Combinatorial Approach to Generalization Bounds Tightening

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Outline

1. **Theory of Empirical Prediction**
   - Weak Probability Axioms and Empirical Prediction
   - Example: Transductive Form of the Law of Large Numbers
   - Discussion

2. **Theory of Generalization Ability**
   - Classical Generalization Bounds
   - Data-Dependent Bounds
   - Measuring Effective Local Shatter

3. **Experiments with Rule Induction System**
   - The Rule Induction Classifier
   - Experimental results: shatter coefficients
   - Experimental estimation of rules overfitting

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Dear Colleagues, I shall speak about one of the most challenging problem in computational learning theory — the problem of generalization ability of learning algorithms. I shall start from a general theoretical framework, then consider a fundamental problem of learning theory — the looseness of generalization bounds. I shall finish by some empirical results.
Weak Probability Axioms

1. \( X^L = \{x_i\}_{i=1}^{L} \) — a given finite subset from a set of objects \( X \).

2. All partitions \( X^L = X^\ell_n \cup X^k_n \), \( n = 1, \ldots, N \),
where \( N = C^k_L \), \( L = \ell + k \), are equally probable.

Then...
Let us start from two very simple axioms. First, we assume that in data analysis one can observe only a finite set of objects $X^L$. The set can be unknown, but it never can be infinite.

Second, we assume that objects appear at random, so that all partitions of our set into two subsets have equal chances to realize. The order of objects is the only source of randomness in our framework.
1  \( X^L = \{x_i\}_{i=1}^L \) — a given finite subset from a set of objects \( \mathbb{X} \).

2  All partitions \( X^L = X^\ell_n \cup X^k_n, \ n = 1, \ldots, N \), where \( N = C^k_L, \ L = \ell + k \), are equally probable.

Then

- Consider an event \( A \) as a function \( A: \{1, \ldots, N\} \rightarrow \{0, 1\} \)
- The fraction of partitions \( n: A(n) = 1 \) can be interpreted as probability or expectation:

\[
P_n A(n) \equiv E_n A(n) = \frac{1}{N} \sum_{i=1}^{N} A(n).
\]

Here “probability” \( P_n \) is simply averaging operator \( \frac{1}{N} \sum_{i=1}^{N} \).
Under these assumptions the probability of an event $A(n)$ [see. . .] is defined as the fraction of partitions $n$ [see. . .] for which $A(n)$ is true. Note that there is no difference between expectation [see. . .] and probability [see. . .] in this framework.
A general problem of Empirical Prediction

- Empirical framework:
  some partition $(X_n^\ell, X_n^k)$ realizes, $n \in \{1, \ldots, N\}$;
  subsample $X_n^\ell$ is observable,
  subsample $X_n^k$ is hidden.

- Given a function $T : \mathbb{X}^k \times \mathbb{X}^\ell \rightarrow \mathbb{R}$
  1. Chose a function $\hat{T} : \mathbb{X}^\ell \rightarrow \mathbb{R}$ such that
     the value $\hat{T}_n = \hat{T}(X_n^\ell)$ predicts the value $T_n = T(X_n^k, X_n^\ell)$.
  2. Estimate prediction accuracy (obtain an upper bound):
     $$P_n\left[ d(\hat{T}_n, T_n) > \epsilon \right] \leq \eta(\epsilon),$$

where $d(\hat{r}, r)$ — discrepancy function, e.g. $d(\hat{r}, r) = |\hat{r} - r|$. 

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The problem of empirical prediction arise when some of the equiprobable partitions [see. . . ] realizes, but one observe only a first subset $X_{n}^{\ell}$ [see. . . ] whereas $X_{n}^{k}$ [see. . . ] remains hidden. We want to predict a value of a given function $T$ [see. . . ] that depends on both parts, having an information $\hat{T}$ [see. . . ] computed from the observed part only. Also we want to know in advance how accurate our prediction can be [see. . . ] . Then, the problem is to upper bound the fraction of partitions for which our prediction fails. To formulate exactly what means “fails” we introduce a discrepancy function $d$ [see. . . ] (that can be simply a difference in most cases) and a threshold of exactness $\epsilon$ [see. . . ].
Why we call this framework “Weak Probability Axioms”?

Insight:
Weak PA leads to “simplified Probability Theory”

Indeed...
Why we call this framework “the weak probability axioms” and even the weak Probability Theory?
Why we call this framework “Weak Probability Axioms”? 

Insight:

Weak PA leads to “simplified Probability Theory”

Indeed... 

- Weak PA is sufficient to prove fundamental facts: 
  - the Law of Large Numbers, with exact convergence rate; 
  - Kolmogorov-Smirnov criterion, also exact;  
  - many statistical hypothesis tests (order statistics etc.);  
  - Vapnik-Chervonenkis generalization bounds (see later);  
  - etc.
First, because it is very general. Empirical Prediction is one of the central problems in Probability Theory, Statistics and Learning Theory. In practice predictions are interesting only for finite sets of objects, because nobody can observe an infinite set of objects. Empirical Prediction is transductive by its nature. Many fundamental results in Statistics and Learning Theory can be reformulated in transductive form.
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  - many statistical hypothesis tests (order statistics etc.);
  - Vapnik-Chervonenkis generalization bounds (see later);
  - etc.

- Weak PA is based on less restrictive assumptions:
  - no need of probability measure on \( \mathbb{X} \);
  - no need of frequentist definition of probability via \( L \to \infty \);
On the other hand, Weak Probability Axioms are less restrictive if compared with classical Kolmogorov’s Axioms. Here we don’t need of probability measure on object space and we don’t define a probability via passage to the limit.
Why we call this framework “Weak Probability Axioms”?

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Indeed...

- Weak PA is sufficient to prove fundamental facts:
  - the Law of Large Numbers, with exact convergence rate;
  - Kolmogorov-Smirnov criterion, also exact;
  - many statistical hypothesis tests (order statistics etc.);
  - Vapnik-Chervonenkis generalization bounds (see later);
  - etc.

- Weak PA is based on less restrictive assumptions:
  - no need of probability measure on $\mathbb{X}$;
  - no need of frequentist definition of probability via $L \to \infty$;
  - no need of “probability” at all!
Frankly speaking, we don’t need of the notion of probability at all. In our framework “probability” is no more than a synonym of “fraction of partitions”.
Def. The frequency of $S \subset \mathbb{X}$ in a finite sample $U \subset \mathbb{X}$:

$$\nu_S(U) = \frac{1}{|U|} \sum_{u \in U} [u \in S].$$

Theorem (common knowledge)

The frequency $\nu_S(X^n_k)$ predicts the frequency $\nu_S(X^n_\ell)$.

Prediction accuracy is given by an exact bound

$$P_n[\nu_S(X^n_k) - \nu_S(X^n_\ell) \geq \epsilon] = H(\ell \frac{s(\epsilon)}{L m}),$$

where $H(\frac{\ell s}{L m})$ is a tail of hypergeometric distribution,

$$s(\epsilon) = \left\lfloor \ell \frac{(m - \epsilon k)}{L} \right\rfloor,$$

$m = L \nu_S(X^L)$.

Remark. Here $\hat{T}(U) = T(U) = \nu_S(U)$. 

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Let us consider the Law of Large Numbers as an example. It is commonly known that the frequency of an event $S$ in the hidden sample [see...] can be predicted by its frequency in the observed sample [see...] . In classical Probability Theory one predict not frequency but a probability of the event $S$ and one use inequalities of Hoeffding, Bernstein and Chernoff to give asymptotical bounds of the prediction accuracy. In our transductive framework the bound is given by hypergeometric distribution [see...] . Remarkable than it is an exact [see...] non-asymptotic bound.
Example: transductive form of the Law of Large Numbers

The tail of hypergeometric distribution:

\[ H^{\binom{L}{m}, s(\epsilon)} = \sum_{t=s_0}^{s(\epsilon)} \frac{C_m C_{L-m}^t}{C_L^t}, \quad s_0 = \max\{0, m - k\} \]

Here \( \epsilon = 0.05, L = 300, \ell = 200, m = 30. \)
In this case convergence is a trivial consequence of the fact that the relative width [see...] of hypergeometric peak tends to zero when the sample size $L$ tends to infinity.
Links with classical Kolmogorov’s Probability Axioms

**Theorem (Correspondence principle)**

*If we have obtained a bound under Weak PA*

\[ Q_\epsilon(X^L) = P_n \left[ d(\hat{T}(X^\ell_n), T(X^\ell_n, X^k_n)) > \epsilon \right] \leq \eta(\epsilon, X^L) \]

*then analogous bound will be true under Strong PA (classical Kolmogorov’s PA)*

\[ E_{X^L} Q_\epsilon(X^L) = P_{X^L} \left[ d(\hat{T}(X^\ell), T(X^\ell, X^k)) > \epsilon \right] \leq E_{X^L} \eta(\epsilon, X^L) \]
Each bound obtained under Weak Axioms can be easily restated under classical Kolmogorov’s Axioms. To do this one may take expectation [see...] of both sides of the inequality.
Links with classical Kolmogorov’s Probability Axioms

**Theorem (Correspondence principle)**

*If we have obtained a bound under Weak PA*

\[
Q_\epsilon(X^L) = P_n \left[ d(\hat{T}(X^\ell_n), T(X^\ell_n, X^K_n)) > \epsilon \right] \leq \eta(\epsilon, X^L)
\]

*then analogous bound will be true under Strong PA (classical Kolmogorov’s PA)*

\[
E_{X^L} Q_\epsilon(X^L) = P_{X^L} \left[ d(\hat{T}(X^\ell), T(X^\ell, X^K)) > \epsilon \right] \leq E_{X^L} \eta(\epsilon, X^L)
\]

When a transduction becomes an induction

If \( \eta(\epsilon, X^L) \equiv \eta(\epsilon) \) then the same bound is true for any sample.
Philosophic remark. It is commonly to think that transduction is more restrictive that induction. This is not the case when one write a functional in transductive form and obtain its bound that is true for any sample $X^L$ [see...] . Really this means than transduction transforms into induction.
Cross-Validation

If we did not succeed to obtain a bound theoretically, or if we obtained very loose (overestimated) bound:

\[ Q_\epsilon(X^L) = P_n \left[ d(\hat{T}_n, T_n) > \epsilon \right] \leq ???, \]

then we can measure it empirically:

\[ Q_\epsilon(X^L) \approx \frac{1}{|N'|} \sum_{n \in N'} \left[ d(\hat{T}_n, T_n) > \epsilon \right] \]

where \( N' \subset \{1, \ldots, N\} \) is small enough to compute the sum and big enough to estimate be accurate.
The keystone advantage of Weak Probability Axioms. It provides the unique starting point for both theoretical and empirical consideration. If we did not succeed to obtain a bound theoretically, we can measure the prediction functional empirically, replacing the average of all partitions [see...] by the average of some partitions [see...] . This leads to the well known empirical technique — Cross Validation.

The main idea of the further part of presentation is that Cross Validation can help to understand the causes of bounds looseness.
Weak Probability Axioms: pro and con

+ Weak PA is suitable for data analysis, statistics, COLT
Intermediate summary. Our framework is suitable for data analysis, statistics and learning theory, where all samples are finite and all variables can be calculated from data. In classical Probability Theory one operate with hypothetical asymptotic values such that probabilities, expectations, distribution functions, etc. We intend to manage without this hypothetical values. For what reason? Because asymptotic considerations are often the cause of bound looseness and can lead to numerous misunderstandings that are very difficult to reveal.
Weak Probability Axioms: pro and con

+ Weak PA is suitable for data analysis, statistics, COLT
− ...but not suitable for physics
The same time I think that Weak PA will not suitable for physics and other applications where continuity is crucial... so every theory may have its own limitations.
Weak Probability Axioms: pro and con

+ Weak PA is suitable for data analysis, statistics, COLT
- … but not suitable for physics
+ Weak PA gives non-asymptotic, exact bounds
- … represented by complicated combinatorial formula
It can give exact bounds but we are to elaborate fast algorithms to calculate them effectively.
Weak Probability Axioms: pro and con

+ Weak PA is suitable for data analysis, statistics, COLT
− ... but not suitable for physics
+ Weak PA gives non-asymptotic, exact bounds
− ... represented by complicated combinatorial formula
+ Weak PA satisfies a “correspondence principle”
− ... but not all theorems in classical Probability Theory have analogs in Weak PA
Each bound obtained under Weak PA can be restated under classical PA. But many measure-specific theorems cannot be transferred to the weak form.
Weak Probability Axioms: pro and con

+ Weak PA is suitable for data analysis, statistics, COLT
− ... but not suitable for physics
+ Weak PA gives non-asymptotic, exact bounds
− ... represented by complicated combinatorial formula
+ Weak PA satisfies a “correspondence principle”
− ... but not all theorems in classical Probability Theory have analogs in Weak PA

Open problem:
What part of mathematical statistics can be restated in Weak PA?
A big open problem is “what part of mathematical statistics can be restated in Weak PA?” My opinion is that a sufficiently big part.
The learning problem (classification, regression, etc.)

- $X$ — set of objects, $Y$ — set of outputs
- Binary loss function $\mathcal{L} : X \times Y \rightarrow \{0, 1\}$
  - In classification: $\mathcal{L}(x, y) = [y \neq y^*(x)]$,
  - In regression: $\mathcal{L}(x, y) = [|y - y^*(x)| > \delta]$ where $y^*(x)$ — unknown target function
- Training set $X^\ell = \{x_i\}_{i=1}^\ell \subset X$ with known losses $\mathcal{L}(x_i, y)$
- Learning algorithm
  $\mu : X^\ell \mapsto f$, where $f : X \rightarrow Y$ — some function
- Average error of a function $f : X \rightarrow Y$ on a set $U \subset X$
  $\nu(f, U) = \frac{1}{|U|} \sum_{u \in U} \mathcal{L}(u, f(u))$
- Generalization ability:
  $\nu(\mu X^\ell, U)$ must be sufficiently small for most $U \in X^*$
Now I pass to the second part of my presentation — the Learning Problem. Given a training set $X^\ell$ [see...] one must learn a function $f$ [see...] that approximates the unknown target function $y^*(x)$ [see...] as well as possible. The approximation quality on a finite sample $U$ is measured by the average error $\nu(f, U)$ [see...] also called the frequency of errors.

The most challenging problem in Learning Theory — how to guarantee that the learned function $f$ will have a small frequency of errors [see...] out of the training set.
Vapnik and Chervonenkis (1974):

$$P_\epsilon(F) = P_{X^L} \left[ \sup_{f \in F} (\nu(f, X^k) - \nu(f, X^\ell)) > \epsilon \right]$$

$$\leq \Delta^F(L) \cdot 1.5 \ e^{-\epsilon^2 \ell}; \ (if \ \ell = k)$$

where $F$ is the entire functions set (search space);
$\Delta^F(L)$ — *Global Shatter Coefficient* of the set $F$,
the number of functions $f$ from $F$ distinguishable on $X^L$;
$\Delta^F(L) \leq 1.5 \ \frac{L^h}{h!}, \ h = \ VC \ dimension \ (growth \ function) \ of \ F.$
The classical generalization bound is a product of two terms: complexity term called *shatter coefficient* [see...] and convergence term [see...] that tends to zero when the sample length tends to infinity.

Let us remind that shatter coefficient of the functions set $A$ is defined as the maximal number of functions from $A$ pairwise indistinguishable on a set $X^L$.

If the set $X^L$ is fixed then the shatter coefficient is called *local*.

If the set $X^L$ is arbitrary then the shatter coefficient depends on sample size $L$ only (and not on concrete objects) and is called *global* [see...].

In classic VC theory only global shatter coefficient was used.
Vapnik and Chervonenkis (1974):

\[
P_\epsilon(F) = P_{X^L} \left[ \sup_{f \in F} (\nu(f, X^k) - \nu(f, X^\ell)) > \epsilon \right] 
\leq \Delta^F(L) \cdot 1.5 e^{-\epsilon^2 \ell}; \quad (\text{if} \ \ell = k)
\]

where \( F \) is the entire functions set (search space); \( \Delta^F(L) \) — Global Shatter Coefficient of the set \( F \), the number of functions \( f \) from \( F \) distinguishable on \( X^L \); \( \Delta^F(L) \leq 1.5 \frac{L^h}{h!}, \quad h = \text{VC dimension (growth function)} \) of \( F \).

The bound is extremely loose, as it doesn’t take into account:

- the distribution of objects \( X^\ell \);
- the target \( y^*(x) \);
- the learning algorithm \( \mu \);
- \( 1.5 e^{-\epsilon^2 \ell} \) is an asymptotic approximation.
The main shortcoming of this bound [see...] is that the uniform convergence taken in VC theory as an axiom results in extremely overestimated complexity term [see...]. Taking supremum [see...] one neglect many important peculiarities [see...] of the given task.
Data-dependent bounds

- **Localization effect:**
  If target $y^*$, learning algorithm $\mu$, and sample $X^L$ are fixed then not all functions from $F$ can be obtained.

- Uniform convergence bound (Vapnik & Chervonenkis, 1969)
- Theory of learnable (Valiant, 1982)
- First data-dependent bound (?)
- The most tight general VC-like bound (M. Talagrand)
- Self-bounding (Freund)
- Algorithmic luckiness ()
- Computationally tight sample complexity bounds [J. Langford]
When a particular task is fixed, only a little part of functions [see...] can be obtained. This localization effect was understood long ago and several frameworks for data-dependent bound was proposed.

... 

Our proposition distinguishes by the total change of Probability Axioms.
Data-dependent bounds under Weak PA

Weak PA:

\[ Q_\epsilon(\mu, X^L) = P_n \left[ \nu(f_n, X^k_n) - \nu(f_n, X^\ell_n) > \epsilon \right] \]
\[ \leq \Delta^\ell_L(\mu, X^L) \cdot \max_m H(L_m) \]
\[ \left( \leq \Delta^F(L) \cdot 1.5 e^{-\epsilon^2 \ell} \right); \]

where \( f_n = \mu X^\ell_n \) is a result of learning;
\( \Delta^\ell_L(\mu, X^L) \) — Local Shatter Coefficient of the set of functions obtainable by learning: \( \{ f_n \mid n = 1, \ldots, N \} \).
The bound can be obtained under Weak PA [see...] . We will see later that this bound is still very loose. The only advantage of this bound is that this functional [see...] can be measured effectively to understand the causes of bound looseness.

Note that the older data-independent VC bound can be derived from this one.

The next idea is to eliminate maximum on $m$ [see...] in convergence term.
Further tightening data-dependent bound

- **Idea:** the scalar complexity contains too small information about the learning process.
- Splitting the local shatter coeff into *Shatter Profile* $\{D_m\}_{m=0}^L$:

$$\Delta_L^\ell(\mu, X^L) = \sum_{m=1}^{L} D_m(\mu, X^L)$$

$D_m(\mu, X^L)$ — local shatter coefficient of the set of functions having $m$ errors on $X^L$: $\{f_n \mid \nu(f_n, X^L) = \frac{m}{L}, \ n = 1, \ldots, N\}$. 

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This idea has yet another interpretation. One scalar characteristic of complexity contains too small information about learning process. It’s not sufficient to know, how many different functions can be obtained as a result of learning. Also it is worth to take into account how many functions of a given quality can be obtained.

For this reason we split the local shatter coefficient into $L + 1$ components [see...] . Each component can be considered as a local shatter coefficient of the set of functions that make exactly $m$ errors on the full sample $X^L$. So we obtain the non-scalar characteristic of complexity that we call *Shatter Profile* [see...] .
Further tightening data-dependent bound

- **Idea:** the scalar complexity contains too small information about the learning process.
- Splitting the local shatter coeff into *Shatter Profile* \( \{D_m\}_{m=0}^L \):

\[
\Delta^\ell_L(\mu, X^L) = \sum_{m=1}^{L} D_m(\mu, X^L)
\]

\( D_m(\mu, X^L) \) — local shatter coefficient of the set of functions having \( m \) errors on \( X^L \): \( \{f_n \mid \nu(f_n, X^L) = \frac{m}{L}, \ n = 1, \ldots, N\} \).

- Weak PA:

\[
Q_\epsilon(\mu, X^L) \leq \sum_{m=1}^{L} D_m(\mu, X^L) \cdot H(s(\epsilon))
\]
Further tightening data-dependent bound

Idea: the scalar complexity contains too small information about the learning process.

Splitting the local shatter coeff into Shatter Profile $\{D_m\}_{m=0}^{D_m}$:

$$\Delta L(\mu, X_L) = \sum_{m=1}^{L} D_m(\mu, X_L)$$

$D_m(\mu, X_L)$ — local shatter coefficient of the set of functions having $m$ errors on $X_L$: $\{f_n | \epsilon(f_n, X_L) = q, n = 1, \ldots, N\}$.

Weak PA:

$$Q(\mu, X^1) \leq \sum_{m=1}^{L} D_m(\mu, X^1) \cdot H(\epsilon_{m+1}^{(m)})$$

With shatter profile we obtain a more tight bound [see...]. The previous one could be obtained from this one taken the maximum of convergence term by $m$ [see...].
The Effective Local Shatter Profile and Coefficient

- **Ideally accurate but “unfair” bound** answers a question: what would be the shatter profile $D_m$ to bound be exact?

- **Theorem**

$$ Q_{\epsilon, m}(\mu, X^L) = P_n \left[ \nu(f_n, X_n^k) - \nu(f_n, X_n^\ell) > \epsilon \right] \left[ \nu(f_n, X^L) = \frac{m}{L} \right] $$

$$ \leq D_m(\mu, X^L) \cdot H\left(\frac{s(\epsilon)}{L_m}\right); $$

Let us change “≤” by “=” and express $D_m$:
To compare proposed bounds empirically we must have an ideally accurate bound as a reference point. Then we introduce a subsidiary functional $Q_m$ [see... ] which helps to estimate all components of the Shatter Profile separately [see... ] . We call this bound unfair because usually one estimate the complexity term to get an upper bound of the quality functional. Whereas here we do a contrary thing: we estimate the quality functional via Cross-Validation [see... ] in order to answer a question: what would be the shatter profile $D_m$ [see... ] to bound be exact?
The Effective Local Shatter Profile and Coefficient

- **Ideally accurate but “unfair” bound** answers a question: what would be the shatter profile $D_m$ to bound be exact?

- **Theorem**

  $$Q_{\epsilon,m}(\mu, X^L) = P_n \left[ \nu(f_n, X^L_n) - \nu(f_n, X^L_\ell) > \epsilon \right] \left[ \nu(f_n, X^L) = \frac{m}{L} \right]$$

  $$\leq D_m(\mu, X^L) \cdot H^{L \cdot s(\epsilon)};$$

  Let us change “$\leq$” by “$=$” and express $D_m$:

  - **Effective Local Shatter Profile** $\hat{D}_m, \ m = 0, \ldots, L$:

    $$\hat{D}_m = \frac{1}{|N'|} \sum_{n \in N'} \left[ \nu(f_n, X^L_n) - \nu(f_n, X^L_\ell) > \epsilon \right] \left[ \nu(f_n, X^L) = \frac{m}{L} \right] \cdot H^{L \cdot s(\epsilon)}.$$ 

  - **Effective Local Shatter Coefficient**: $\hat{\Delta}^L = \hat{D}_0 + \cdots + \hat{D}_L$. 

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We call the unfair estimate of shatter profile the \textit{Effective Local Shatter Profile} [see...] . Then we define the \textit{Effective Local Shatter Coefficient} as the sum of all profile components [see...] .
Empirical study

- The goal of experiment — to understand:
  - how tight these bounds are?
  - what effect is more important to bound tightening?

- Bounds to be compared:
  - The worst: classical VC bound
  - Local shatter coefficient bound
  - Local shatter profile bound
  - The best: effective (unfair) local shatter profile bound

- Testing area — the Rule Induction algorithm, because:
  - Global SC is well known
  - Local SC can be easily estimated during rule search

- Testing area — 7 tasks from UCI repository
The goal of our empirical study was to check the looseness of our bounds and to understand, what effects must be taken into account additionally? We compare all three bounds [see...] with an ideal [see...] . We choose a Rule induction System as a Testing Area because global and local shatter coefficients can be easily obtained for this kind of learning algorithms.
The rule definition

- Features $f_1(x), \ldots, f_n(x)$.

- Rule $\phi$ is a conjunction: $\phi(x, \theta) = \bigwedge_{f \in \Omega} \left[ f(x) \leq \theta_f \right]$, where $\Omega \subset \{f_1, \ldots, f_n\}$, $\theta_f$ — threshold for feature $f(x)$.

- Rule $\phi(x, \theta)$ is well interpretable while $|\Omega| \lesssim 5$.

- $\phi(x) = 1 \iff$ Rule $\phi(x)$ covers object $x \in X$.

- Rule $\phi_y(x)$ of class $y$ covers many objects from $y$ and none or a few objects from $\mathbb{Y} \setminus y$.

![Diagram showing rule coverage]

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A few words about what is rule induction. A rule is a well interpretable predicate, usually a conjunction, that covers many objects of one class and none [see...] or a few [see...] objects of other classes. For example this [see...] is not a rule because it covers both classes significantly.
Classifier is a combination of rules

- Decision List of rules:
  
  \[
  \begin{align*}
  \text{If } & \phi^1_{y_1}(x) \rightarrow f(x) := y_1; \\
  \text{If } & \phi^2_{y_2}(x) \rightarrow f(x) := y_2; \\
  & \ldots \\
  \text{If } & \phi^T_{y_T}(x) \rightarrow f(x) := y_T;
  \end{align*}
  \]

- Weighted Voting of rules:
  
  \[
  f(x) = \arg \max_{y \in \mathbb{Y}} \sum_{t=1}^{T_y} w^t_y \phi^t_y(x),
  \]

  where

  \[
  \phi^t_y(x) \text{ — } t\text{-th rule of class } y; \\
  w^t_y \text{ — its weight.}
  \]
A rule-based classifier can be considered as an ensemble of rules based on the principle of seniority voting [see...] or majority voting [see...].
### Results

<table>
<thead>
<tr>
<th>Task</th>
<th>$L$</th>
<th>Glob.SC</th>
<th>Loc.SC</th>
<th>Loc.S.Profile</th>
<th>Eff.Loc.SC</th>
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<td>$2.8 \cdot 10^8$</td>
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<tr>
<td>german</td>
<td>1000</td>
<td>$5.2 \cdot 10^8$</td>
<td>$3.1 \cdot 10^4$</td>
<td>$1.8 \cdot 10^4$</td>
<td>$47 \pm 38$</td>
</tr>
<tr>
<td>hepatitis</td>
<td>155</td>
<td>$5.5 \cdot 10^6$</td>
<td>$1.8 \cdot 10^4$</td>
<td>$8.4 \cdot 10^3$</td>
<td>$58 \pm 46$</td>
</tr>
<tr>
<td>horse-colic</td>
<td>300</td>
<td>$1.9 \cdot 10^6$</td>
<td>$1.3 \cdot 10^4$</td>
<td>$6.3 \cdot 10^3$</td>
<td>$5 \pm 3$</td>
</tr>
<tr>
<td>hypothyroid</td>
<td>3163</td>
<td>$5.3 \cdot 10^8$</td>
<td>$2.2 \cdot 10^4$</td>
<td>$9.2 \cdot 10^3$</td>
<td>$43 \pm 28$</td>
</tr>
<tr>
<td>liver</td>
<td>345</td>
<td>$1.5 \cdot 10^7$</td>
<td>$2.9 \cdot 10^4$</td>
<td>$9.5 \cdot 10^3$</td>
<td>$12 \pm 8$</td>
</tr>
<tr>
<td>promoters</td>
<td>106</td>
<td>$4.4 \cdot 10^9$</td>
<td>$5.3 \cdot 10^4$</td>
<td>$2.4 \cdot 10^4$</td>
<td>$13 \pm 4$</td>
</tr>
</tbody>
</table>

### Interpretation

In all tasks effective local SC $\ll L$.
Then the “effective local VC dimension” newer exceeds 1!
We measured complexity terms for seven real tasks from UCI repository [see...] and four types of bounds. The result is choking. None of the bounds can be called tight! Moreover, the effective local shatter coefficient [see...] is always significantly less than the sample size [see...] . This means that the effective local VC dimension never exceeds 1. So the VC dimension bears no relation to generalization bounds.
Directions of further investigation

- So, what causes of bound looseness did we miss?
  I have two ideas:

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So, what causes of bound looseness did we miss?

I have two ideas:

- Functions from \( \{ f_n \mid n = 1, \ldots, N \} \) have different chance to be obtained as a result of learning.
- Two similar functions from \( \{ f_n \mid n = 1, \ldots, N \} \) really are not 2 but \( \approx 1 \) function.
Define the *overfitting* of a rule $\phi(x)$ as

$$\delta(\phi, X^\ell, X^k) = \nu(\phi, X^k) - \nu(\phi, X^\ell)$$

Estimate (nonparametric) regression: how $\delta$ depends on rules properties measured on training set $X_n^\ell$ during inductive search:

- number of errors made on $X_n^\ell$;
- number of objects covered on $X_n^\ell$;
- number of terms in a rule;
- entropy of a rule on $X_n^\ell$;
- breadth and width of the inductive search;

We use these technique for:

- Controlling rules overfitting
- Risk Assessment — estimation of PD (probability of default) in Credit Scoring application: $PD(x) = \nu(\phi, X^\ell) + \delta(\phi)$
Example 1

<table>
<thead>
<tr>
<th>error rate</th>
<th>number of rules found</th>
</tr>
</thead>
<tbody>
<tr>
<td>av r train err</td>
<td>av r test err</td>
</tr>
<tr>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>100</td>
</tr>
</tbody>
</table>

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Example 2

**Error Rate**

- **Average training error (av r train err)**
- **Average test error (av r test err)**
- **Their difference (their diff)**
- **Confidence interval (conf.interval)**

**Number of Rules Found**

- **Entropy of rules on training set**

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Combinatorial Approach to Generalization Bounds Tightening
Example 3

The graph shows the error rate and the number of rules found as a function of the rating of the rule. The error rate is represented by four lines:
- **Blue line**: average training error (av r train err)
- **Green line**: average test error (av r test err)
- **Red line**: difference between training and test error (their diff)
- **Blue dotted line**: confidence interval (conf.interval)

The x-axis represents the number of rules found, and the y-axis represents the error rate. The graph also includes a shaded area indicating the confidence interval for the error rate.

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