Temporal Difference Learning

Reinforcement Learning

September 29, 2020

MSU

Reminder: Value functions

State value function (V-function): $V(s) : \mathbb{E}_T \left[\sum_{t \geq 0} {}^t r_t \mid s_0 = s \right]$

State-action value function (Q-function): $Q(s, a) : \mathbb{E}_T \left[\sum_{t \geq 0} {}^t r_t \mid s_0 = s, a_0 = a \right]$

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Optimal V-function: $V(s) : \max V(s)$

Optimal Q-function: $Q(s, a) : \max Q(s, a)$

Reminder: Value functions

State value function (V-function): $V^{\pi}(s) = \mathbb{E}_{\pi}\left[\sum_{t \geq 0} \gamma^t r_t \mid s_0 = s\right]$

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Optimal V-function: $V^*(s) = \max_{\pi} V^{\pi}(s)$

Optimal Q-function: $Q^*(s, a) = \max_{\pi} Q^{\pi}(s, a)$

Bellman equations: $\forall s, a:$

$$Q^{\pi}(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a)} \mathbb{E}_{a' \sim \pi(a'|s')} Q^{\pi}(s', a')$$

$$Q^*(s, a) = r(s, a) + \gamma \mathbb{E}_{s' \sim p(s'|s,a)} \max_{a'} Q^*(s', a')$$

Reminder: Dynamic Programming

Setup: $|S| ! 8; |A| ! 8$

Assumption: model is known

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Policy Iteration

Policy Evaluation:

compute V^{k} for current policy $_{k}$;

Policy Improvement:

$_{k \;\; 1}psq$ Đ $\underset{a}{\text{argmax}}\, Q \; ^{k}ps; aq$;

repeat;

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Setup: $|S| < \infty$; $|A| < \infty$
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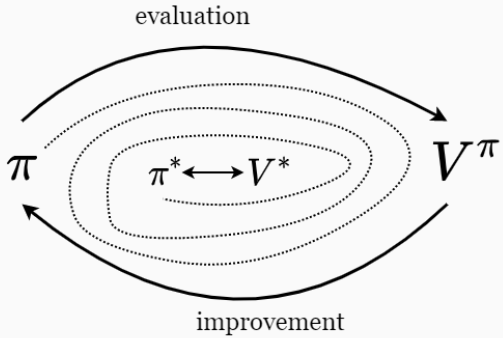
Policy Evaluation:

compute V^{π_k} for current policy π_k;

Policy Improvement:

$\pi_{k+1}(s) \doteq \arg\max_a Q^{\pi_k}(s,a)$;

repeat;



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Value Iteration

- solve Bellman optimality equation
  (e. g. for V):

$$V_{k+1}(s) \leftarrow \max_a r(s,a) + \gamma \mathbb{E}_{s'} V_k(s')$$

$$\pi(s) \leftarrow \operatorname{argmax}_a Q(s,a) \text{ is optimal;}$$

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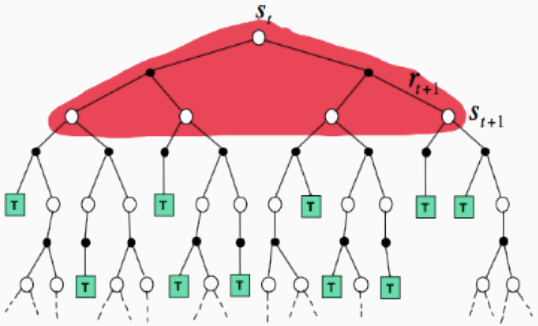

Trial and Error learning

Setup: $|S| \; ! \; 8 ; |A| \; ! \; 8$

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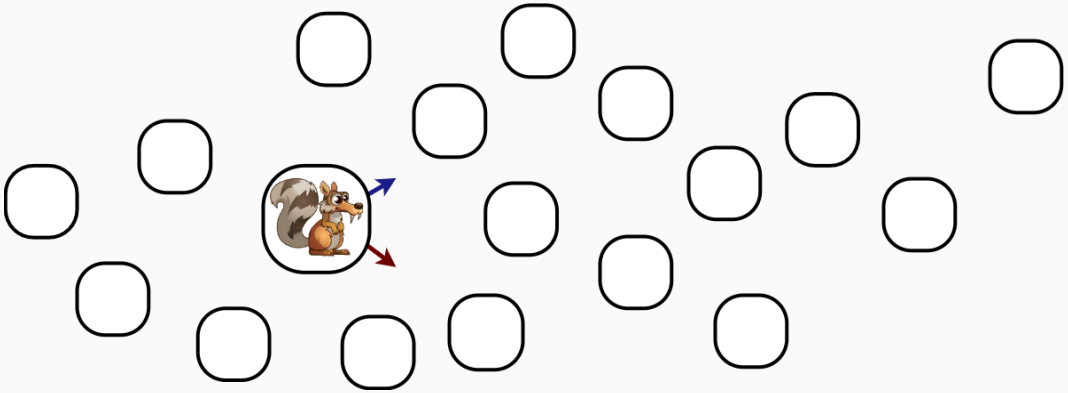

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Learn model through experience

Use VI / PI with learned model to find optimal policy

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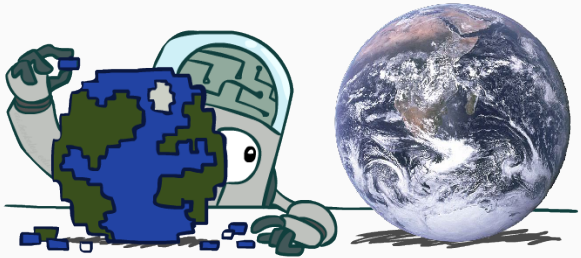
a) universe

b) brain



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$S \quad A \tilde{N} P_{\mathfrak{p}} S_{\mathfrak{q}}$

X (almost) supervised learning

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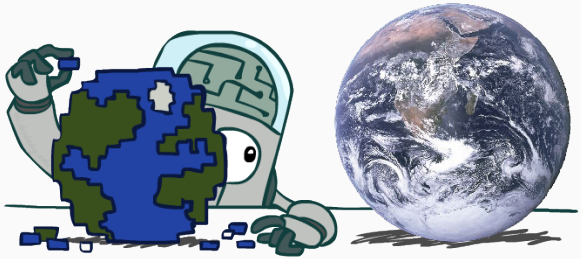

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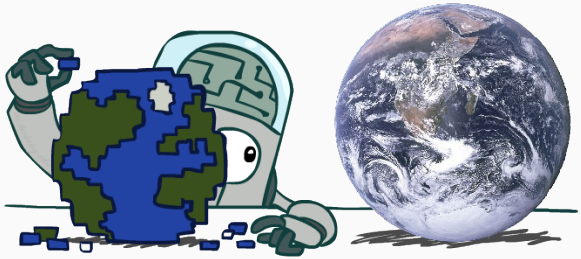
dataset is not provided :(



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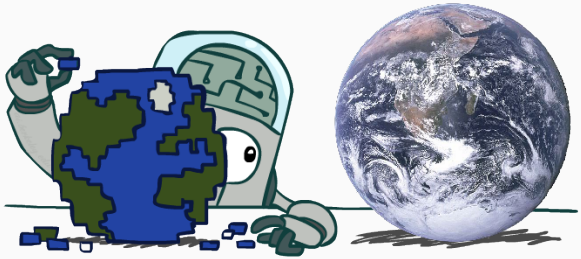

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Map states and actions to reward directly

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Where do we use the model?

Policy Iteration

evaluate policy
e. g. by solving Bellman equation:

$$V^{k}(s) \leftarrow \mathbb{E}_{a}\, r(s,a) \quad \mathbb{E}_{s'} V^{k}(s')$$

$$k+1(s) \leftarrow \operatorname{argmax}_{a} Q^{k}(s,a);$$

repeat;

Value Iteration

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$$Q^{k}(s,a) \leftarrow r(s,a) + \mathbb{E}_{s'}\mathbb{E}_{a'}\left[Q^{k}(s',a')\right]$$

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$\pi(s) \leftarrow \arg\max_{a} Q(s,a)$ is optimal;

$$x \leftarrow \mathbb{E}_{s'}\left[f(s',x)\right]; \quad x - ?$$

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Can we derive *some* model-free PI / VI?

Monte-Carlo Backup

How to solve this: x $\mathbb{E}_{s^1} f(s^1)$

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Monte-Carlo estimation:

$$Q(s,a) \approx \frac{1}{N} \sum_{i=0}^{N} R(T_i)$$

where $T_i \sim (s_0, a_0, \ldots)$

Monte-Carlo Backup

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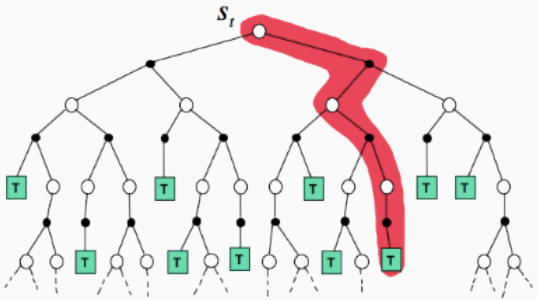
Monte-Carlo estimation:

$$Q(s,a) \quad \frac{1}{N} \sum_{i=0}^{N} R(T_i)$$

where $T_i \quad | s_0 \quad s; a_0 \quad a$

Can we have i. i. d. samples?



Monte Carlo Algorithm

Monte Carlo Algorithm

Initialize $Q(s,a)$ and policy $\pi(s)$ arbitrarily.

for $k = 0, 1, 2, \ldots$

play several games using π;

estimate $Q(s,a)$ using Monte Carlo;

$\pi(s) \leftarrow \operatorname{argmax}_a Q(s,a)$

! requires infinite samples for each pair s, a;

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- still needs full episodes;
- still do not utilize MDP structure;
- wastes information about state connections;
- high variance of collected samples;
- no clear theoretical guarantees;

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update $Q(s,a)$ with new samples;

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Reuse samples from previous policy
(for example, with smaller weight)

Closer look at Monte Carlo

Suppose we want to compute online mean m of i. i. d. samples $x_1; x_2 :::$. On step k:

$$m_k : \quad \frac{1}{k} \overset{k}{\underset{i \quad 1}{\text{ }}} x_i$$

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$$! \qquad \qquad)$$

$$_k : \frac{1}{k} \qquad \underbrace{1 \quad \frac{1}{k} m_{k \quad 1}}_{\text{exponential smoothing}} \quad \frac{1}{k} x_k$$

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$$! \qquad \left(k : \frac{1}{k} \right)$$

$$p1 \quad q\, m_{k-1} \qquad x_k \qquad m_{k-1} \quad p x_k \qquad m_{k-1} \; q$$

exponential smoothing

stoch. gradient descent

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exponential smoothing

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Closer look at Monte Carlo

Suppose we want to compute online mean m of i. i. d. samples $x_1, x_2, \ldots$. On step k:

$$m_k = \frac{1}{k}\sum_{i=1}^{k} x_i = \frac{k-1}{k} m_{k-1} + \frac{1}{k} x_k = m_{k-1} + \frac{1}{k}(x_k - m_{k-1})$$

$$m_k = (1-\alpha) m_{k-1} + \alpha x_k = m_{k-1} - \alpha (m_{k-1} - x_k)$$

exponential smoothing — stoch. gradient descent

Convergence:

$m_k \to \mathbb{E}x$ if learning rate satisfies Robbins–Monro conditions:

$$\sum_{k=1}^{\infty} \alpha_k = \infty, \qquad \sum_{k=1}^{\infty} \alpha_k^2 < \infty$$

Solving equations

Equation	Method	Update
$x \quad f(x)$	Point Iteration	$x_{k+1} : \quad f(x_k)$
$x \quad E_{s^{-1}} f(s^{-1})$	Exponential smoothing	$x_{k+1} : \quad x_k \quad {}_k(f(s^{-1}) \quad x_k) \quad s^{-1} \quad p(s^{-1})$
$x \quad E_{s^{-1}} f(s^{-1}; x)$	Stochastic approximation	

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$x \quad E_{s^1} f(s^1; x)$	Stochastic approximation	$x_{k+1} : \quad x_k \quad \frac{\alpha_k}{s^1} \frac{f(s^1; x_k)}{p(s^1)} \quad x_k$

Temporal Difference: intuition

View 1: solving Bellman expectation equation using Robbins-Monro scheme:

$$Q(s,a) \leftarrow \mathbb{E}_{s'}\Big[\underbrace{r(s,a) + \gamma \mathbb{E}_{a'} Q(s',a')}_{f(s',x)}\Big]$$

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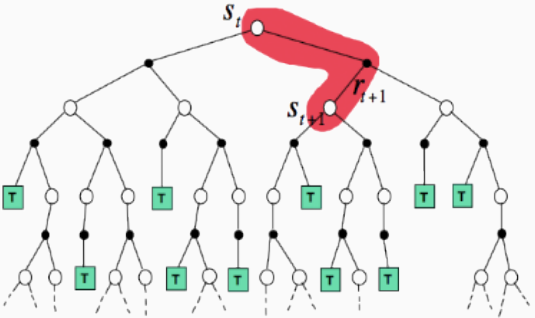

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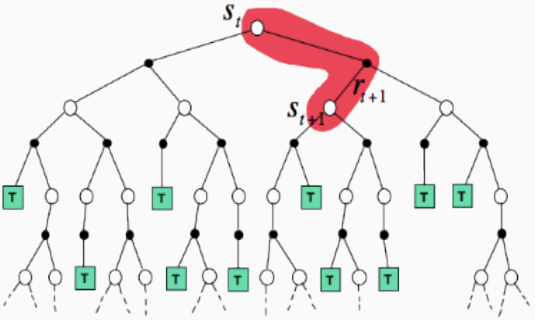
View 2: one-step bootstrapping: approximating future rewards using current approximation.

$$r + \underbrace{\gamma r^1 + \gamma^2 r^2 + \gamma^3 r^3 + \ldots}_{Q_k(s^1,a^1)}$$



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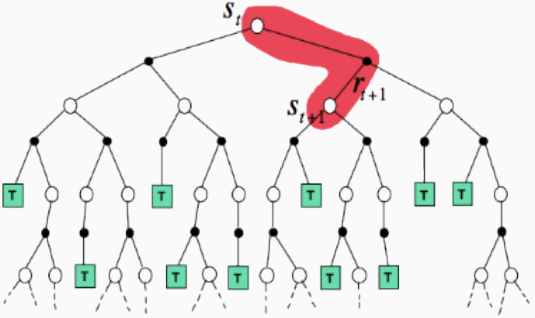

our own predictions used as targets!

Temporal Di□erence

For transition ps; a; r; s^1; a^{1}q:

$$Q_{k\ \ 1}(s,a) \leftarrow Q_k(s,a) + \alpha_k \underbrace{\overbrace{\left[r + Q_k(s^1,a^1)\right]}^{\text{Bellman target}} - Q_k(s,a)}_{\text{temporal di□erence}}$$

Temporal Di□erence

For transition (s, a, r, s', a'):

$$Q_{k+1}(s,a) \leftarrow Q_k(s,a) + \alpha_k \underbrace{\Big[\overbrace{r + \gamma Q_k(s',a')}^{\text{Bellman target}} - Q_k(s,a)\Big]}_{\text{temporal di□erence}}$$

(s, a) — considered fixed.

$s' \sim p(s' \mid s, a)$ — random variable from interaction experience.

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(s', a') — random variable from interaction experience.

a' — random variable from algorithm.

Temporal Di□erence vs Monte Carlo

$$Q_{k\ \ 1}ps;aq \text{ Đ } Q_k ps;aq \qquad {}_k pyps;aq \quad Q_k ps;aqq$$

Which is better?

Temporal Di□erence

$yps;aq : \ r \quad Q_k ps^1;a^1q$

Monte Carlo

$yps;aq : \ r \quad r^1 \quad {}^2r^2 \quad :::$

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$yps;aq : \quad r \qquad Q_k ps^1;a^1q$

X updates after each step

slow reward propagation

Monte Carlo

$yps;aq : \quad r \qquad r^1 \qquad {}^2r^2 \qquad :::$

updates at the end of the episode

X fast reward propagation

Temporal Di□erence vs Monte Carlo

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$yps;aq : \quad r \qquad Q_k ps^1;a^1q$

X updates after each step

slow reward propagation

X low variance

introduces bias

Monte Carlo

$yps;aq : \quad r \qquad r^1 \qquad {}^2r^2 \qquad :::$

updates at the end of the episode

X fast reward propagation

high variance

X unbiased

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X low variance

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Monte Carlo

$yps;aq : \quad r \qquad r^1 \qquad {}^2r^2 \qquad :::$

updates at the end of the episode

X fast reward propagation

high variance

X unbiased

Best estimation is somewhere in between (see TD( )).

Value Iteration and Policy Iteration connection

What if we improve our policy after every step?

$$\pi_k(s) \leftarrow \operatorname*{argmax}_a Q_k(s, a)$$

Value Iteration and Policy Iteration connection

What if we improve our policy after every step?

$$\pi_k(s) = \arg\max_a Q_k(s,a)$$

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What if we improve our policy after every step?

$$\pi_k(s) = \arg\max_a Q_k(s,a)$$

$$Q_{k+1}(s,a) = Q_k(s,a) + \alpha\left[r + Q_k(s',a') - Q_k(s,a)\right]$$

$$Q_k(s,a) + \alpha\left[r + Q_k(s',\pi_k(s')) - Q_k(s,a)\right]$$

Value Iteration and Policy Iteration connection

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View 1: Policy Iteration with policy improvement after *each* policy evaluation update;

View 2: solving Bellman optimality equation (model-free Value Iteration);

Convergence of temporal di□erence

Convergence of Q-learning updates

Let $Q_0(s,a)$ be initialized arbitrary, and for every s,a the following update is used:

$$Q_{k+1}(s,a) \leftarrow Q_k(s,a) + \alpha_k(s,a)\left[r(s,a) + \max_{a'} Q_k(s'_{k,s,a}, a') - Q_k(s,a)\right]$$

Convergence of temporal di□erence

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Then, if:

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Then, if:

$s'_{k,s,a} \sim p(s'|s,a)$;

with probability 1 for every s,a learning rate $\alpha_k(s,a) \in [0,1)$ satisfies Robbins-Monro conditions:

$$\sum_{k=0}^{\infty} \alpha_k(s,a) = \infty \qquad \sum_{k=0}^{\infty} \alpha_k^2(s,a) < \infty$$

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Then, if:

- $s'_{k;s,a} \sim p(s' \mid s,a)$;
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$$\sum_{k=0}^{\infty} \alpha_k(s,a) = \infty \qquad \sum_{k=0}^{\infty} \alpha_k^2(s,a) < \infty$$

then $Q_k(s,a) \xrightarrow[k\to\infty]{w.p.1} Q^*(s,a)$

Exploration vs Exploitation

Analogy 1

Global optimization

Exploration vs Exploitation

Analogy 1

Global optimization

Exploration:

random search

Exploration vs Exploitation

Analogy 1

Global optimization

Exploration:
random search

Exploitation:
local optimization

Exploration vs Exploitation

Analogy 1
Global optimization

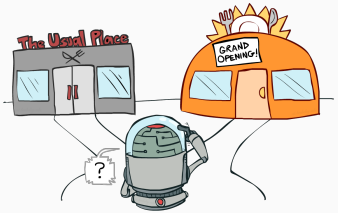
Exploration:
random search

Exploitation:
local optimization

Analogy 2
Multi-armed bandits
(regret minimization)

Exploration:
try something new

Exploitation:
try your favourite



Exploration vs Exploitation

Analogy 1
Global optimization

Exploration:
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Analogy 2
Multi-armed bandits
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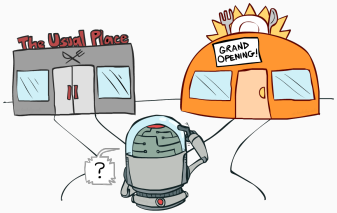

Analogy 3
General MDP

Exploration:
explore the environment

Exploitation:
solve the task

″-greedy exploration

Do something random *sometimes*

Let's call policy ″-greedy with respect to some Q-function, if it behaves:

randomly with probability ″

greedy with probability 1 ″

ε-greedy exploration

Do something random *sometimes*

Let's call policy ε-greedy with respect to some Q-function, if it behaves:

randomly with probability ε

greedy with probability $1 - \varepsilon$

$$\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|A|} & \text{when } a = \operatorname{argmax}_a Q(s, a) \\ \frac{\varepsilon}{|A|} & \text{otherwise} \end{cases}$$

″-greedy exploration

Do something random *sometimes*

Let's call policy ″-greedy with respect to some Q-function, if it behaves:

randomly with probability ″

greedy with probability 1 ″

$$\pi(a \mid s) = \begin{cases} 1 - \varepsilon + \frac{\varepsilon}{|A|} & \text{when } a = \operatorname{argmax}_a Q(s,a) \\ \frac{\varepsilon}{|A|} & \text{otherwise} \end{cases}$$

Decaying exploration:

$\varepsilon_k \to 0; \quad \varepsilon_k > 0$

Persistent exploration:

$\varepsilon_k = \text{const}(k)$

Q-learning

Q-learning (online)

Initialize $Q(s, a)$ arbitrarily;

observe s_0;
for $k = 0, 1, 2, \ldots$
    take action a_k ε-greedy$(Q(s_k, a))$;
    observe r_k, s_{k+1};

Q-learning

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$y = r_k + \max_{a_{k+1}} Q(s_{k+1}, a_{k+1})$;

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$Q(s_k, a_k) \leftarrow (1 - \alpha_k) Q(s_k, a_k) + \alpha_k y$

Q-learning

Q-learning (online)

Initialize $Q\ ps; aq$ arbitrarily;

observe s_0;

for $k \quad 0; 1; 2 : : :$

take action $a_k \quad "$-greedy$pQ\ ps_k; aqq$;

observe $r_k; s_{k\ 1}$;

$y : \quad r_k \quad \max_{a_{k\ 1}} Q\ ps_{k\ 1}; a_{k\ 1}q$;

$Q\ ps_k; a_kq \text{ Đ } p1 \quad {}_kqQ\ ps_k; a_kq \quad {}_ky$

Q-learning (with replay buffer)

Initialize $Q\ ps; aq$ arbitrarily, $D \quad H$;

Q-learning

Q-learning (online)

Initialize $Q(s, a)$ arbitrarily;

observe s_0;
for $k = 0, 1, 2, \ldots$
  take action a_k ε-greedy$(Q(s_k, a))$;
  observe r_k, s_{k+1};
  $y = r_k + \max_{a_{k+1}} Q(s_{k+1}, a_{k+1})$;
  $Q(s_k, a_k) \leftarrow (1 - \alpha_k) Q(s_k, a_k) + \alpha_k y$

Q-learning (with replay buffer)

Initialize $Q(s, a)$ arbitrarily, $D \leftarrow \emptyset$;

observe s_0;
for $k = 0, 1, 2, \ldots$
  take action a_k ε-greedy$(Q(s_k, a))$;
  observe r_k, s_{k+1};
  store (s_k, a_k, r_k, s_{k+1}) in D;

Q-learning

Q-learning (online)

Initialize $Q(s, a)$ arbitrarily;

observe s_0;
for $k = 0, 1, 2, \ldots$
  take action a_k ε-greedy$(Q(s_k, a))$;
  observe r_k, s_{k+1};
  $y \leftarrow r_k + \gamma \max_{a_{k+1}} Q(s_{k+1}, a_{k+1})$;
  $Q(s_k, a_k) \leftarrow (1 - \alpha) Q(s_k, a_k) + \alpha y$

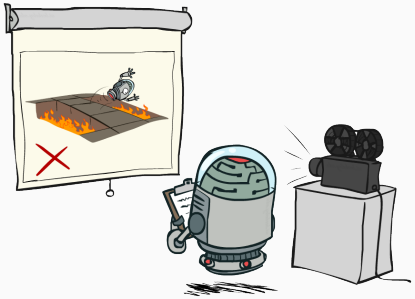
Q-learning (with replay buffer)

Initialize $Q(s, a)$ arbitrarily, $D \leftarrow \emptyset$;

observe s_0;
for $k = 0, 1, 2, \ldots$
  take action a_k ε-greedy$(Q(s_k, a))$;
  observe r_k, s_{k+1};
  store (s_k, a_k, r_k, s_{k+1}) in D;
  sample (s, a, r, s') from D;
  $y \leftarrow r + \gamma \max_{a'} Q(s', a')$;
  $Q(s, a) \leftarrow (1 - \alpha) Q(s, a) + \alpha y$

Q-learning is o□-policy

Data source	Interaction policy	Convergence
Online experience	Infinite visitation	Q
Expert data	-	Q
Experience replay	Infinite visitation	Q



Expectation

Safe RL

Expectation

Reality

Safe RL

Expectation

Reality

Consider the family of «ε-soft policies»:

$$\pi_\varepsilon(s,a) = \pi(a|s) \cdot (1-\varepsilon) + \frac{\varepsilon}{|A|}$$

"-soft Policy Improvement

In the family of «"-soft policies» "-greedy is policy improvement:

$$\sim : \quad \text{"-greedy}pQ \quad ps; aqq \quad \tilde{n} \quad \sim \copyright$$

Q-learning vs SARSA

"-soft Policy Improvement

In the family of «"-soft policies» "-greedy is policy improvement:

$$\sim : \quad \text{"-greedy}(Q\ ps; aqq \quad \tilde{n} \quad \sim \copyright$$

$$Q_{k\ 1}ps; aq \text{ Đ } Q_k ps; aq \qquad {}_k\ rps; aq \qquad Q_k ps^1; a^1q \qquad Q_k ps; aq$$

Q-learning

a^1 taken from target policy

$${}_k psq : \quad \operatorname*{argmax}_a Q_k ps; aq$$

SARSA

a^1 taken from behavior policy

$${}_k : \quad \text{"-greedy}(Q_k ps; aqq$$

Q-learning vs SARSA

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a^1 taken from target policy

$$_k psq : \quad \operatorname{argmax}_a Q_k ps; aq$$

SARSA

a^1 taken from behavior policy

$$_k : \quad \text{"-greedy}(Q_k ps; aqq$$

SARSA convergence properties

SARSA converges to optimal "-soft policy, satisfying

$$\text{"-greedy}(Q \ ps; aqq$$

SARSA

SARSA

Initialize $Q(s, a)$ arbitrarily.

observe s_0, select a_0 randomly;
for $k = 0, 1, 2, \ldots$

take action a_k, observe r_k, s_{k+1};

sample a_{k+1} ε-greedy$(Q(s_{k+1}, a))$

$y = r_k + Q(s_{k+1}, a_{k+1})$;

$Q(s_k, a_k) \leftarrow (1 - \alpha_k) Q(s_k, a_k) + \alpha_k y$

SARSA

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  $y \leftarrow r_k + Q(s_{k+1}, a_{k+1})$;
  $Q(s_k, a_k) \leftarrow (1 - \alpha_k) Q(s_k, a_k) + \alpha_k y$

Expected-SARSA

Initialize $Q(s, a)$ arbitrarily.

observe s_0;
for $k = 0, 1, 2, \ldots$
  sample a_k ε-greedy$(Q(s_k, a))$;
  take action a_k, observe r_k, s_{k+1};
  $y \leftarrow r_k + \mathbb{E}_{a_{k+1}} Q(s_{k+1}, a_{k+1})$;
  $Q(s_k, a_k) \leftarrow (1 - \alpha_k) Q(s_k, a_k) + \alpha_k y$;

SARSA is on-policy

For transition $(s; a; r; s^1; a^1)$:

$$Q_{k+1}(s; a) \leftarrow Q_k(s; a) + \alpha_k \left(r + Q_k(s^1; a^1) - Q_k(s; a) \right)$$

Data source	Interaction policy	Convergence
Online experience	Decaying exploration	Q^*
	Persistent exploration	Q^* -greedy
Expert data (a^1 taken from buffer)	-	

SARSA is on-policy

For transition $ps; a; r; s^1; a^1q$:

$$Q_{k\ \ 1}ps; aq \text{ Đ } Q_k ps; aq \qquad {}_k pr \qquad Q_k ps^1; a^1q \qquad Q_k ps; aqq$$

Data source	Interaction policy	Convergence
Online experience	Decaying exploration Persistent exploration	Q Q_{ε}-greedy
Expert data (a^1 taken from bu□er)	-	Q^{expert}
Experience replay (a^1 taken from bu□er)		

SARSA is on-policy

For transition $ps; a; r; s^1; a^1q$:

$$Q_{k\ \ 1}ps; aq \text{ Đ } Q_k ps; aq \qquad {}_k pr \qquad Q_k ps^1; a^1q \qquad Q_k ps; aqq$$

Data source	Interaction policy	Convergence
Online experience	Decaying exploration	Q
	Persistent exploration	$Q_{\text{"-greedy}}$
Expert data (a^1 taken from bu□er)	-	Q^{expert}
Experience replay (a^1 taken from bu□er)	(arbitrary)	divergence

«Fixing» SARSA (not really sarsa now)

For transition (s, a, r, s^1) generate a^1 $(a^1 \mid s^1)$:

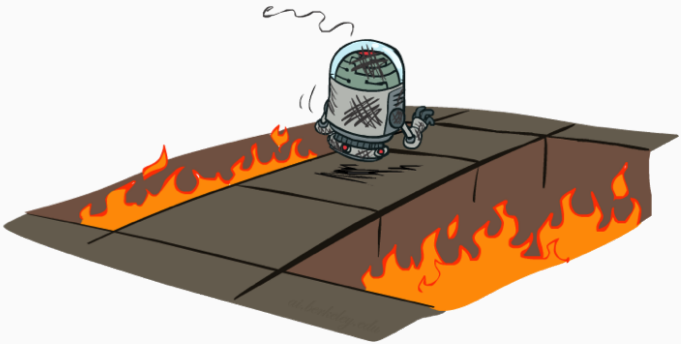
$$Q_{k\ 1}(s, a) \leftarrow Q_k(s, a) + \alpha r \quad Q_k(s^1, a^1) \quad Q_k(s, a))$$

«Fixing» SARSA (not really sarsa now)

For transition $ps; a; r; s^1q$ generate a^1 $pa^1 \mid s^1q$:

$$Q_{k\ \ 1}ps; aq \text{ Đ } Q_k ps; aq \qquad {}_k pr \qquad Q_k ps^1; a^1q \qquad Q_k ps; aqq$$

Data source	Interaction policy	Convergence
(any, but a^1 generated online)	Decaying exploration	Q
	Persistent exploration	$Q_{\text{„ -greedy}}$



Literature

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