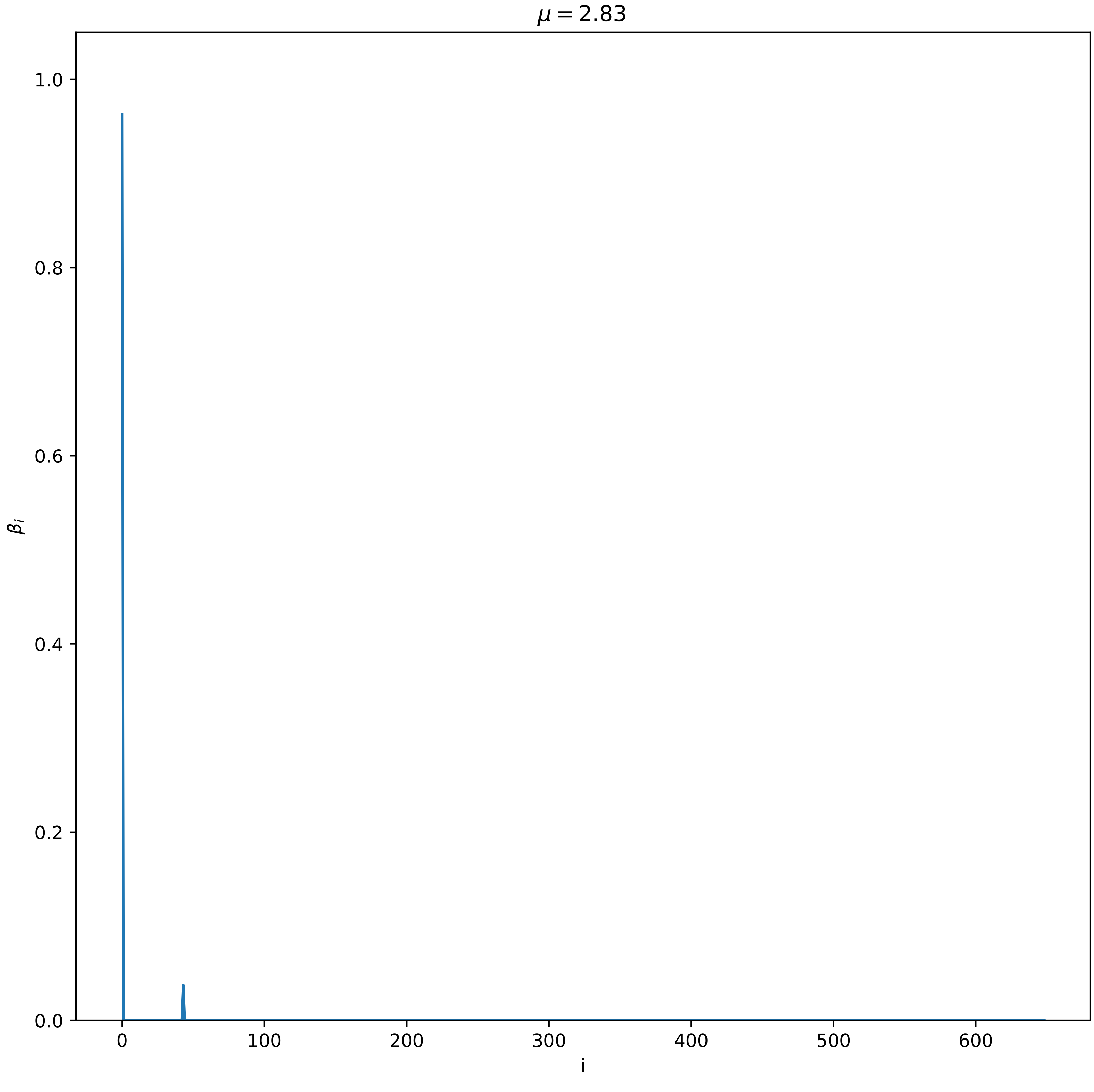
$\mu = 2.80$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



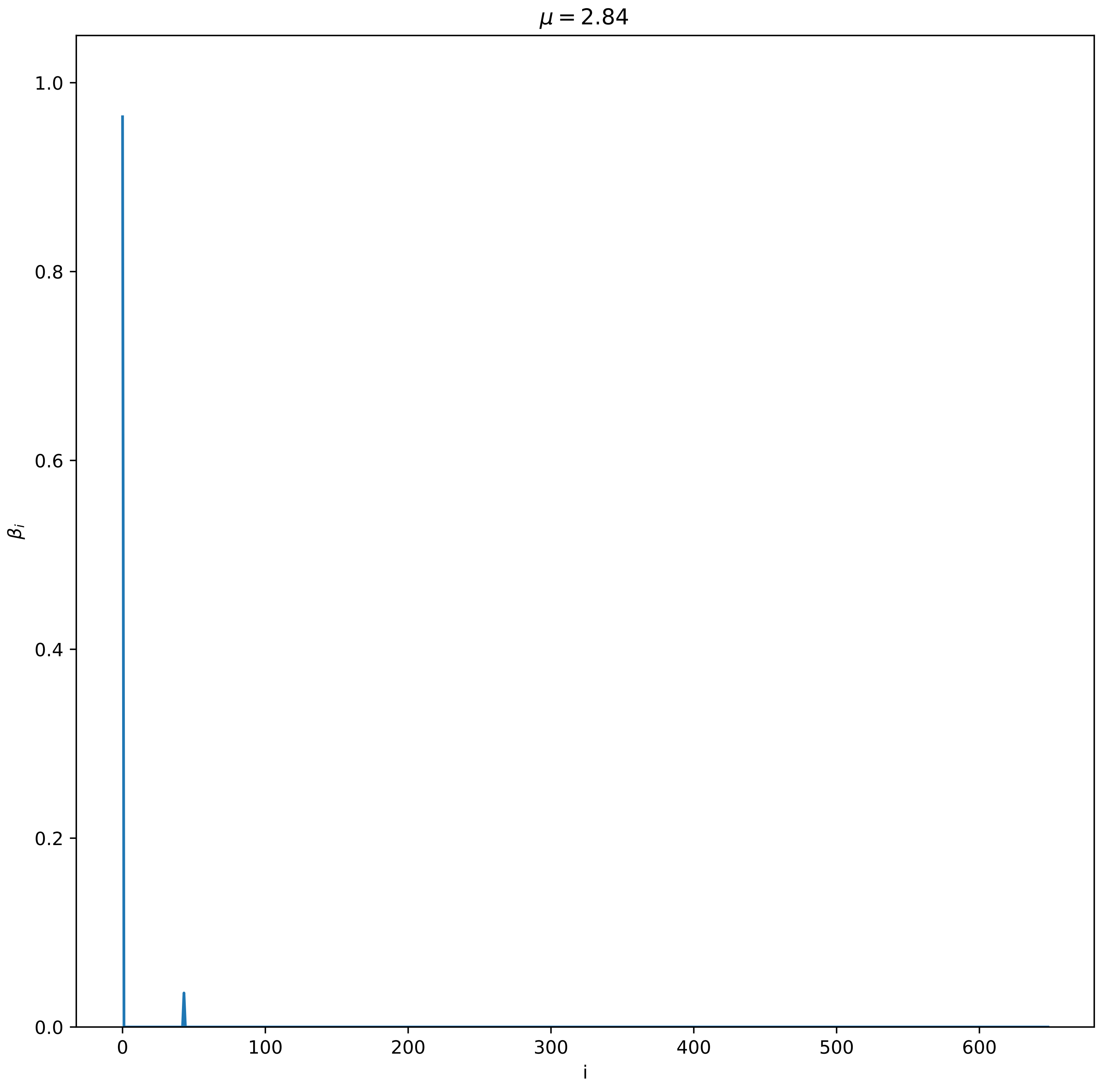
$\mu = 2.81$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

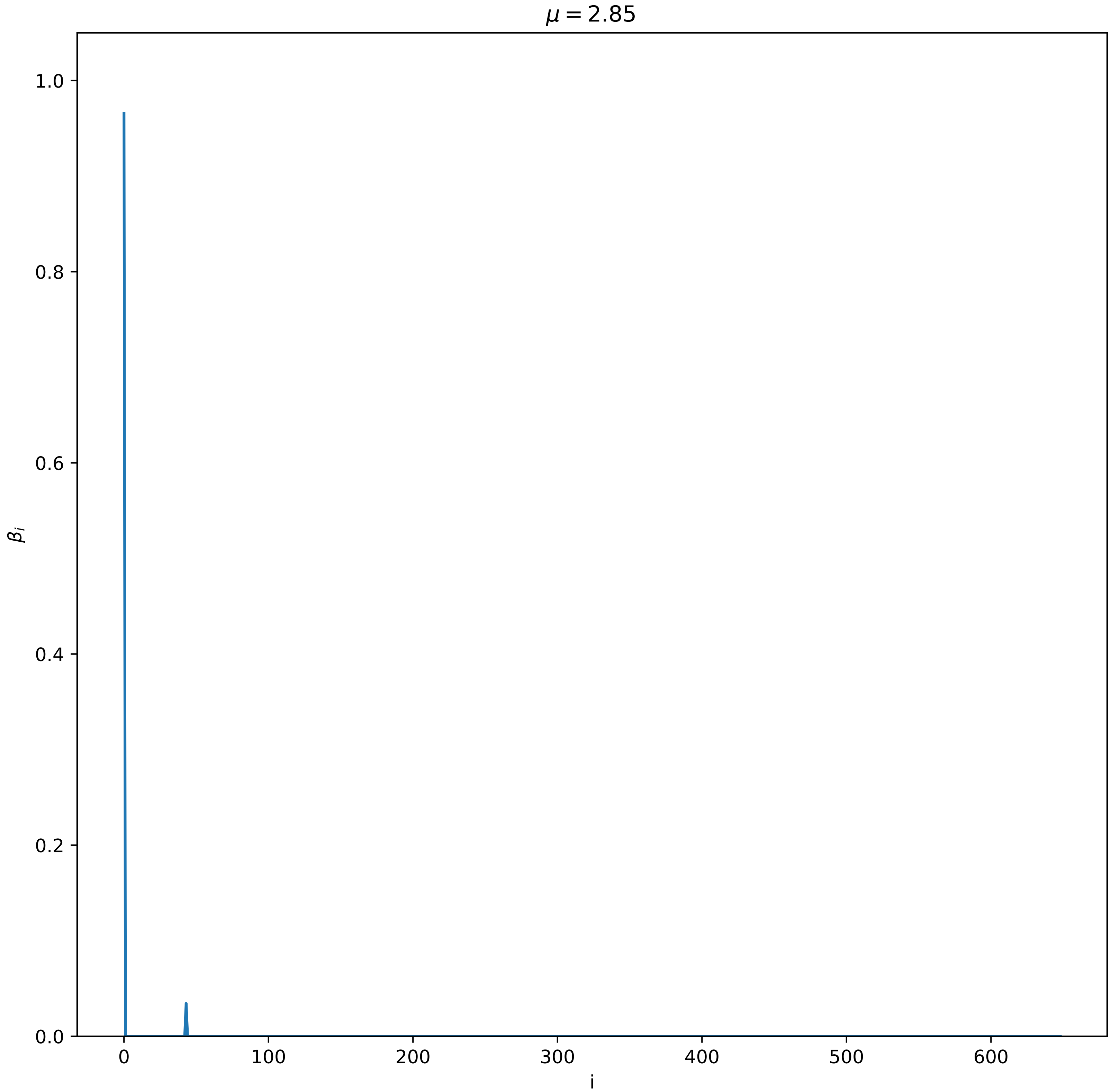


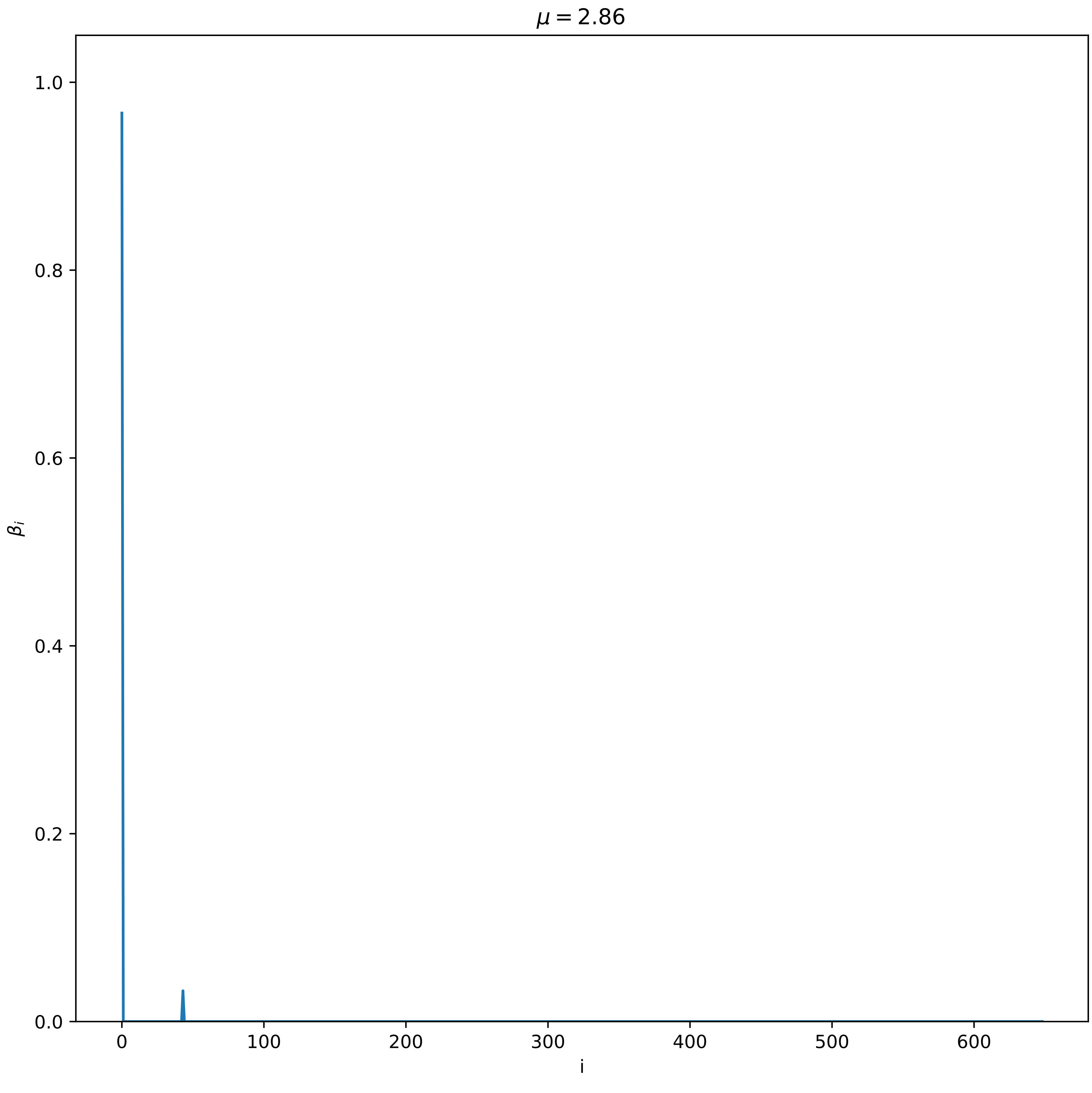
$\mu = 2.82$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



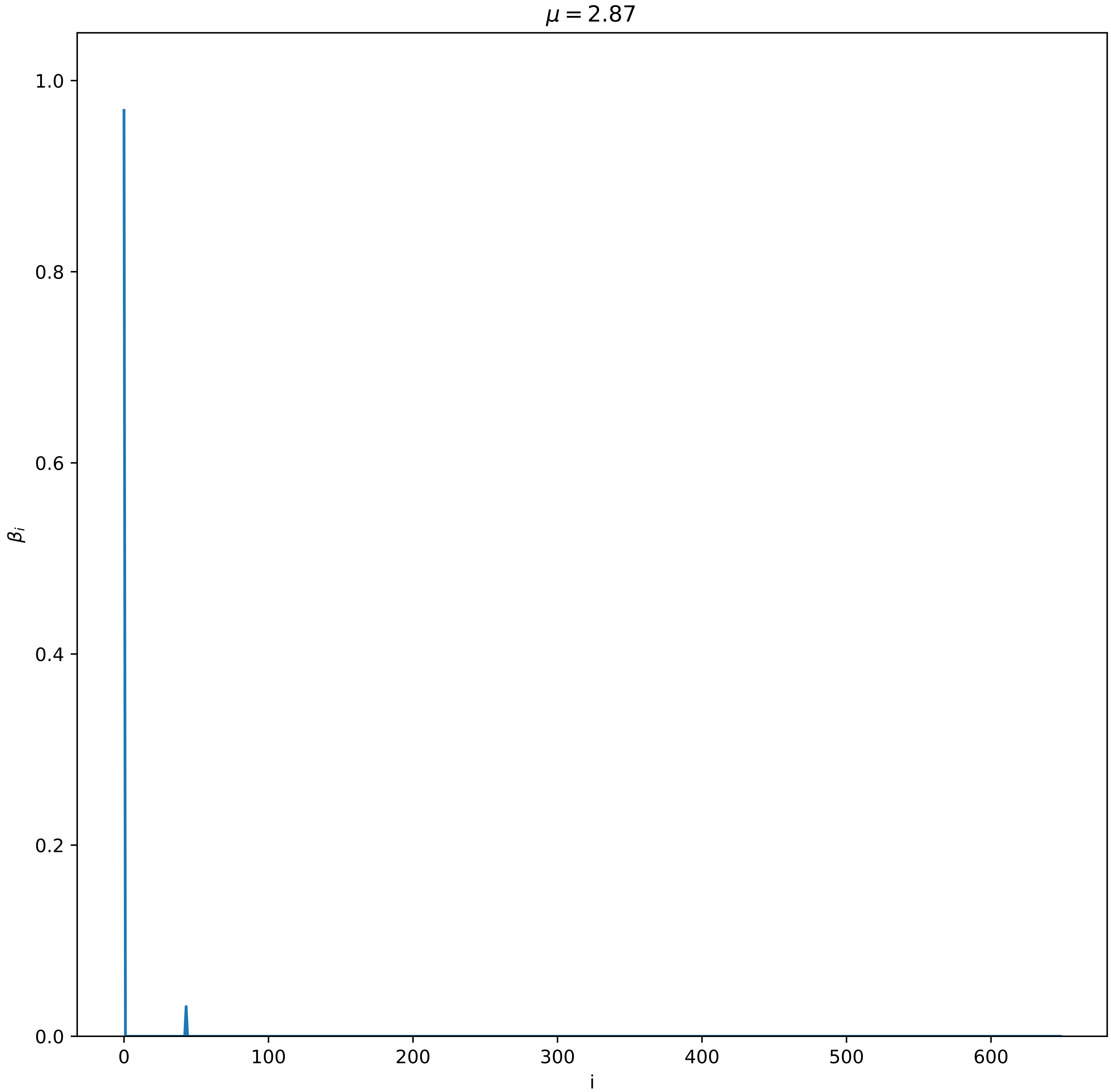
$\mu = 2.83$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 2.84$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

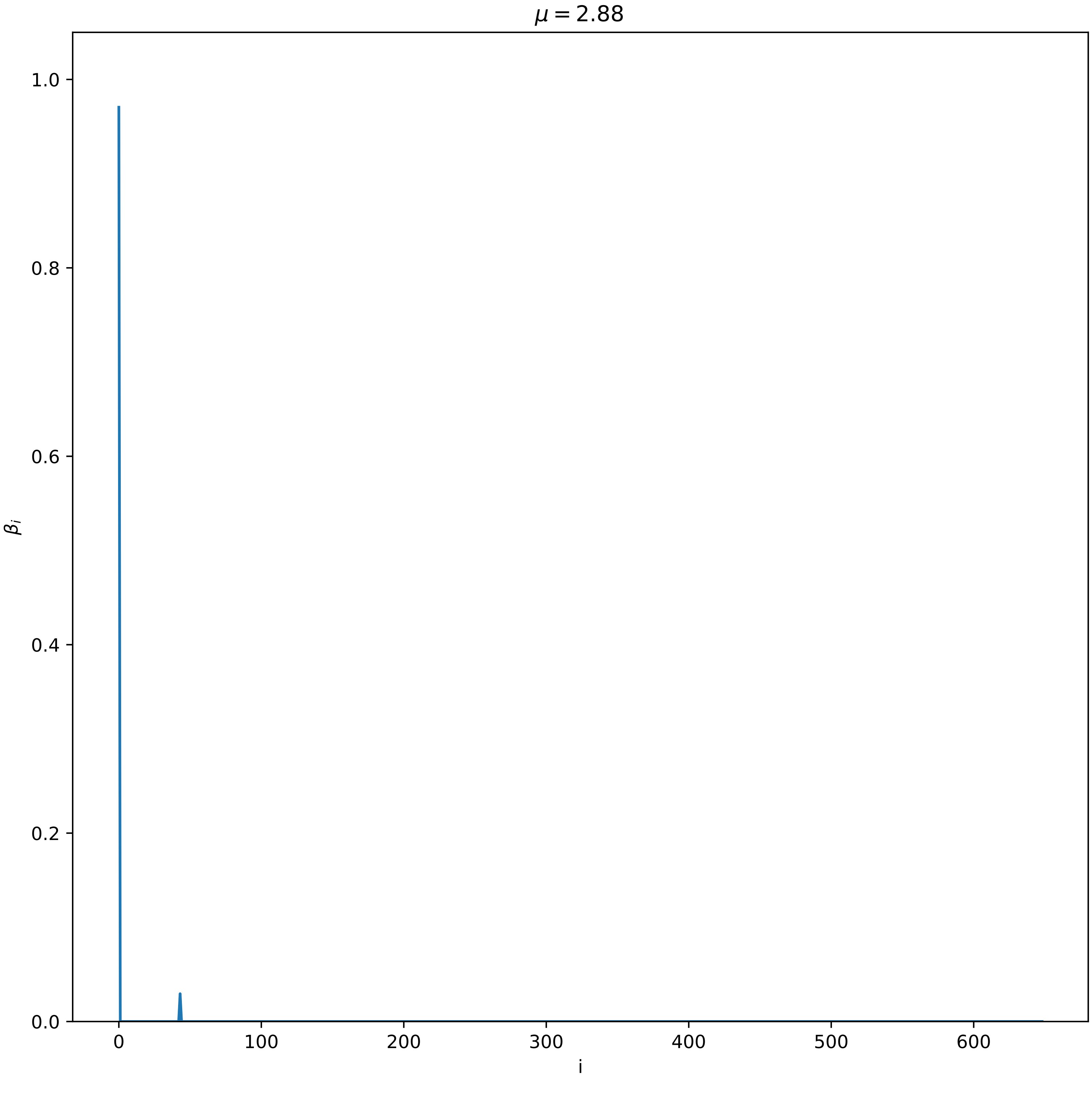

$\mu = 2.85$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



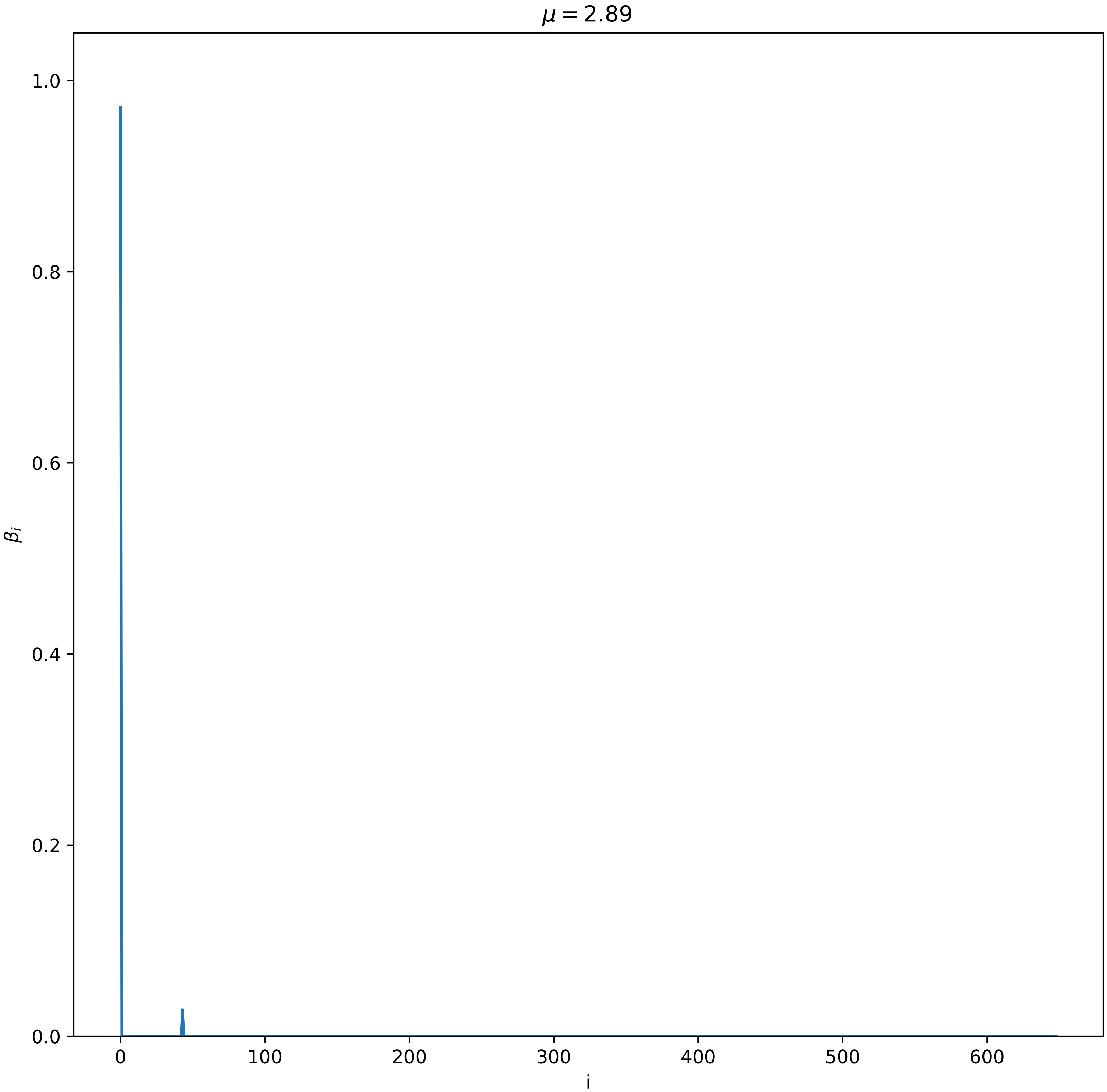
$\mu = 2.86$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



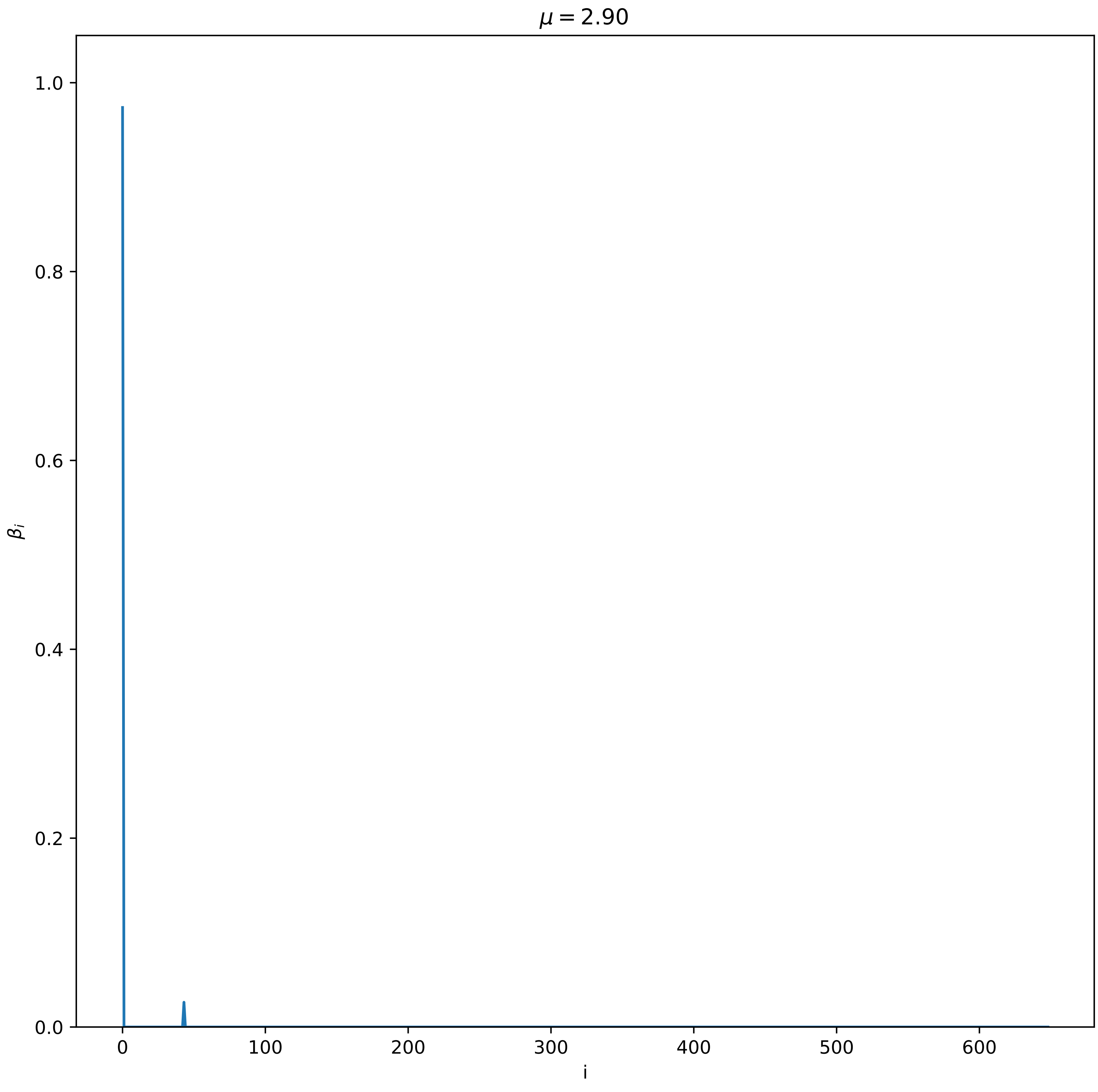
$\mu = 2.87$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

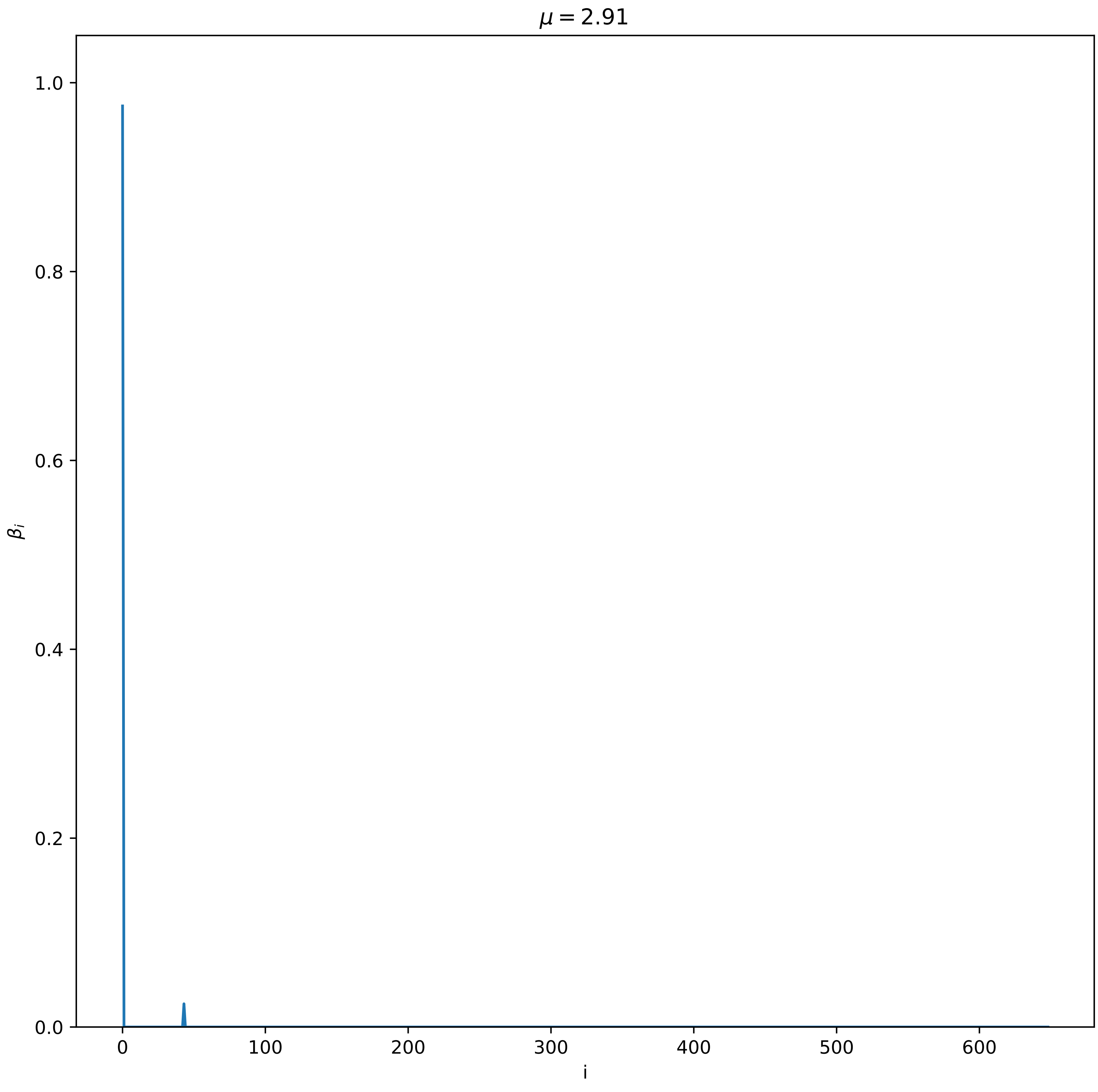


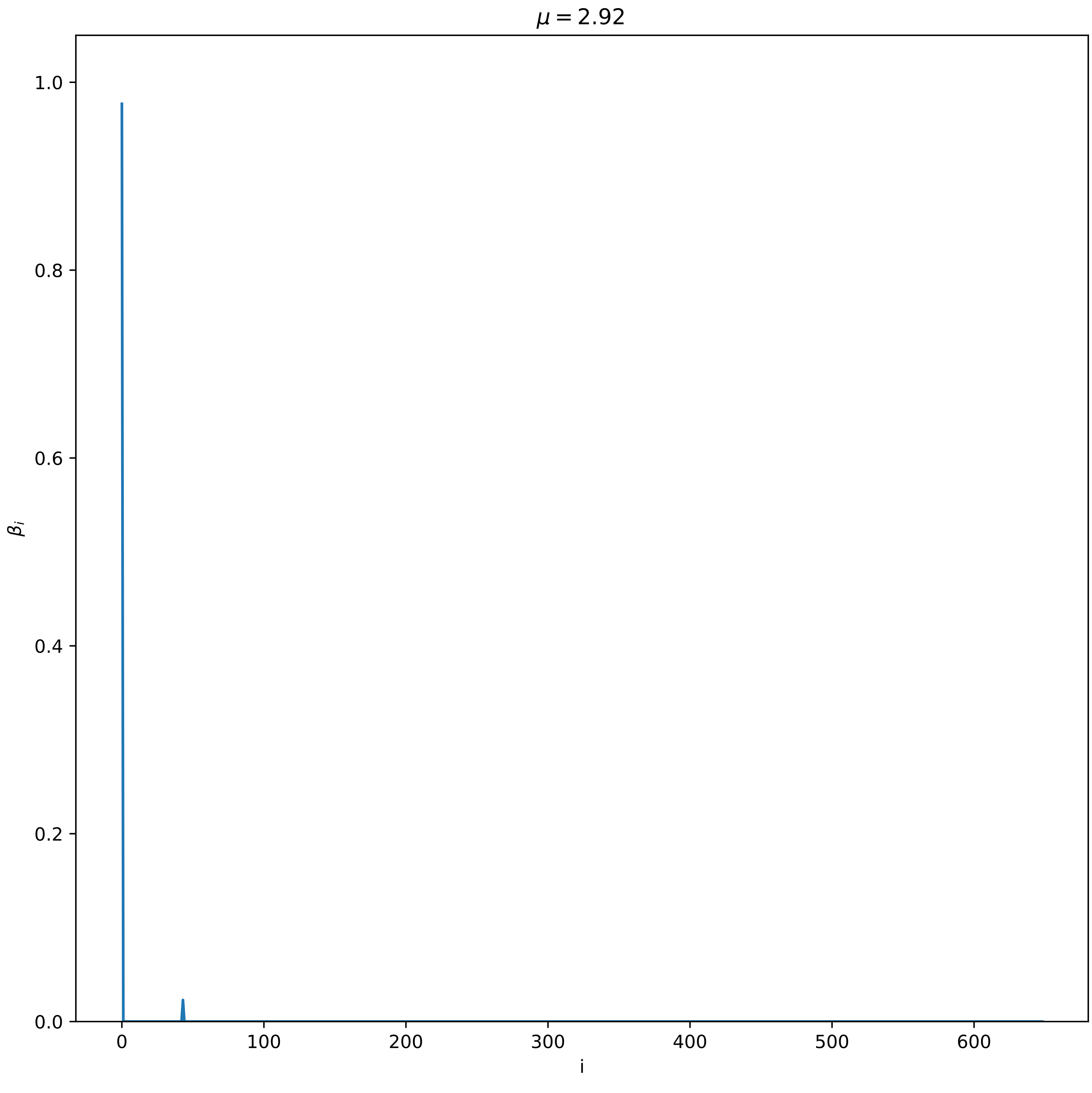
$\mu = 2.88$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



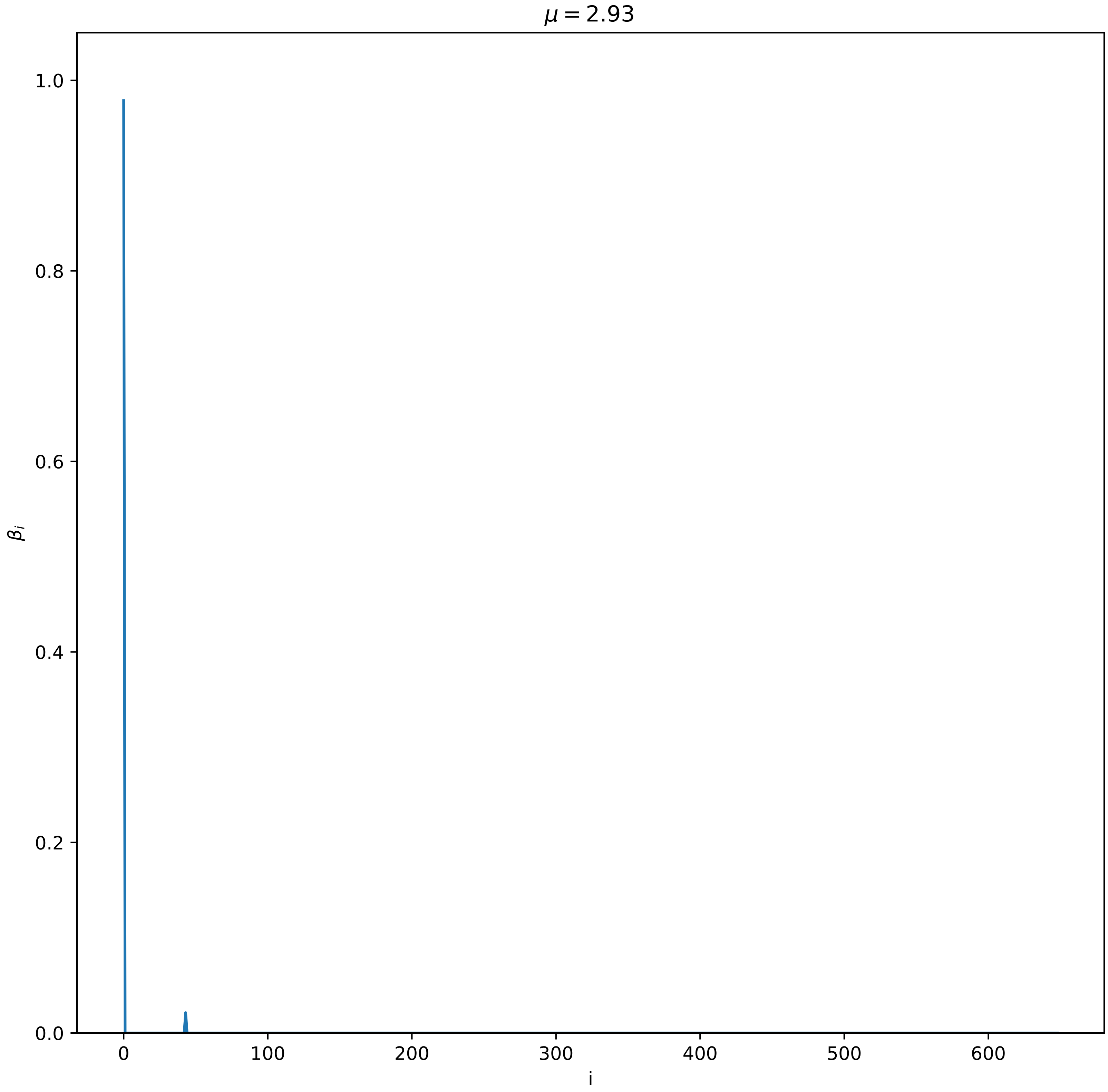
$\mu = 2.89$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

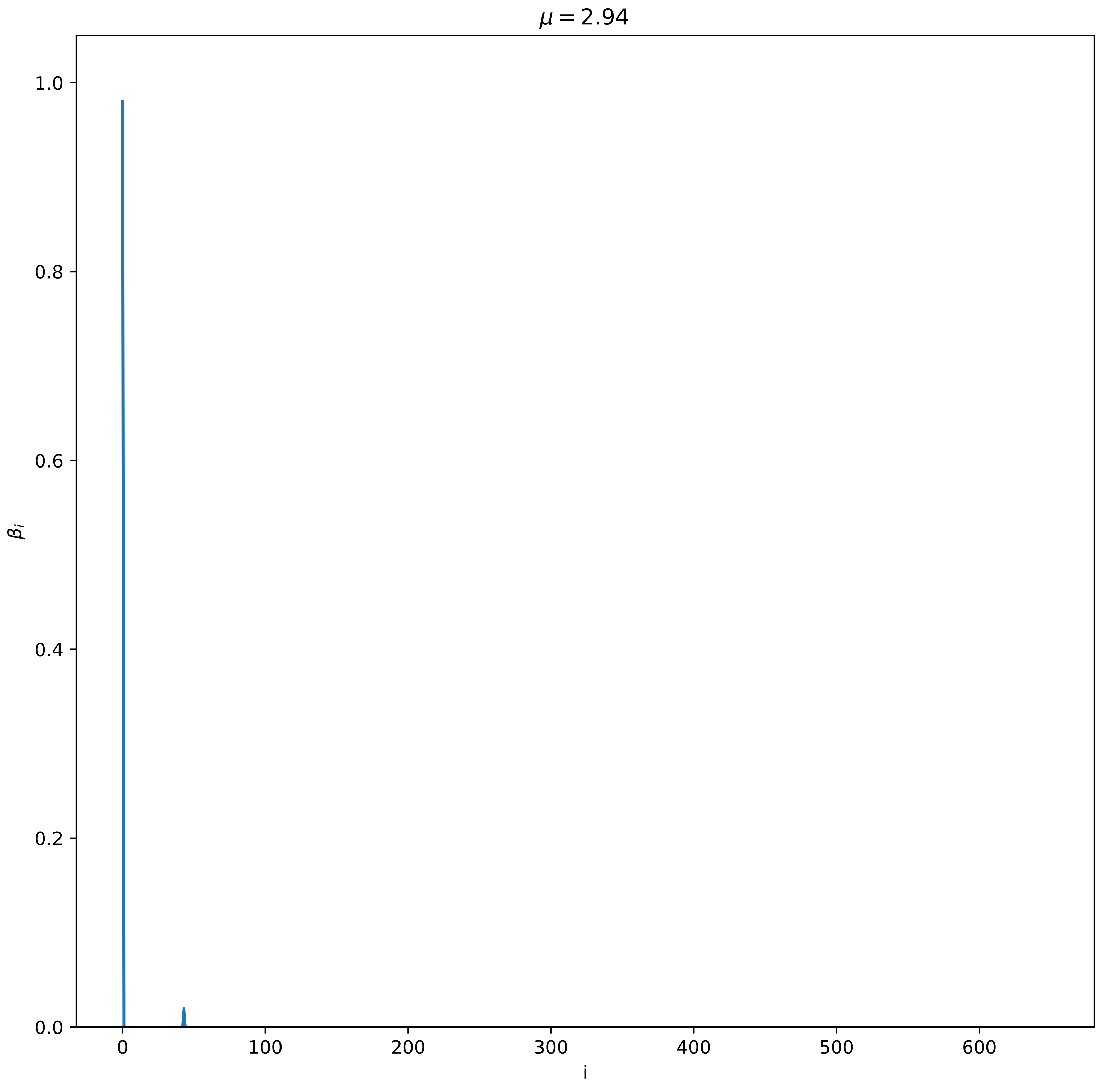

$\mu = 2.90$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

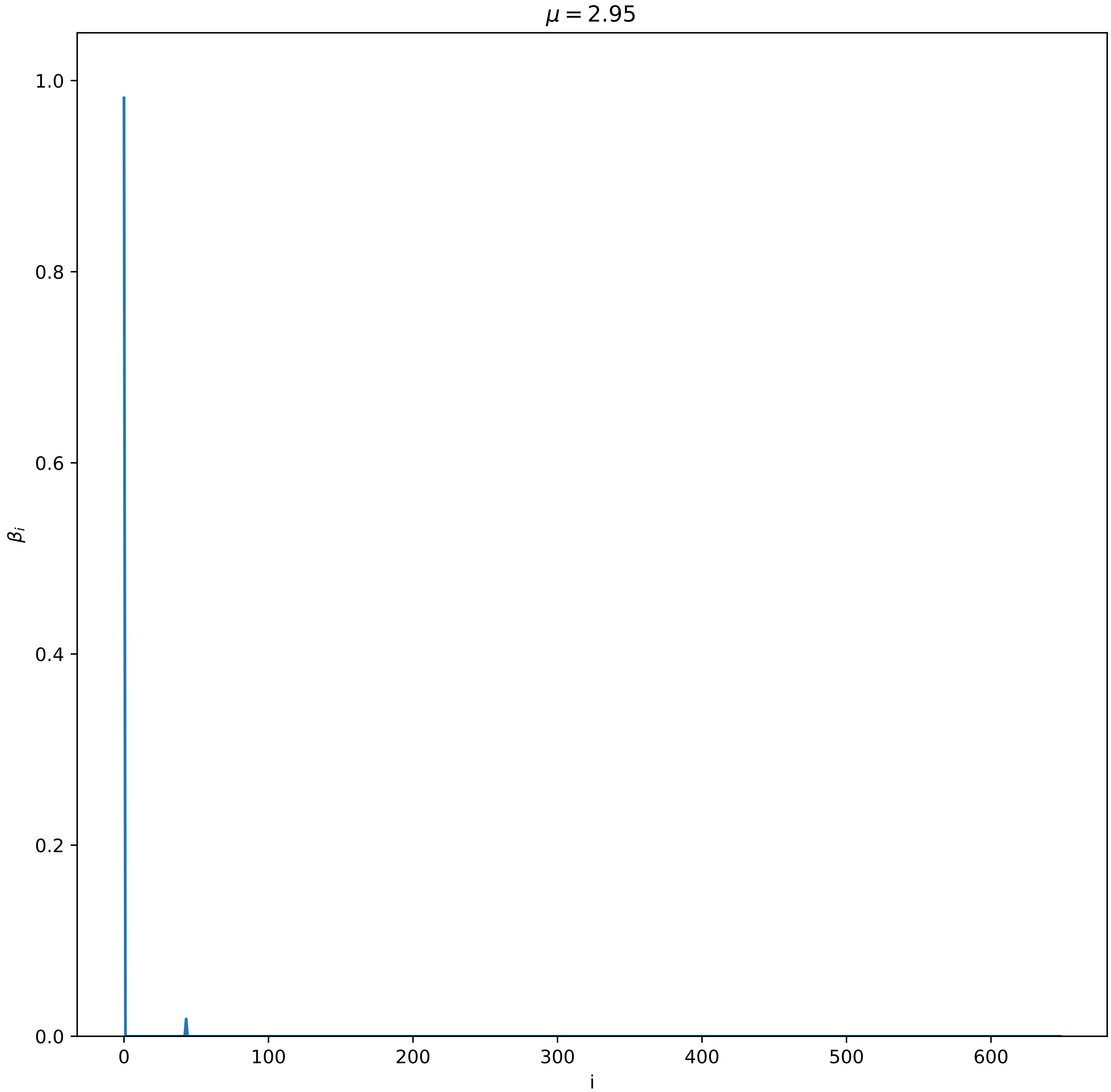

$\mu = 2.91$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



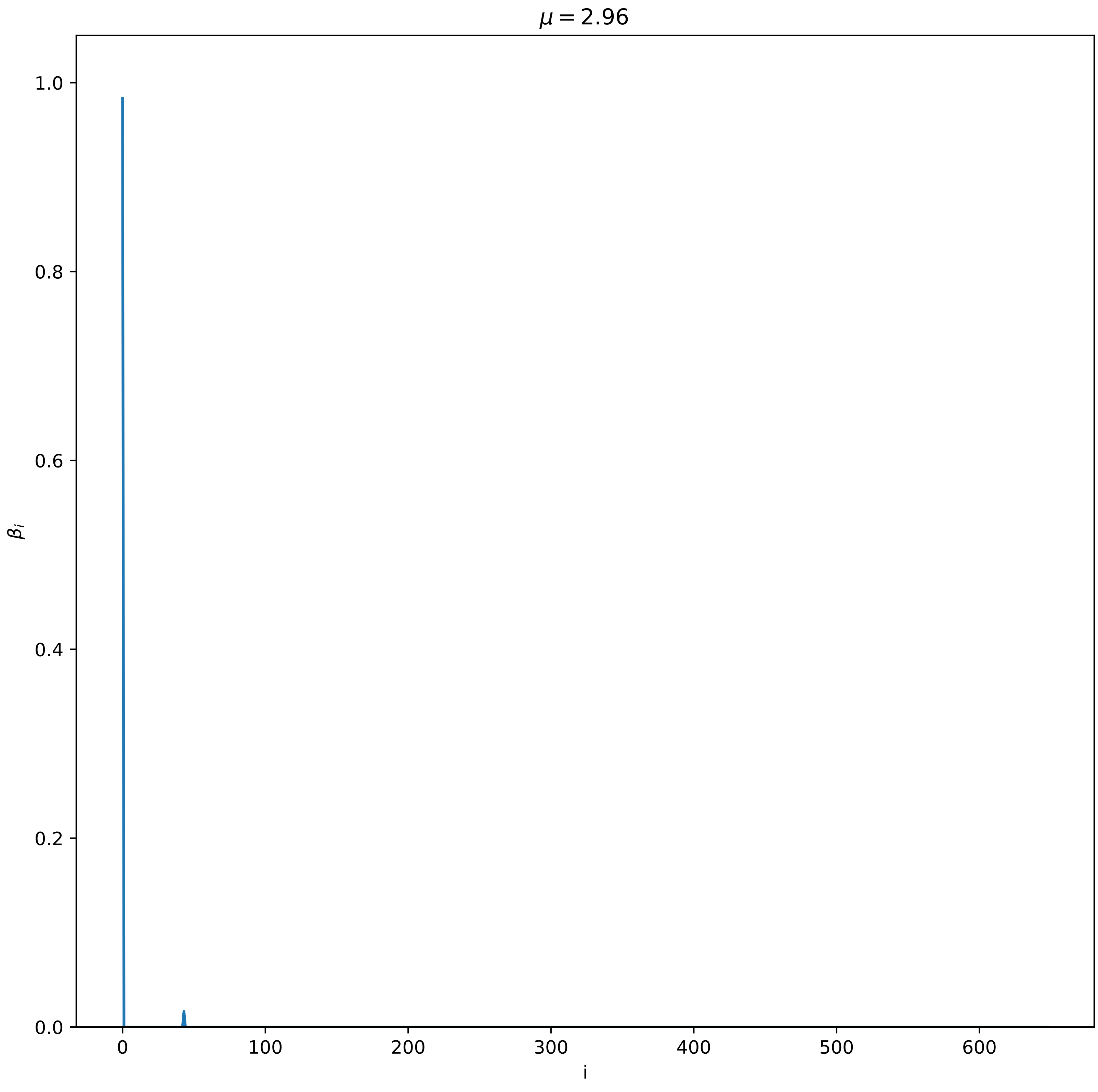
$\mu = 2.92$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 2.93$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

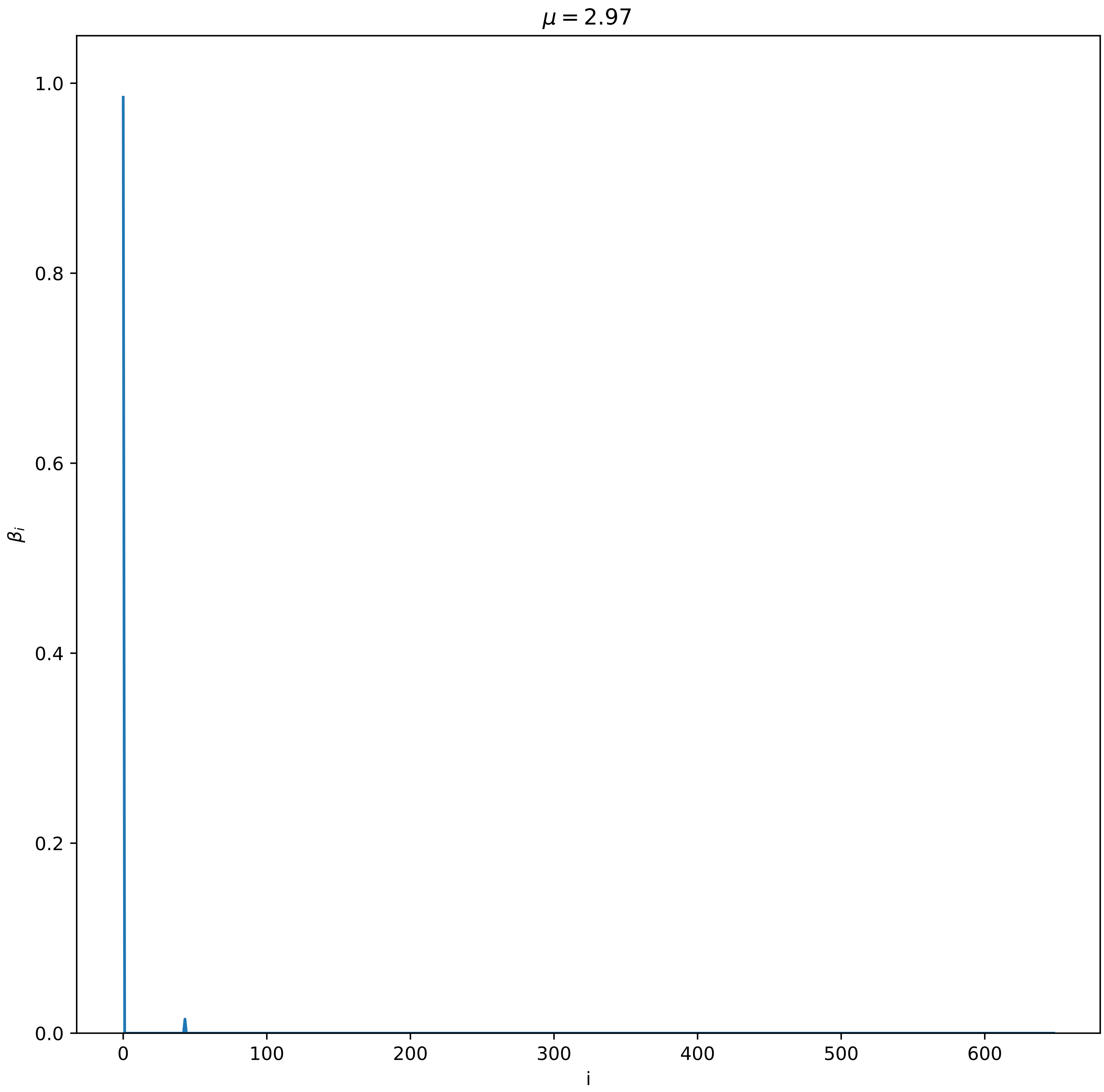

$\mu = 2.94$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



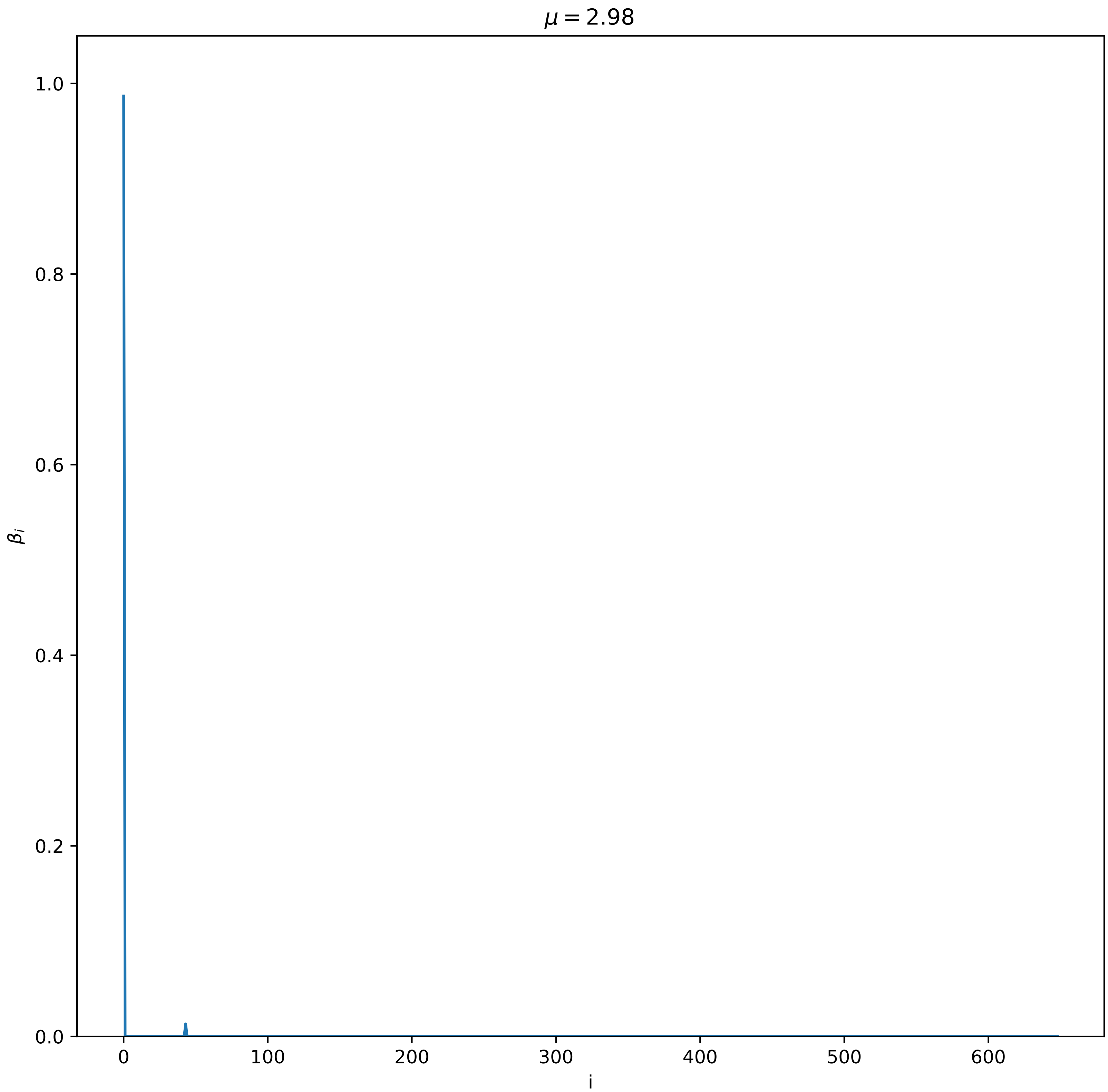
$\mu = 2.95$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

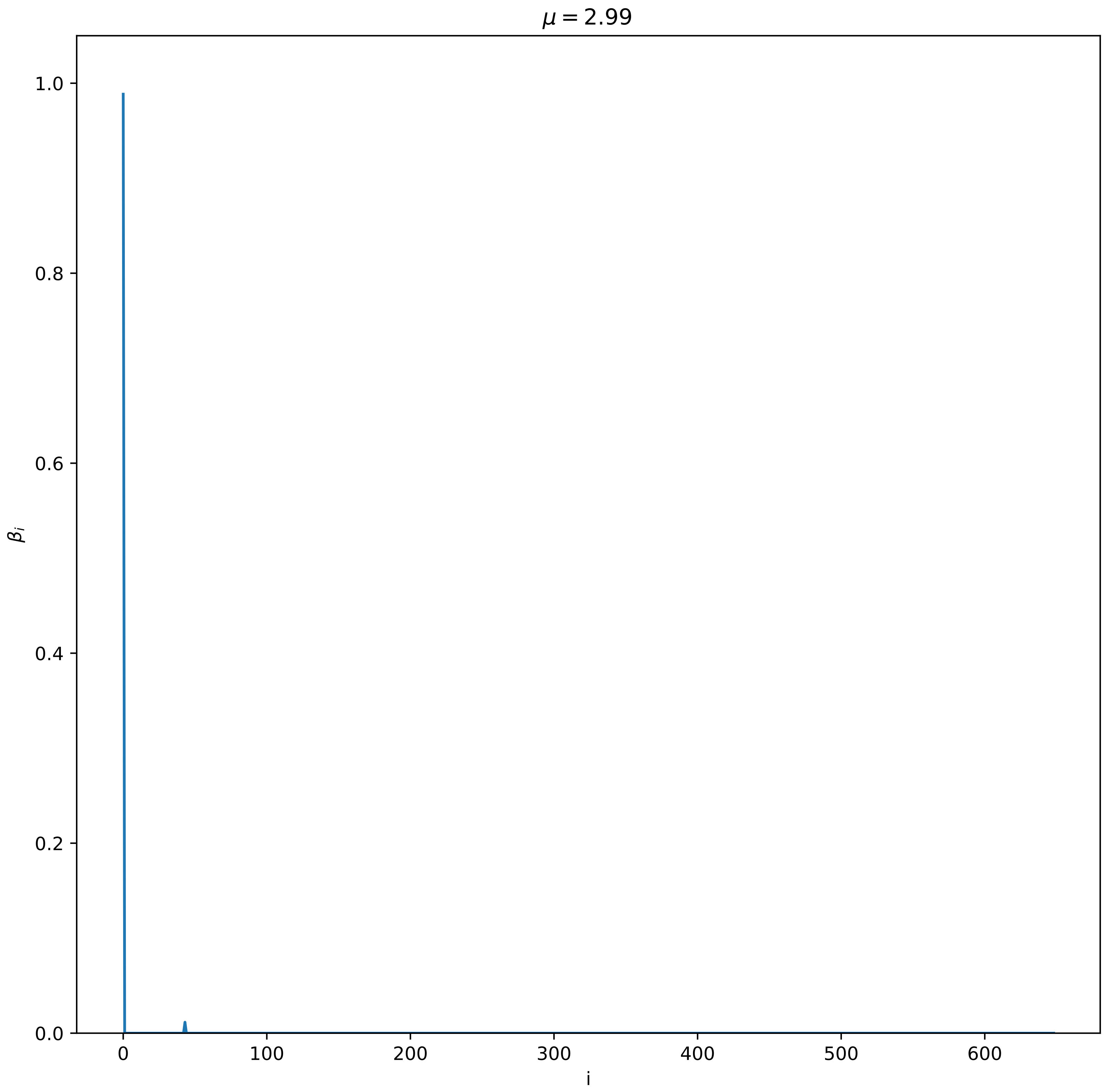


$\mu = 2.96$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

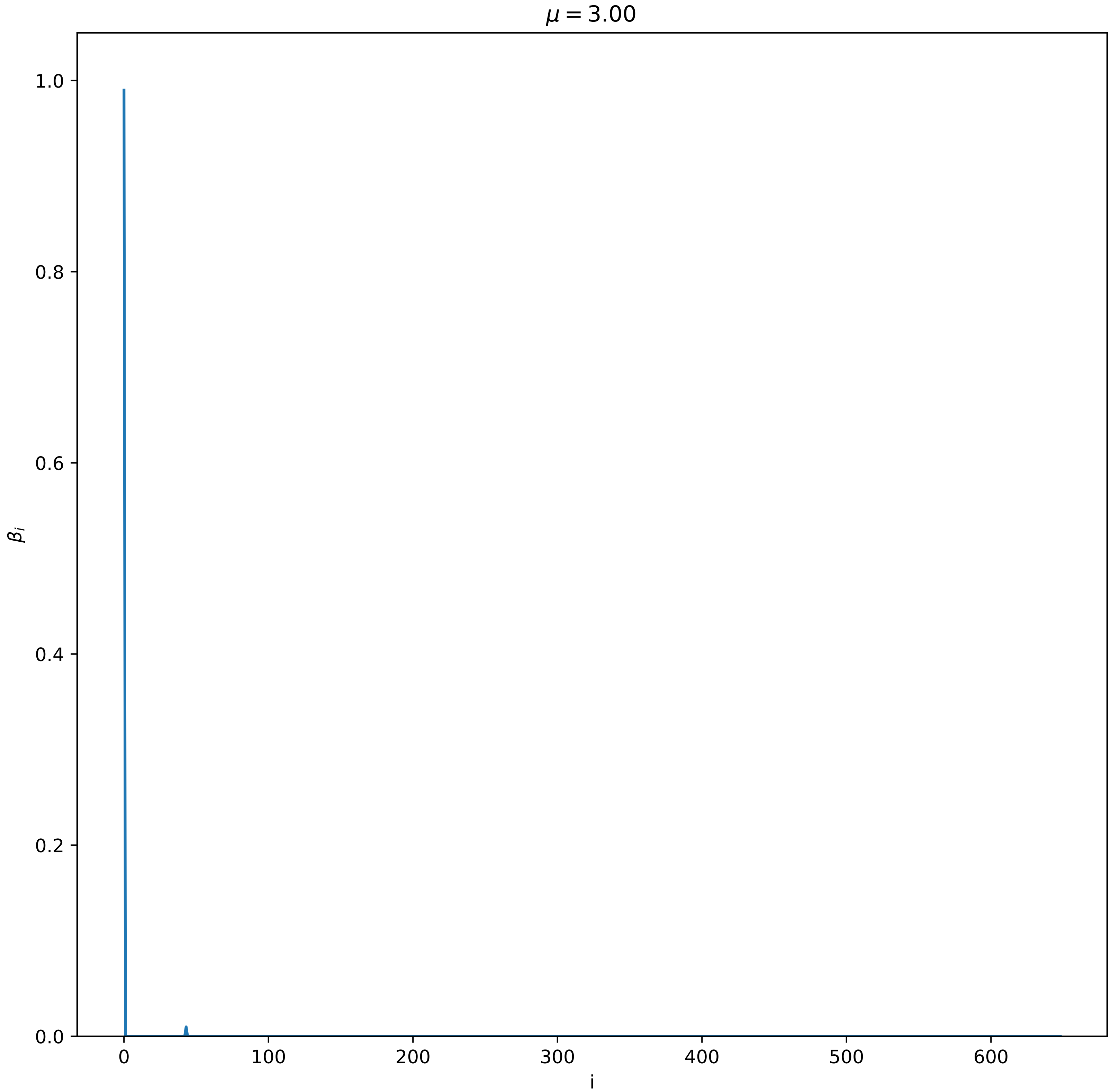


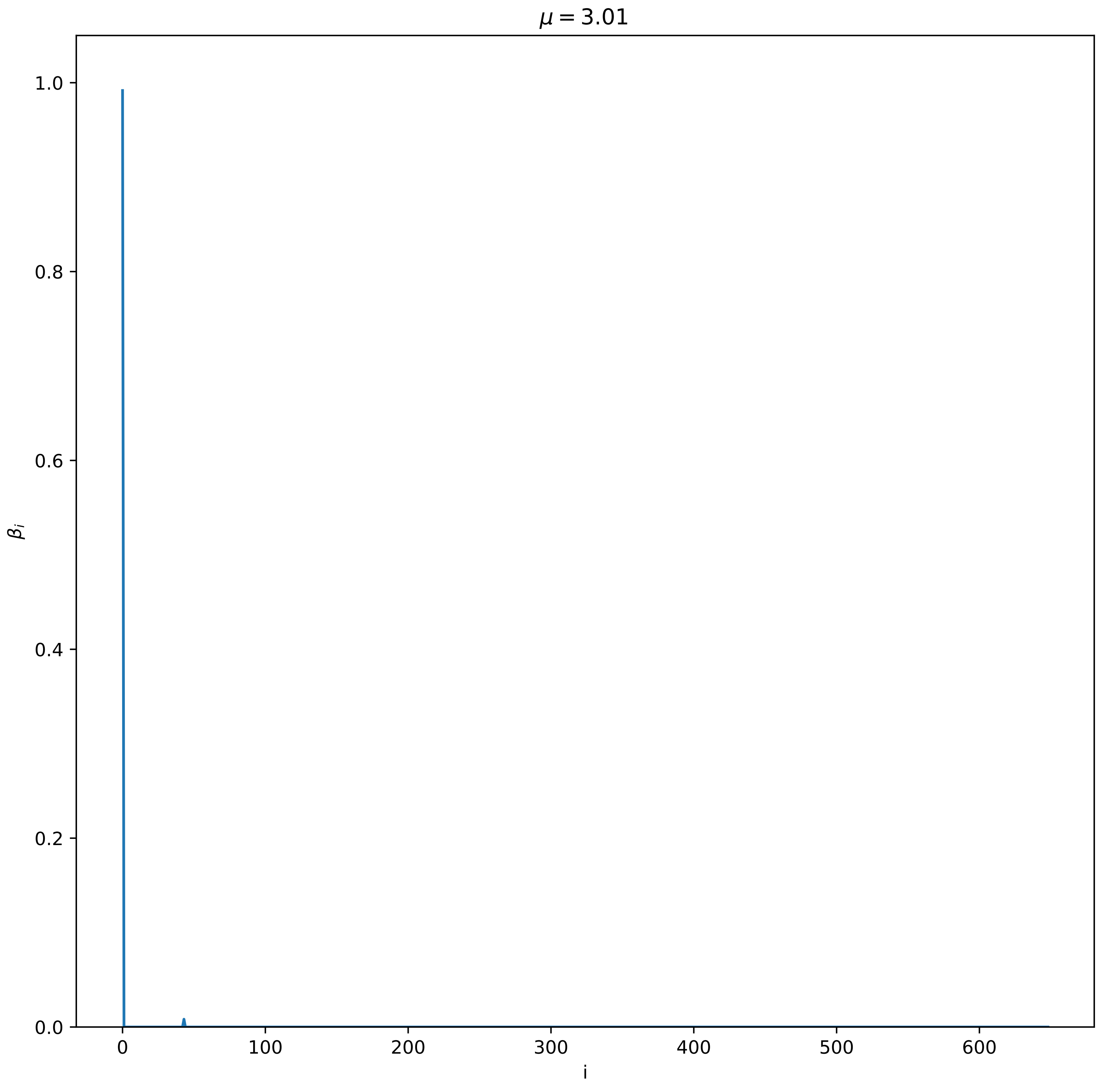
$\mu = 2.97$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 2.98$
β_i
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
600
i

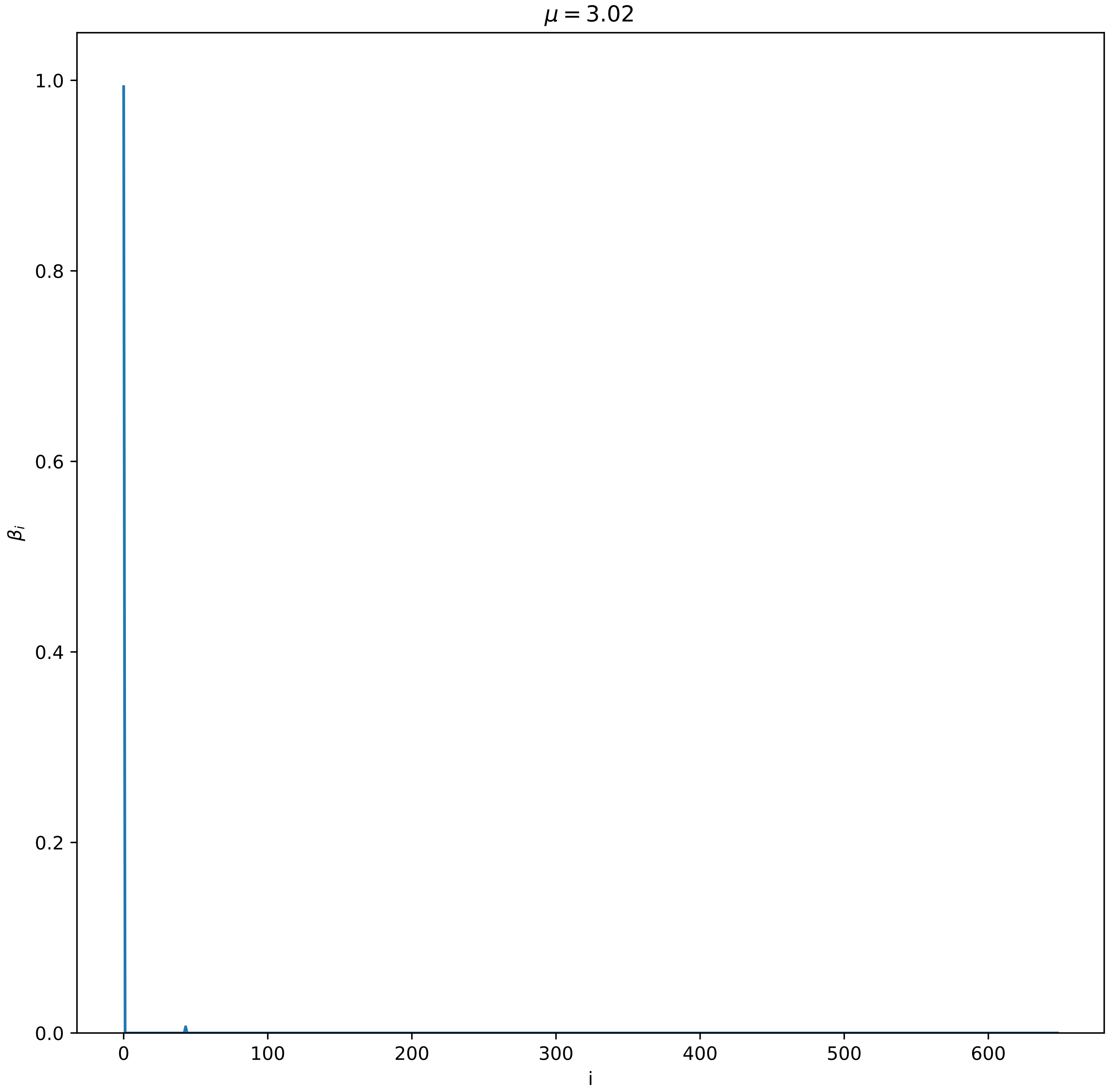


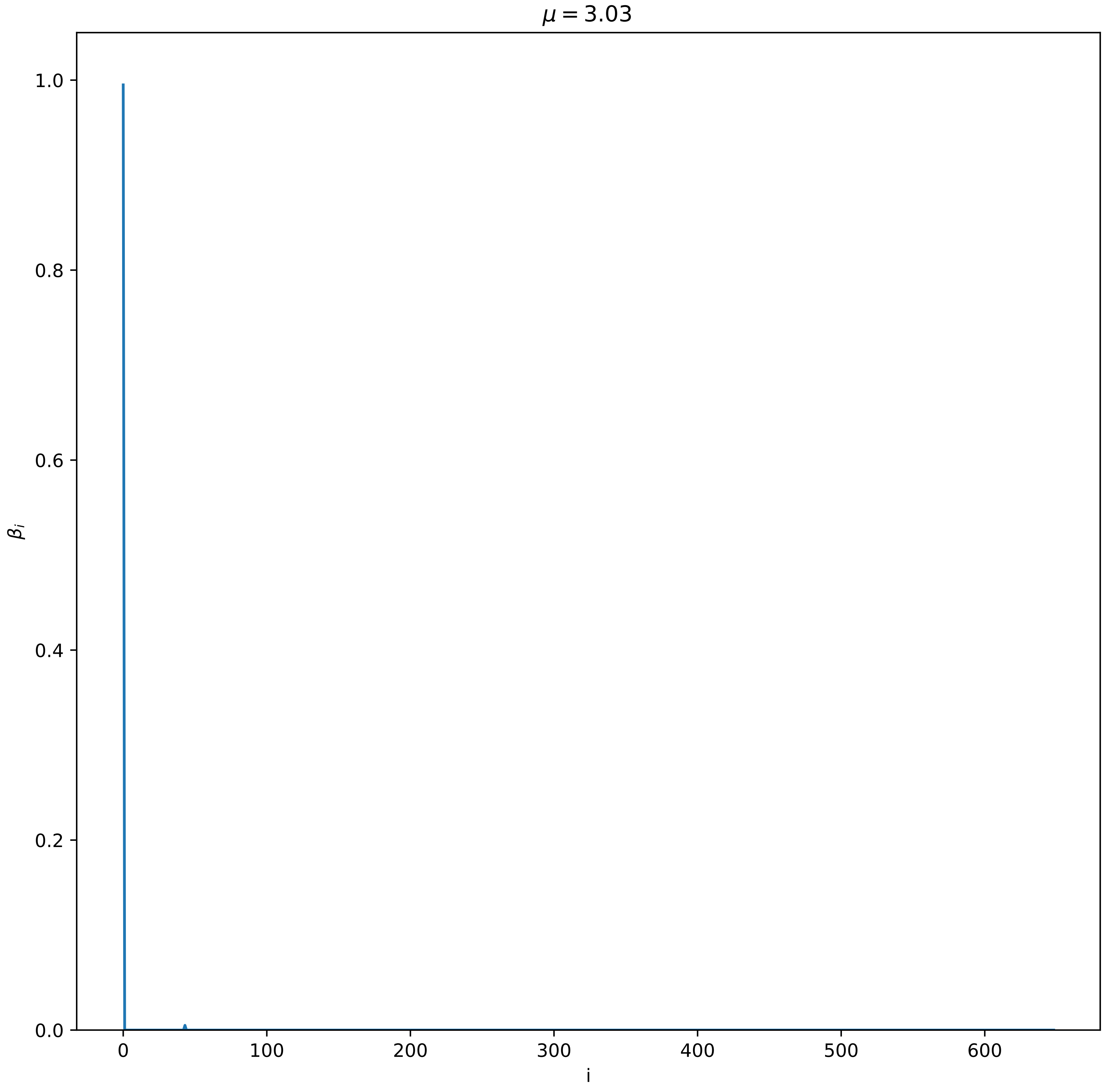
$\mu = 2.99$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 3.00$
β_i
i
0.0
0.2
0.4
0.6
0.8
1.0
0
100
200
300
400
500
600

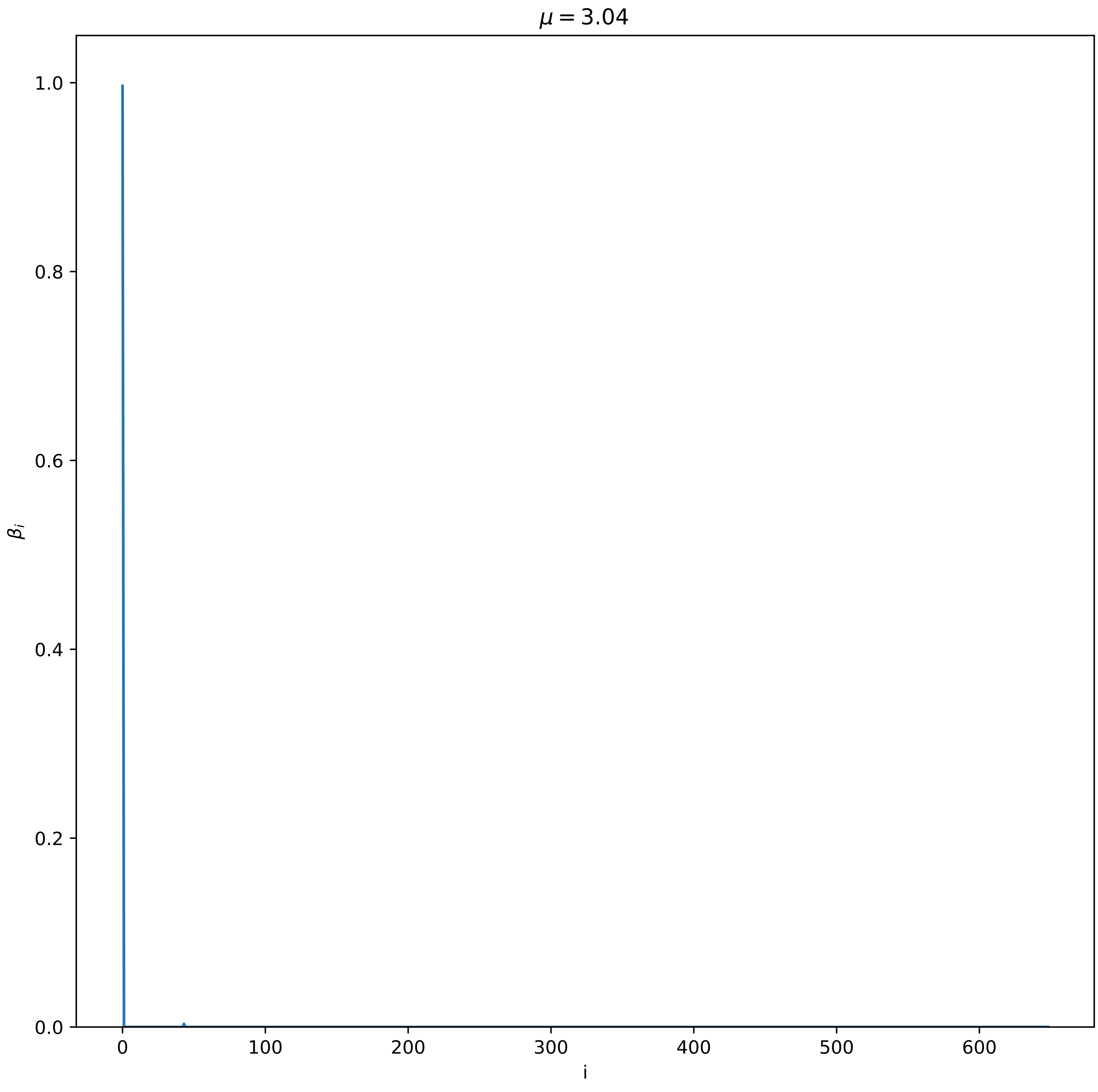


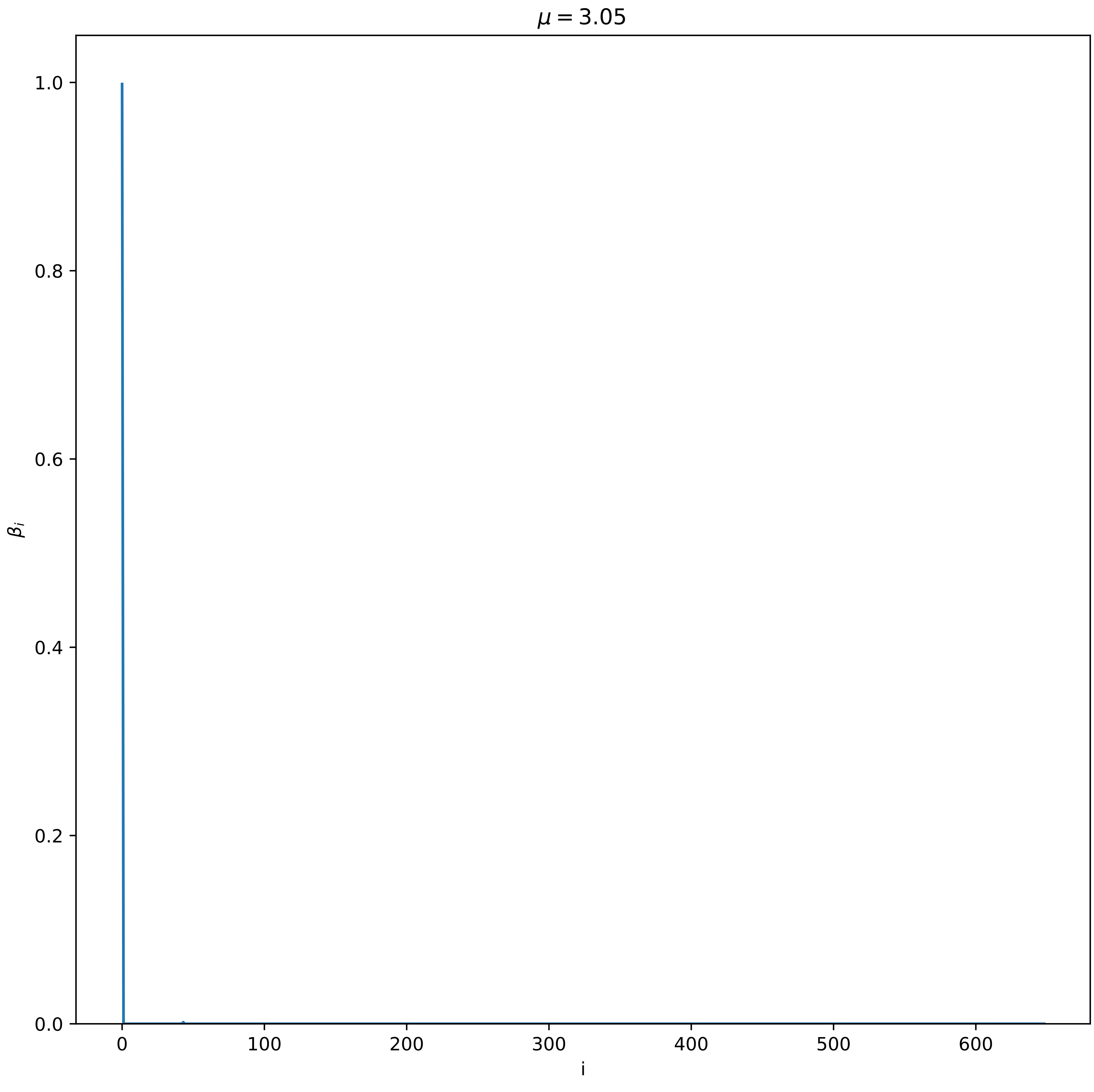
$\mu = 3.01$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 3.02$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

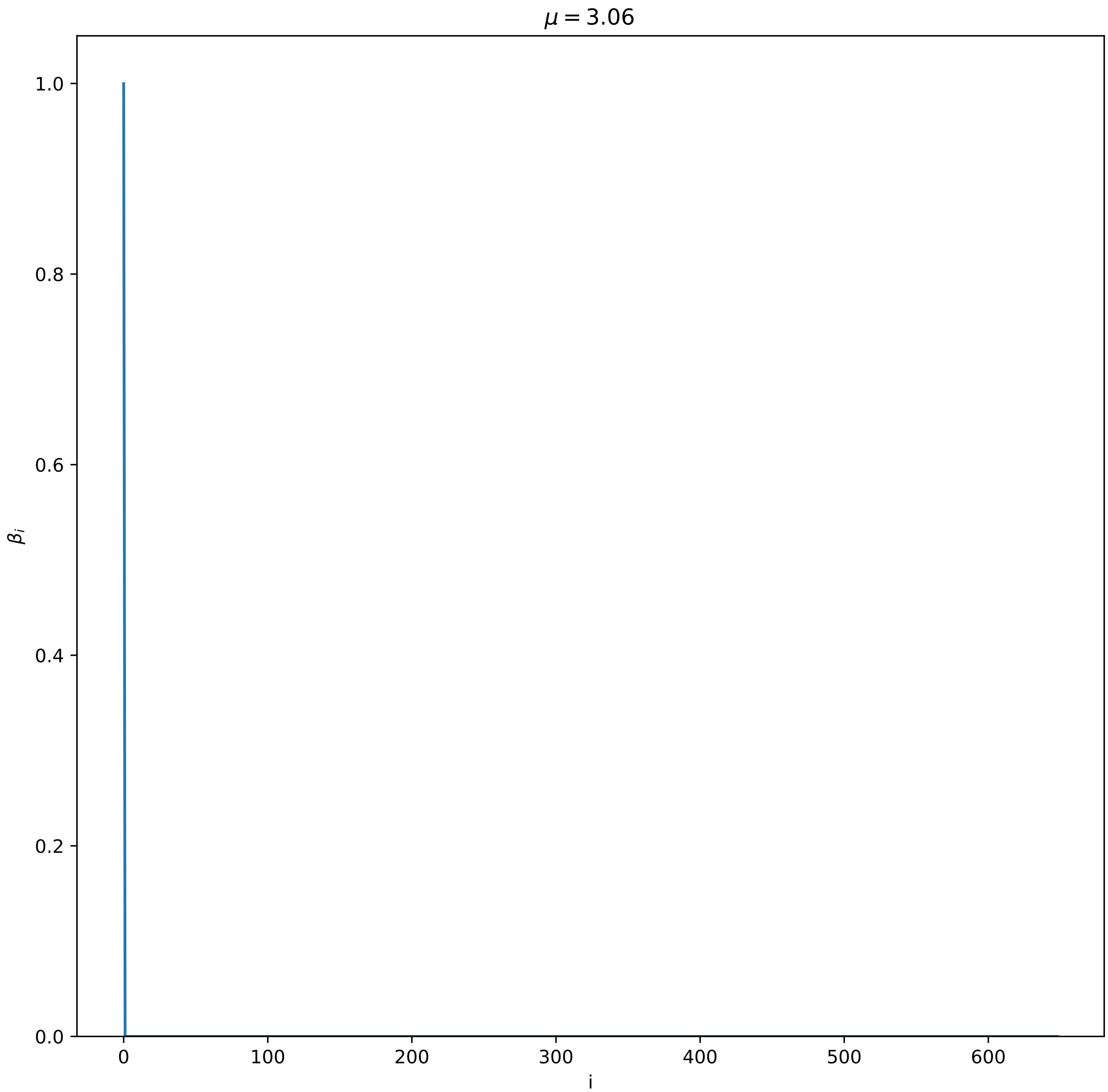


$\mu = 3.03$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

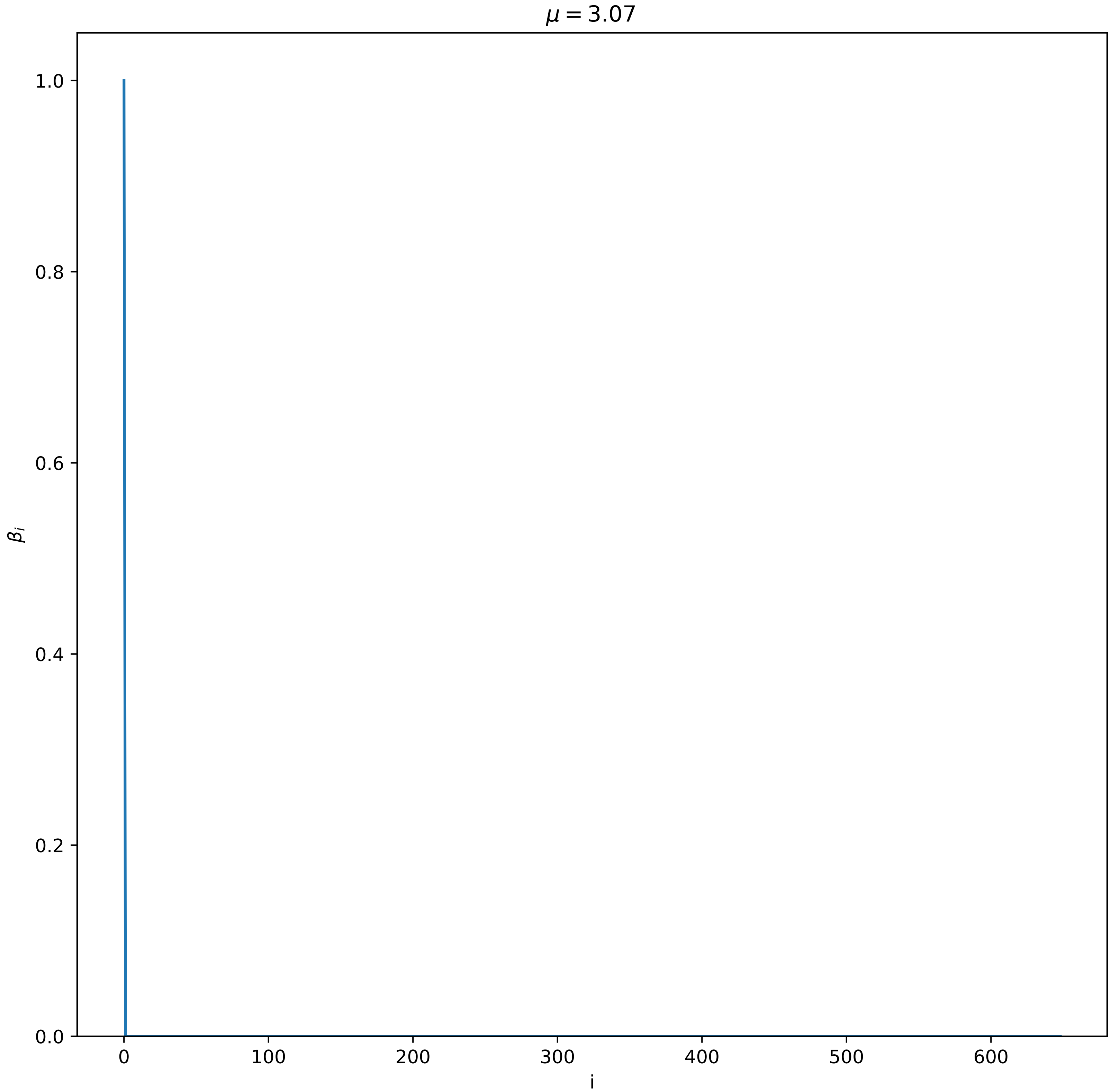

$\mu = 3.04$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



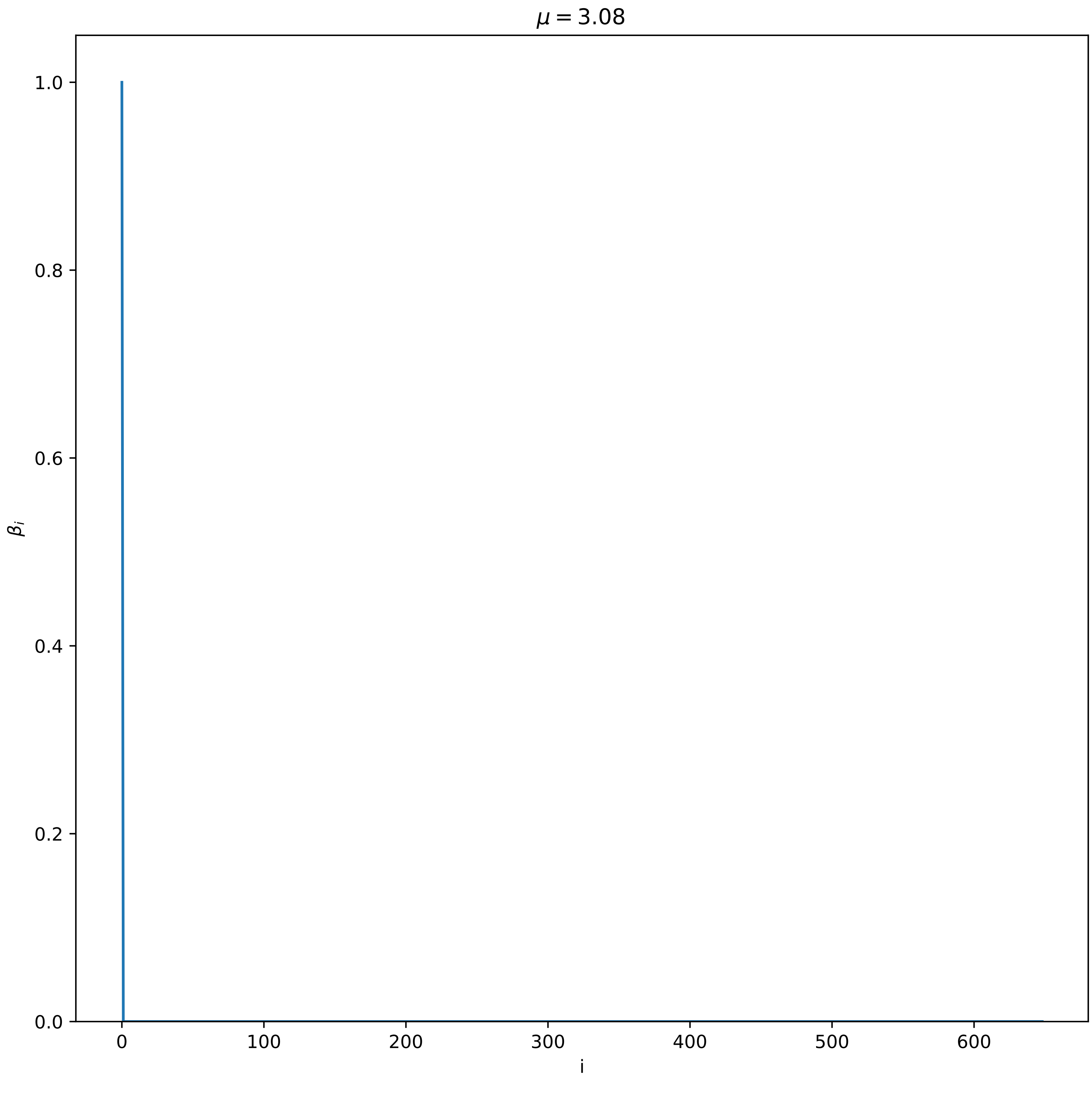
$\mu = 3.05$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

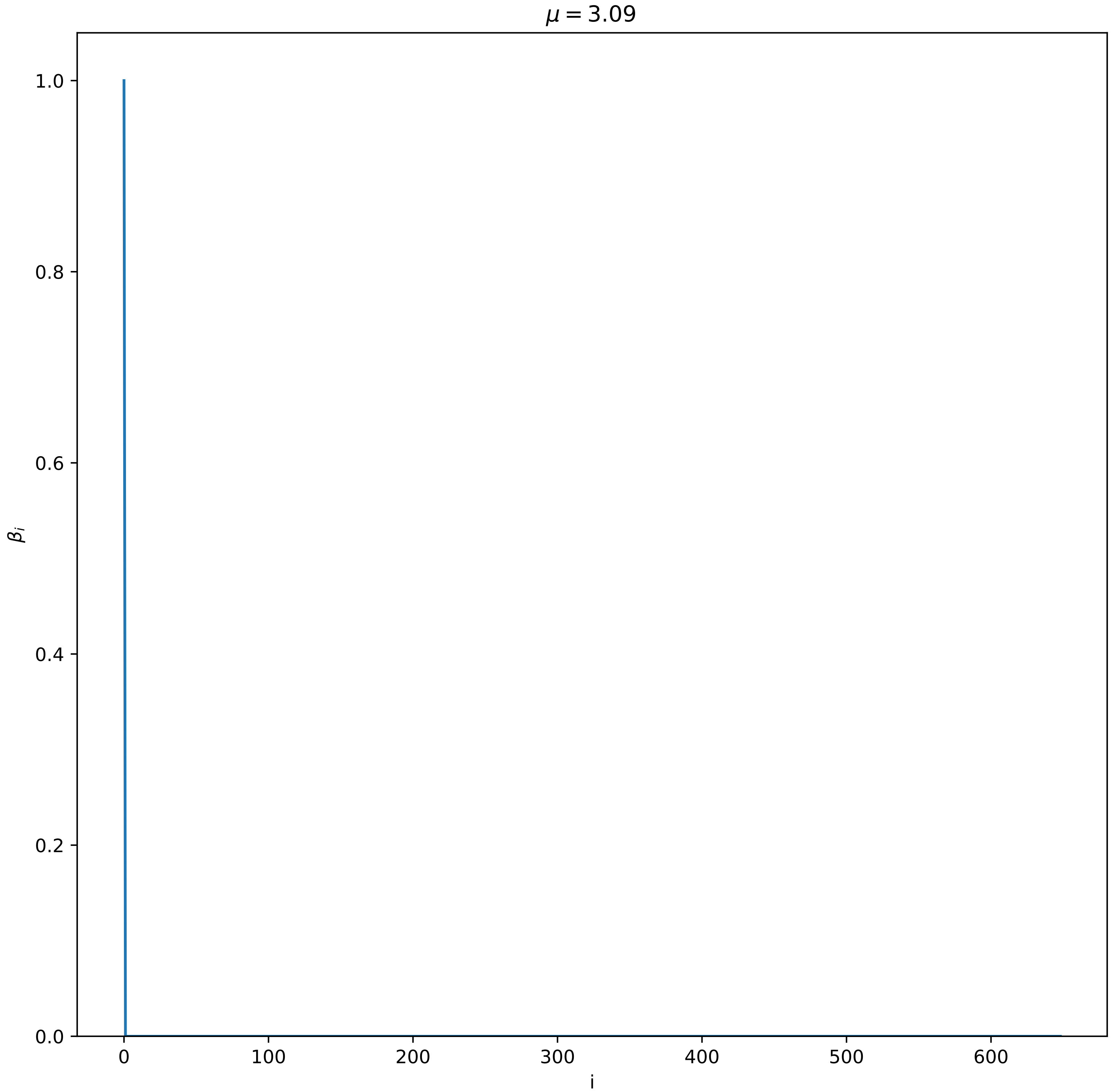


$\mu = 3.06$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

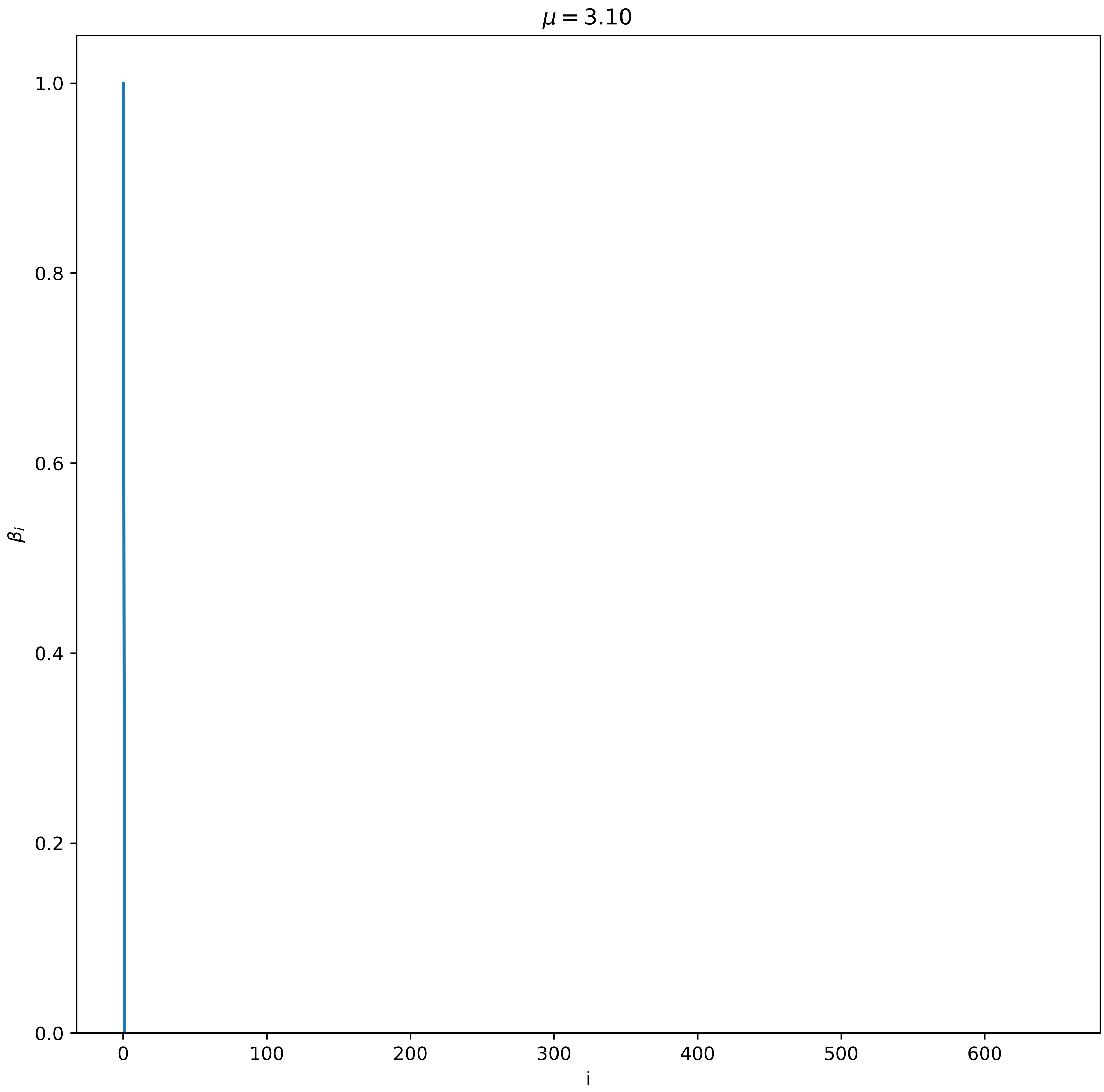


$\mu = 3.07$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i


$\mu = 3.08$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 3.09$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i



$\mu = 3.10$
1.0
0.8
0.6
0.4
0.2
0.0
β_i
0
100
200
300
400
500
600
i

Expert Based Factor Search

Factor Serch на основе технического анализа (Пугач, Морозов):

$$\begin{cases} \boldsymbol{\beta}^T \underbrace{\boldsymbol{\Sigma}} \boldsymbol{\beta} - \mu \underbrace{\bar{\mathbf{x}}^T} \boldsymbol{\beta} + c \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta}\right)^T \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta}\right) \to \min(\boldsymbol{\beta}) \\ \mathbf{1}^T \boldsymbol{\beta} = 1; \qquad \boldsymbol{\beta} \geqslant \mathbf{0} \end{cases} \tag{6}$$

Наш Factor Serch с учетом экспертоного мнения:

$$\begin{cases} \boldsymbol{\beta}^T \underbrace{\Big((1-\alpha)\boldsymbol{\Sigma} + \alpha \mathbf{B}\Big)} \boldsymbol{\beta} - \mu \underbrace{\Big((1-\alpha)\bar{\mathbf{x}} + \alpha \mathbf{z}\Big)^T} \boldsymbol{\beta} \\ \qquad + c \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta}\right)^T \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta}\right) \to \min(\boldsymbol{\beta}) \\ \mathbf{1}^T \boldsymbol{\beta} = 1; \qquad \boldsymbol{\beta} \geqslant \mathbf{0} \end{cases} \tag{7}$$

$$\begin{cases} \boldsymbol{\beta}^T \Big((1-\alpha)\boldsymbol{\Sigma} + \alpha \mathbf{B} \Big) \boldsymbol{\beta} - \mu \Big((1-\alpha)\overline{\mathbf{x}} + \alpha \mathbf{z} \Big)^T \boldsymbol{\beta} + \\ \qquad\qquad + c \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta} \right)^T \left(\mathbf{y} - \mathbf{X}^T \boldsymbol{\beta} \right) \to \min(\boldsymbol{\beta}) \\ \mathbf{1}^T \boldsymbol{\beta} = 1; \qquad \boldsymbol{\beta} \geqslant \mathbf{0} \end{cases} \tag{8}$$

Если $\mathbf{y} = (y_t, t = 1, \ldots, T)$ — анализируемый портфель, то всякая комбинация параметров (α, μ, c) дает решение этой задачи как совокупность коэффициентов $\hat{\boldsymbol{\beta}} = (\beta_i, i = 1, \ldots, n)$ и следовательно состав портфеля:
$\hat{\mathbb{I}} = \{i : \beta_i > 0\} \in \mathbb{I}, \quad \hat{n} = |\hat{\mathbb{I}}|$ — размер портфеля

Это ничто иное как параметры регуляризации модели.
Для выбора их значений для данного наблюденного сигнала $\mathbf{y}$ необходимо использовать некоторый критерий верификации модели (обощающей способности). Например Leave One Out (LOO).

Критерий Leave One Out для Factor Search

Итак, пусть $\mathbf{y} = (y_t, t = 1, \ldots, T)$ — анализируемый портфель.
В предположении, что администратор построил его по принципу рассмотренному ранее, остается найти значения параметров α, μ, c

Критерий LOO дает количественную оценку всякому варианту α, μ, c.

$$LOO(\alpha, \mu, c) = \frac{1}{T} \sum_{t=1}^{T} \left(y_t - \sum_{i=1}^{n} \beta^{(t)}_{\alpha,\mu,c,i} x_{t,i} \right)^2 \to \min(\alpha, \mu, c) \qquad (9)$$

Одна трудность — вычисление $\beta^{(t)}$ с одним пропущенным наблюдением нужно повторить очень много раз (T раз).

Для преодоления этой проблемы Пугач и Морозов разработали специальную форму критерия LOO, учитывающую специфику задачи Factor Search.

Критерий Leave One Out для Factor Search

Специальная форма критерия LOO для Factor Search:

$$LOO(\alpha, \mu, c) = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{y_t - \sum_{i=1}^{n}\beta_{\alpha,\mu,c,i}x_{t,i}}{1 - \mathbf{x}^T_{\mathbb{I}_{\alpha,\mu,c},t}\left(\mathbf{X}_{\mathbb{I}_{\alpha,\mu,c}}\mathbf{X}^T_{\mathbb{I}_{\alpha,\mu,c}}\right)^{-1}\mathbf{x}_{\mathbb{I}_{\alpha,\mu,c},t}}\right)^2 \to \min(\alpha,$$

Здесь для каждого комбинаций набора (α, μ, c) достаточно один раз найти вектор коэффицентов регрессии, обратив соответствующую матрицу.

Экспериментальная часть

$n = 650; \qquad T = 240$

Строим портфели:

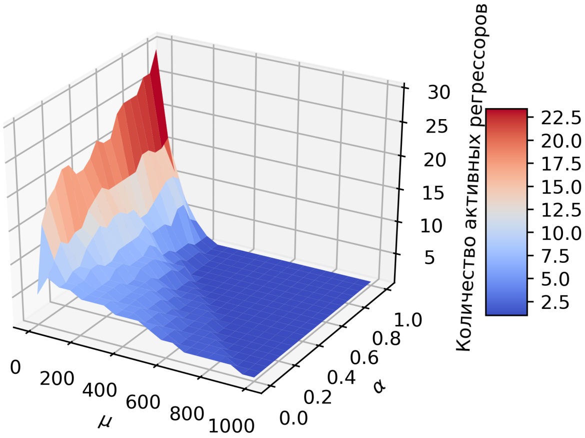
$$\begin{cases} \boldsymbol{\beta}^*_{\mu,\alpha} = \arg\min\left(\alpha\boldsymbol{\beta}^T\boldsymbol{\Sigma}\boldsymbol{\beta} + (1-\alpha)\boldsymbol{\beta}^T\mathbf{B}\boldsymbol{\beta} + \mu\bar{\mathbf{x}}^T\boldsymbol{\beta}\right) \\ \mathbf{1}^T\boldsymbol{\beta} = 1 \\ \boldsymbol{\beta} \geqslant \mathbf{0} \end{cases} \tag{10}$$

Множество активных индексов: $\mathbb{I}^*_{\mu,\alpha} = \{i : \beta^*_{\mu,\alpha;i} > 0\}$

$$y_{\mu,\alpha,t} = \sum_{i \in \mathbb{I}^*_{\mu,\alpha}} \beta^*_{\mu,\alpha,i} x_{t,i} + \xi_t \tag{11}$$

$\sigma^2(\xi) = 10\%$

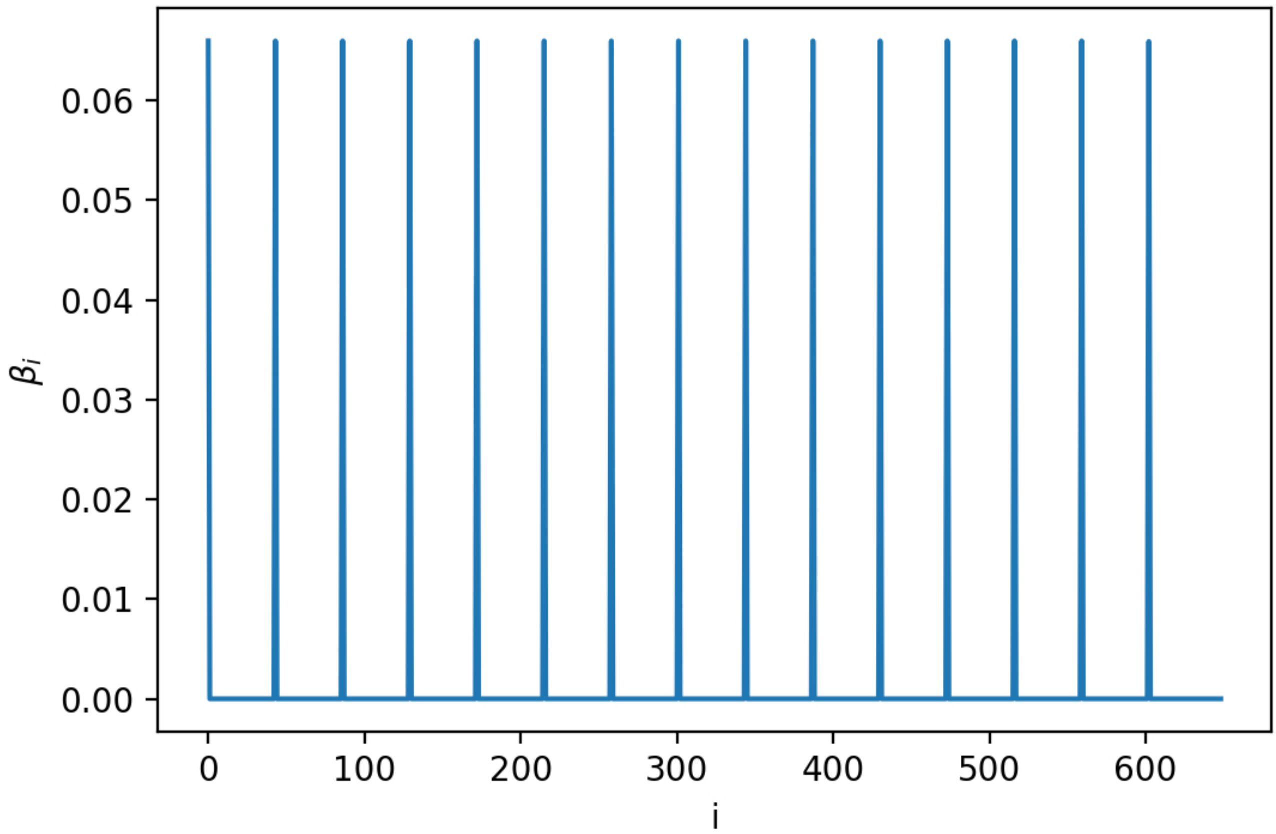
Зависимость размера портфеля : $|\mathbb{I}^*_{\mu,\alpha}|$ от α, μ :

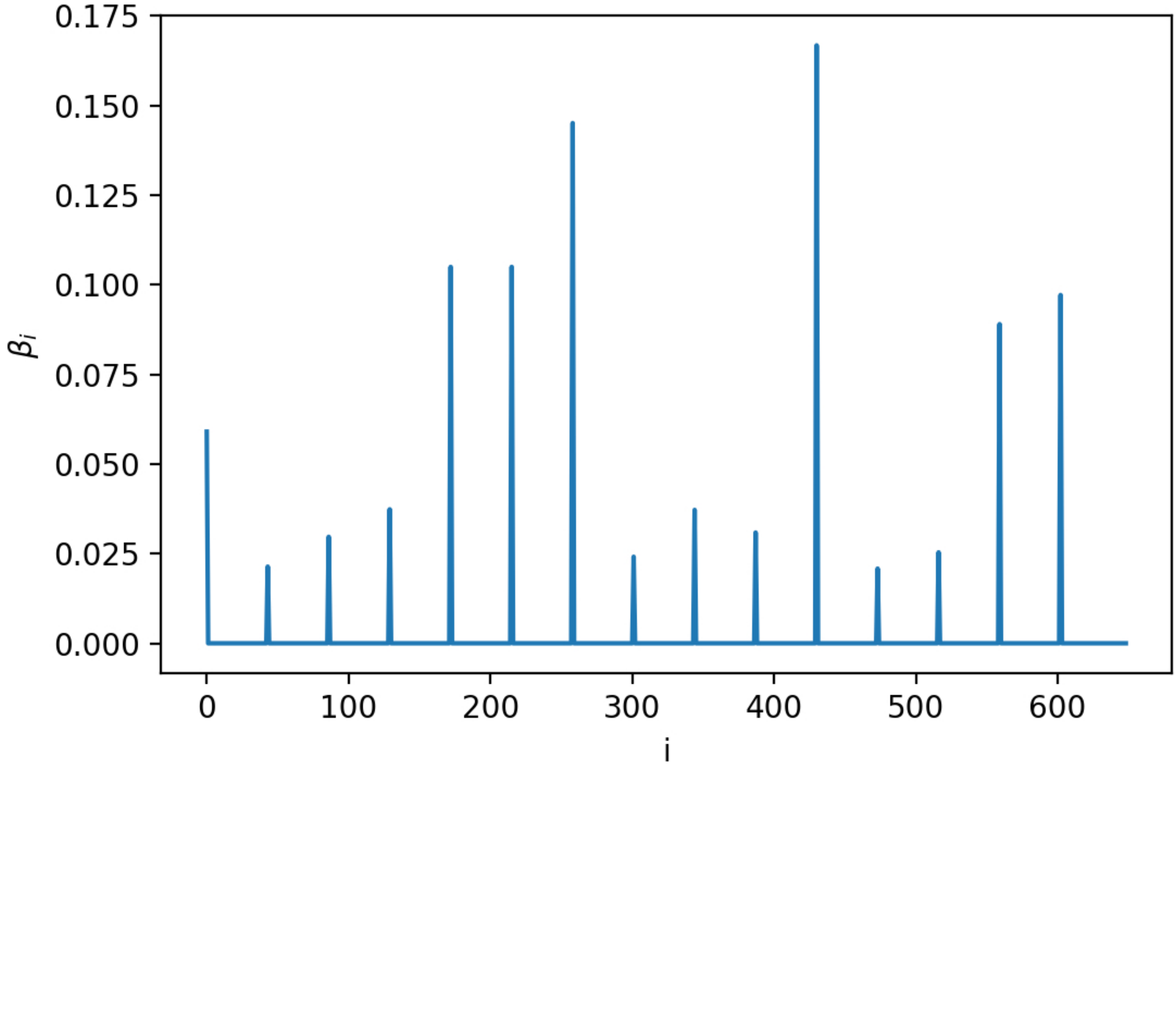


Экспериментальная часть

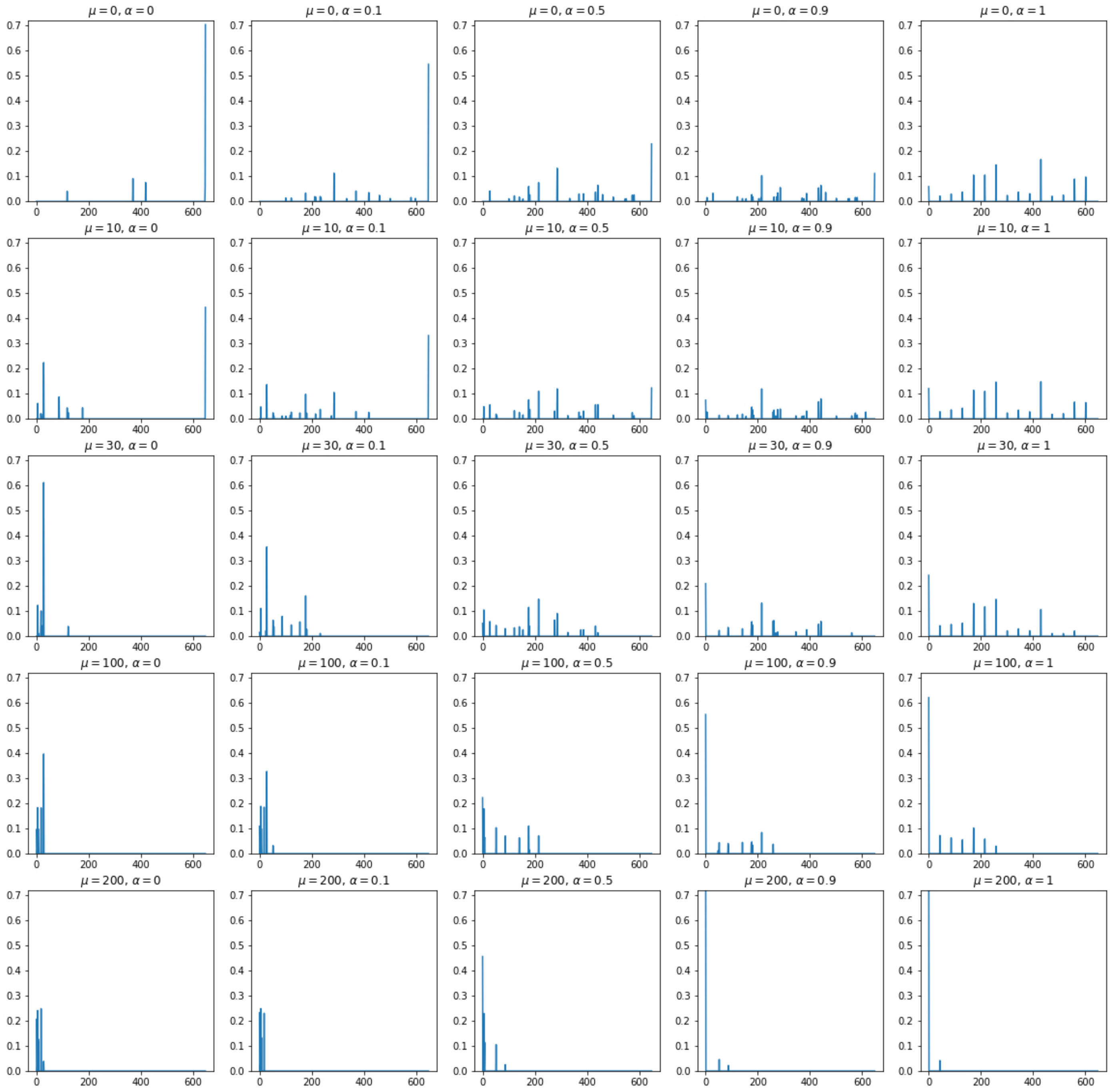
Спасибо за внимание!

$$
\begin{cases}
\boldsymbol{\beta}^T\Big((1-\alpha)\boldsymbol{\Sigma}+\alpha\mathbf{B}\Big)\boldsymbol{\beta}-\mu\Big((1-\alpha)\bar{\mathbf{x}}+\alpha\mathbf{z}\Big)^T\boldsymbol{\beta}+ \\
\qquad +c\left(\mathbf{y}-\mathbf{X}^T\boldsymbol{\beta}\right)^T\left(\mathbf{y}-\mathbf{X}^T\boldsymbol{\beta}\right)\to\min(\boldsymbol{\beta}) \\
\mathbf{1}^T\boldsymbol{\beta}=1; \\
\boldsymbol{\beta}\geqslant\mathbf{0}
\end{cases}
\tag{12}
$$

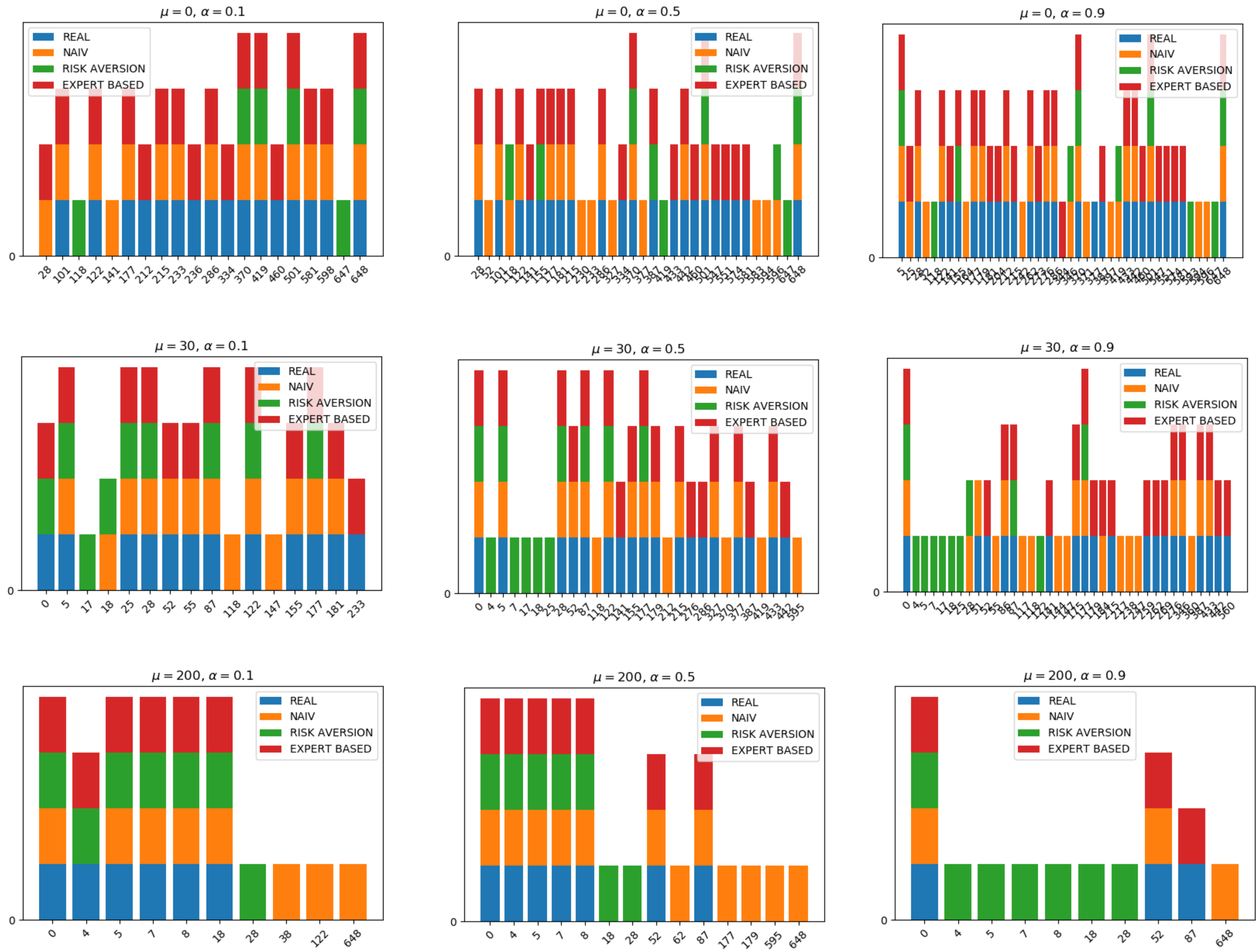

0.06
0.05
0.04
0.03
0.02
0.01
0.00
β_i
0
100
200
300
400
500
600
i



0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
β_i
0
100
200
300
400
500
600
i



$\mu = 0, \alpha = 0$
$\mu = 0, \alpha = 0.1$
$\mu = 0, \alpha = 0.5$
$\mu = 0, \alpha = 0.9$
$\mu = 0, \alpha = 1$
$\mu = 10, \alpha = 0$
$\mu = 10, \alpha = 0.1$
$\mu = 10, \alpha = 0.5$
$\mu = 10, \alpha = 0.9$
$\mu = 10, \alpha = 1$
$\mu = 30, \alpha = 0$
$\mu = 30, \alpha = 0.1$
$\mu = 30, \alpha = 0.5$
$\mu = 30, \alpha = 0.9$
$\mu = 30, \alpha = 1$
$\mu = 100, \alpha = 0$
$\mu = 100, \alpha = 0.1$
$\mu = 100, \alpha = 0.5$
$\mu = 100, \alpha = 0.9$
$\mu = 100, \alpha = 1$
$\mu = 200, \alpha = 0$
$\mu = 200, \alpha = 0.1$
$\mu = 200, \alpha = 0.5$
$\mu = 200, \alpha = 0.9$
$\mu = 200, \alpha = 1$
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
0 200 400 600



$\mu = 0,\ \alpha = 0.1$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
28 101 118 122 141 177 212 215 233 236 286 334 370 419 460 501 581 598 647 648
$\mu = 0,\ \alpha = 0.5$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
$\mu = 0,\ \alpha = 0.9$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
$\mu = 30,\ \alpha = 0.1$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
0 5 17 18 25 28 52 55 87 118 122 147 155 177 181 233
$\mu = 30,\ \alpha = 0.5$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
$\mu = 30,\ \alpha = 0.9$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
$\mu = 200,\ \alpha = 0.1$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
0 4 5 7 8 18 28 38 122 648
$\mu = 200,\ \alpha = 0.5$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
0 4 5 7 8 18 28 52 62 87 177 179 595 648
$\mu = 200,\ \alpha = 0.9$
REAL
NAIV
RISK AVERSION
EXPERT BASED
0
0 4 5 7 8 18 28 52 87 648